A Deterministic Approach To Validate Universality of Collatz Conjecture

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Abstract: The Collatz conjecture suggests that that for any integer, $n \in Z^+$, iterating the function:

$$F(n) = \begin{cases} \frac{n}{2} & \text{if n is even} \\ 3n+1 & \text{if n is odd} \end{cases}$$

eventually leads to 1. In this paper we develop a clear algebraic framework to prove that this convergence occurs universally. We classify all positive integers into 16 distinct modulo residual classes, where all types follow a specific transformation pattern and divisibility behaviour under the Collatz map. This structure forms a finite-state transition system allowing us to discover modular residual non trivial loops using depth first search, out of which fewer looping sequences are increasing and most are decreasing. Our analysis proves and demonstrates that such increasing modulo type based looping sequences are inherently unstable with finite number of cycles. This eventually exhibits net contraction with division consistently dominating multiplicative growth. All together this Framework offers rigorous and structurally complete resolution of the conjecture.

Introduction:

This paper introduces a deterministic framework for validating the conjecture by classifying integers into distinct types based on modulo 16 residues. Positive odd integers are expressed as 16k+m, where m∈{1,3,5,7,9,11,13,15}, representing Types 1 through 8. Positive even integers are expressed as 16k+m′, where m′∈{0,2,4,6,8,10,12,14} representing EV1 through EV8.

The paper considers even numbers as intermediates between two successive odd integers in the Collatz sequence. Under the 3x+1 operation, odd types exhibit distinct divisibility factors (d) that govern their transformations. For instance:

- Types 1 and 5 become divisible by 4.
- Types 2, 4, 6, and 8 become divisible by 2.
- Type 3 becomes divisible by 2^n ($n \geq 4$).
- Type 7 becomes divisible by 8.

These divisibility properties lead to specific transformation rules. For example:

- Type 1 transforms into Types 1, 3, 5, or 7.
- Type 2 transforms into Types 3 or 7.
- Types 3 and 7 can transform into any odd type.
- Type 4 transforms into Type 2 or 6.
- Type 5 transforms into Type 2, 4, 6 or 8.
- Type 6 transforms into Type 1or 5.
- Type 8 transforms into Type 4 or Type 8 further.

Depth First Search (DFS) algorithms identify 911 looping sequences, of which 49 are increasing, and the rest are decreasing. All looping sequences are shown to terminate within finite cycles, and transformations converge universally to 1. The conjecture's universality is established by the absence of infinite looping, unbound growth and by the pigeonhole principle.

Methodology: The paper uses modulo residual classes of 16 as a tool for classification aiming to explore disciplined structures in Collatz sequence. A clarification is needed why other modulo classes are not used.
Modulo 8: We could have four residual classes of 8k + m: m = 1, 3, 5 and 7. Let's assign them as Type A, B, C and D respectively. This system captures less granules than 16k + m and creates confusion. Let's take transformations of Type D:

$8k + 7 \rightarrow 24k + 22$ (by 3x +1)

$24k + 22 \rightarrow 12k + 11$

Now, substituting k by 8k´ + m=> 12k + 11 = 96k´ + 12m + 11
For each values of m = 1, 3, 5, 7, 96k´ + 12m + 11 represents Type D integer. This implies, there is no escape route from Type D which indicates an unrealistic infinite looping or unbound growth. This is where modulo 16 offers more clarity. Each defined modulo 16 types has definite escape route(s) that adds new insight and transparency in understanding Collatz behaviour of all integers.
Modulo 32 or more: It is obvious that, higher modulo classes present more granules. A higher modulo with greater number of classes and much greater number of looping sequences would only aid to the complexity, and do not offer any new insight.
Therefore, by Occam's razor, modulo 16 classes are the optimal choice for this purpose.

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Section 1:
Lemma 0: **Classification**: All odd positive integers can be expressed in a form of 16k +m where m∈{1,3,5,7,9,11,13,15} and all even positive integers can be expressed as 16k + m´

Proof: When an odd positive integer >15 is divided by 16, there can be only eight values of remainder 'm' such that m = 1, 3, 5, 7, 9, 11, 13 and 15.
When an even positive integer >15 is divided by 16, there can be only eight values of remainder 'm´' such that m´ = 0, 2, 4, 6, 8, 10, 12 and 14. (k being the quotient in each case).

Table 1: Odd integers: Core integer 'k': Represents another positive integer – odd or even.

16k+m; m =	1	3	5	7	9	11	13	15
Defined as 'Type'	Type 1	Type 2	Type 3	Type 4	Type 5	Type 6	Type 7	Type 8

Table 2: Even Integers

16k+ m´; m´ =	0	2	4	6	8	10	12	14
Defined as 'Ev'	Ev 1	Ev 2	Ev 3	Ev 4	Ev 5	Ev 6	Ev 7	Ev 8

Lemma 1: Divisibility (d = 2^n): Odd integers belonging to each defined types on 3x+1 operation transform into the next odd integers when divided by 2^n, where n € Z^+
Proof:
Type 1: $16k + 1 \rightarrow 48k + 4$ (by 3x +1 operation).
$48k + 4 \rightarrow 12k + 1$ is an odd integer for any value of k. This gives n = 2 i.e. divisibility 'd' (= 2^n) = 4

Similarly,
Type 2: 16k +3 → 48k+10 => Gives n = 1 i.e. divisibility 'd' = 2
Type 3: 16k + 5 → 48k + 16 => Gives n = 4 or more (if k is odd), i.e. d = 16 or more
Type 4: 16k + 7 → 48k + 22 => Gives n = 1 i.e. divisibility 'd' = 2
Type 5: 16k + 9 → 48k + 28 => Gives n = 2 i.e. divisibility 'd' = 4
Type 6: 16k + 11 → 48k + 34 => Gives n = 1 i.e. divisibility 'd' = 2
Type 7: 16k + 13 → 48k + 40 => Gives n = 3 i.e. divisibility 'd' = 8
Type 8: 16k + 15 → 48k + 46 => Gives n = 1 i.e. divisibility 'd' = 2

Table 3: Divisibility Factors (d) for Each Odd Type Under the 3x+1 Operation:

Types	1	2	3	4	5	6	7	8
Divisibility(d)	4	2	2^n ($n \geq 4$)	2	4	2	8	2

Lemma 2: Integers transformation rules: On 3x + 1 operation followed by division by 'd'
1) Type 1 transforms into Types 1, 3, 5, or 7.
2) Type 2 transforms into Types 3 or 7.
3) Types 3 transforms into any odd type.
4) Type 4 transforms into Type 2 or 6.
5) Type 5 transforms into Type 2, 4, 6 or 8.
6) Type 6 transforms into Type 1or 5.
7) Type 7 transforms into any odd types
8) Type 8 transforms into Type 4 or Type 8 further.

Proof:
Type1 transformation:
Step I: 16k +1 → 48k + 4 (by 3x + 1 operation) → 12k + 1 (division by 4)
Step II: If the core integer 'k' belongs to Type1, substituting 'k' by 16x + 1 ($x \in Z^+$):
12(16x +1) +1 = 192x + 13 = 16(12x) +13 which is a type7 integer. Therefore, Type 1 integers transform into Type 7.
Likewise, if k belongs to Type 2, substituting 'k' by 16x+3:
12(16x + 3) + 1 = 192x + 37 = 16(12x + 2) + 5 => represents a Type 3 integer. Therefore, Type1 integers transform into Type 3 also.
The core integer 'k' when substituted by all even classes (Ev1 to Ev8) and by all odd classes (Type1 to Type 8), summarized results given in the following table:

Table 4: Transformation summary of Type 1:

'k' Substituted by	Ev 1,3,5,7	Type 2, 4,6,8	Ev 2,4,6,8	Type 1,3,5,7
Transforms into	Type 1	Type 3	Type 5	Type 7

With similar treatment on all the rest odd types, the following results are obtained:
Table 5: Transformation rules of Type2:

Type 2	**Type 3**	**Type 7**
Type of core k=>	**EV 1, 2, 3, 4, 5, 6, 7, 8** **(all even integers)**	**Type 1, 2, 3, 4, 5, 6, 7, 8** **(all odd integers)**

Illustrative Examples: Type2 integer = 16k + 3 → 48k + 10 (by 3x +1) → 24k + 5 (division by 2)
If k = 2n (all even integers), 24k + 5 = 48n + 5 = 16x3n + 5 => a Type 3 integer.
If k = 2n + 1 (all odd integers), 24k + 5 = 48n + 29 = 16(3n+1) +13 => Type7 integer.

Table 6: Transformation rules of Table3:

Type 3 to	**Type 1**	**Type 2**	**Type 3**	**Type 4**	**Type 5**	**Type 6**	**Type 7**	**Type 8**
For core integer, k	Ev 1: 16m + 0	Ev 4: 16m + 6	Ev 3: 16m + 4	Ev 2: 16m +2	Ev 5: 16m + 8	Ev 8: 16m+ 14	Ev 3: 16m + 4	Ev 6: 16m + 10
For core integer, k	Type 1: 16m +1 m =4n type even integers	Type 4: 16m +3 m = 1+2n type odd integers	Type 1: 16m +1 m =4n +3 type odd integers	Type 5: 16m + 9 m = 4n type even integers	Type 1: 16m +1 m =4n +2 type even integers	Type 4: 16m +3 m = 2n type even integers	Type 1: 16m +1 m =4n +1 type odd integers	Type 8: 16m+15 m = 2n +1 type odd integers
For core integer, k	Type 3: 16m + 5 m = 16n type even integers	Type 3: 16m + 5 m = 16n+6 type even integers	Type 2: 16m + 3 m = 2n type even integers	Type 3: 16m + 5 m = 16n + 2 type even integers	Type 3: 16m + 5 m = 16n + 8 type even integers	Type 3: 16m + 5 m = 16n + 14 type even integers	Type 2: 16m + 3 m = 2n +1 type odd integers	Type 3: 16m + 5 m = 16n + 10 type even integers
For core integer, k	Type 6: 16m + 11 m = 2n type even integers	Type 5: 16m + 9 m = 4n +1 type odd integers	Type 3: 16m + 5 m = 16n + 12 type even integers	Type 8: 16m+15 m = 2n type even integers	Type 6: 16m + 11 m = 2n+1 type odd integers	Type 5: 16m + 9 m = 4n +3 type odd integers	Type 3: 16m + 5 m = 16n + 4 type even integers	Type 5: 16m + 9 m = 4n +2 type even integers
For core integer, k	Type 7: 16m + 13 m = 8n +2 type even integers	Type 7: 16m + 13 m = 8n +5 type odd integers	Type 7: 16m + 13 m = 8n type even integers	Type 7: 16m + 13 m = 8n +3 type odd integers	Type 7: 16m + 13 m = 8n +6 type even integers	Type 7: 16m + 13 m = 8n +1 type odd integers	Type 7: 16m + 13 m = 8n +4 type even integers	Type 7: 16m + 13 m = 8n +7 type odd integers

Illustrative Examples:
Type 3 integer = 16k + 5 → 48k +16 (by 3x+1) → 3k +1 (division by 16)
If k = Ev4 = 16m + 6, 3k +1 = 48x + 19 = 16(3x +1) + 3 => a Type 2 integer.
If k = Ev5 = 16m + 8, 3k +1 = 48m + 25= 16(3m +1) + 9 => a Type 5 integer

If k = Type 1 = 16m + 1, and m = 4n +1, 3k + 1 = 3(64n +17) + 1 = 192n + 52 = 48n + 13 = 16x3n + 13 => a Type 7 integer.
If k = Type 8 = 16m + 15 and m = 2n, 3k +1 = 3(32m + 15)+1 = 96n + 46 = 48n + 23 = 16x(3n+1) + 7 => a Type 4 integer.

Table 7: Transformation rules of Type 4:

Type 4	Type 2	Type 6
Core integer, k =	Type 1, 2, 3, 4, 5, 6, 7, 8 (all odd integers)	Ev 1, 2, 3, 4, 5, 6, 7, 8 (all even integers)

Illustrative Examples: Type 4 integer = 16k + 7→ 48k + 22 (by 3x + 1) → 24k + 11
If k = 2n +1 (odd integers), 24k + 11 = 48n + 35 = 16 x(3n + 2) + 3 => a Type 2 integer.
If k = 2n (even integers), 24k + 11 = 48n + 11 = 16x3n + 11 => a Type 6 integer.

Table 8: Transformation rules of Type 5:

Type 5	Type 2	Type 4	Type 6	Type 8
Core integer, k =	Type 1, 3, 5. 7	EV 1, 3, 5, 7	Type 2, 4, 6, 8	EV 2, 4, 6, 8

Illustrative Examples: Type 5 integer =16k + 9 → 48k + 28 (by 3x +1) → 12k + 7 (division by 4)
If k = Type 3 = 16m + 5, 12k + 7 = 12(16m+ 5) + 7 = 192m + 67 = 16 (12m + 4) + 3 => a Type 2 integer.
If k = Ev 2 = 16m + 2, 12k + 7 = 12(16m + 2) +7 = 192m + 31 = 16(12m+1) + 15 => a Type 8 integer.

Table 9: Transformation rules of Type 6:

Type 6	Type 1	Type 5
Core integer, k =	Ev 1, 2, 3, 4, 5, 6, 7, 8 (even) (all even integers)	Type 1, 2, 3, 4, 5, 6, 7, 8 (odd) (all odd integers)

Illustrative Examples: Type 6 integer = 16k + 11 → 48k + 34 (by 3x +1) → 24k + 17 (division by 2)
If k = 2n (even), 24k + 17 = 48k + 17 = 16(3n+1) +1 => a Type 1 integer.
If k = 2n +1 (odd), 24k + 17 = 48k + 41 = 16(3n +2) + 9 => a Type 5 integer.

Table 10: Transformation rules of Type 7:

Type 7	Type 1	Type 2	Type 3	Type 4	Type 5	Type 6	Type 7	Type 8
k =	Ev 2, 6	Type 3, 7	Ev 1, 5	Type 2, 6	Ev 4, 8	Type 1, 5	Ev 3, 7	Type 4, 8

Illustrative Examples: Type 7 integer = 16k + 13 → 48k + 40 (by 3x + 1) → 6k + 5 (division by 8)
If k = Type 2 = 16m + 3, 6k + 5 = 96m + 23 = 16x (6m + 1) + 7 => a Type 4 integer.
If k = Ev 7 = 16m + 12, 6k + 5 = 96m + 77 = 16(6m + 4) + 13 => a Type 7 integer.
If k = Type 5 = 16m + 9, 6k + 5 = 96m + 59 = 16(6m + 3) + 11 => a Type 6 integer.

Table 11: Transformation rules for Type 8:

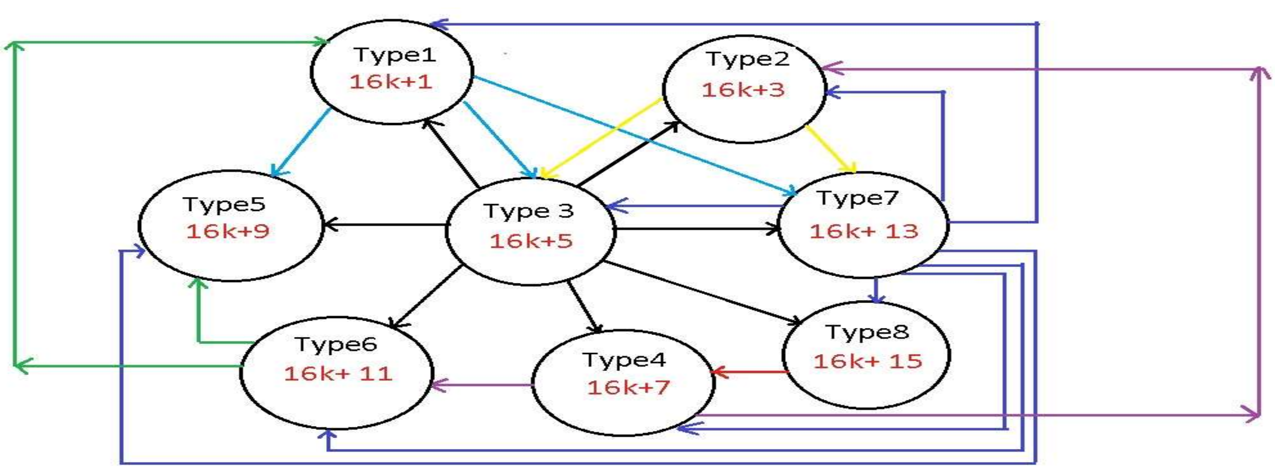
Type 8	Type 4	Type 8
Core integer, k =	Ev 1, 2, 3, 4, 5, 6, 7, 8 (even) (all even integers)	Type 1, 2, 3, 4, 5, 6, 7, 8 (odd) (all odd integer)

Illustrative Examples: Type 8 integer = 16k + 15 → 48k + 46 (by 3x + 1) → 24k + 23
If k = 2n (even), 24k + 23 = 48n + 23 = 16(3n +1) + 7 => a Type 4 integer.
If k = 2n + 1 (odd), 24k + 23 = 48n + 47 = 16(3n +2) + 15 => a Type 8 integer.

Table 12: Transformation Rules Summary of All Odd Integers:

Types	Transformed into	Forbidden Transformations	Maximum divisor (d)	Growth Tendency
Type 1	Type 1, 5 (for even core) Type 3, 7 (for odd core)	Type 2, Type 4, Type 6 and Type 8	4	Decreasing
Type 2	Type 3 (for even core), Type 7	Type 1, Type 2, Type 4, Type 5, Type 6 and Type 8	2	Increasing
Type 3	All	None	16, 32 or more	Decreasing
Type 4	Type 2 (odd), Type 6	Type 1, Type 3, Type 4, Type 5, Type 7 and Type 8	2	Increasing
Type 5	Type 2, Type 4, Type 6 and Type 8	Type 1, Type 3, and Type 5 and Type 7	4	Decreasing
Type 6	Type 5 and Type 1	Type 2, Type 3, Type 4, Type 6, Type7 and Type 8	2	Increasing
Type7	All	None	8	Decreasing
Type 8	Type 4 and Type 8	Type 1, Type 2, Type 3, Type 5, Type 6, Type 7	2	Increasing

Visual of Transformation Rules:



Type 3 and 7 are the most connected integers and Type 8 are the least connected integers.

Section 2: Forbidden Transformations: Some transformations, like Type 1 to Type 2, Type 4 to Type 8 or Type 8 to Type 6 etc. are mathematically impossible. These are 'forbidden transformations.' Following table summarizes all forbidden transformations:

Table 13:

Integer	Type 1	Type 2	Type 3	Type 4	Type 5	Type 6	Type 7	Type 8
Type 1		Forbidden		Forbidden		Forbidden		Forbidden
Type 2	Forbidden	Forbidden		Forbidden	Forbidden	Forbidden		Forbidden
Type 3								
Type 4	Forbidden		Forbidden	Forbidden	Forbidden		Forbidden	Forbidden
Type 5	Forbidden		Forbidden		Forbidden		Forbidden	
Type 6		Forbidden	Forbidden	Forbidden		Forbidden	Forbidden	Forbidden
Type 7								
Type 8	Forbidden	Forbidden	Forbidden		Forbidden	Forbidden	Forbidden	

Examples of Valid Transformation Sequences:

Example 1: **Type2** to **Type3** to **Type5** to **Type6** to **Type1** (non-recursive).

Example 2: **Type 8** to **Type 4** to **Type 2** to **Type 7** to **Type 6** to **Type 5** (non-recursive).

Example 3: **Type 6** to **Type 1** to **Type 3** to **Type 5** to **Type 6** (recursive: **Type 6** to **Type 6**).

Example 4: **Type 4** to **Type 2** to **Type 7** to **Type 8** to **Type 4**(recursive: **Type 4** to **Type 4**).

Section 3: Sequence Analysis: We have 8 odd modulo residual classes from Type 1 to Type 8. Therefore, minimum 1 and maximum 8 allowed transformations required to encounter a repetition or to form a looping sequence.

Let us assume there is a loop with n allowed transformations in which divisibility 'd' of modulo classes involved = 2^a, 2^b, 2^c, 2^d, 2^e, 2^f, 2^g, 2^h. Since there are 4 classes with divisibility, d = 2, two classes with d = 4, one with d = 8 and one with d = 16 or more, at most of the exponents a, b, c, … are equal to 1, at most two exponents = 2, at most one exponent = 3 and at most one exponent ≥ 4. If the starting integer be x, which is sufficiently large and numerically competent for a loop, we can write the following steps:

Step 1: 16k + m → $\frac{48k+3m+1}{2^a}$

Step 2: $\frac{48k+3m+1}{2^a} \rightarrow \frac{3^2\cdot 16k1+3^2m+3+2^a}{2^{a+b}}$

Step 3: $\frac{3^2\cdot 16k1+\ {}^2m+3+2^a}{2^{a+b}} \rightarrow \frac{3^3 16.k+3^3m+3^2+3.2^a+2^{a+b}}{2^{a+b+c}}$

Step 4: $\frac{3^3 16.k+3^3m+3^2+3.2^a+2^{a+b}}{2^{a+b+c}} \rightarrow \frac{3^4 16.k+3^4m+3^3+3^2\cdot 2^a+3.2^{a+b}+2^{a+b+c}}{2^{a+b+c+d}}$

Step 5: $\frac{3^4 16.k+3^4 m+3^3+3^2\cdot 2^a+3.2^{a+b}+2^{a+b+c}}{2^{a+b+c+d}} \rightarrow \frac{3^5 16.k+3^5+3^4+3^3.2^a+3^2 2^{a+b}+3.2^{a+b+c}+2^{a+b+c+d}}{2^{a+b+c+d+e}}$

For a sequence with n steps, if the modular form becomes $16k_n + m_n$, then,

$$16k_n + m_n = \frac{3^n 16.k + 3^n.m + 3^{n-1} + 3^{n-2}.2^a + 3^{n-3}2^{a+b} + \ldots\ldots\ldots + 2^{a+b+c+\cdots(n-1)th\ term}}{2^{a+b+c+d+e+\cdots nth\ term}} \quad \ldots\ldots \text{(i)}$$

Theorem 1: No odd positive integer >1 reappears in the Collatz sequence.

Proof: This deterministic framework provides us with the following constraints:
i) m and m_n is any odd positive integer such that, $1 \leq m, m_n \leq 15$
ii) n is any positive integer such that, $2 \leq n \leq 8$ as there are maximum 8 modular classes of odd numbers available as per lemma 0 and at least two steps are involved to form a looping sequence.
iii) k, k_n are positive integers.
iv) a,b,c,d, ….., n^{th} term are all exponentials as per lemma 1 and at most

four of which can be = 1
two of which can be = 2
one of which can be = 3
one of which can be ≥ 4

Now, if an integer reappears in the above sequence then, $16k + m = 16k_n + m_n$, (i) becomes,

$$16k + m = \frac{3^n 16.k + 3^n.m + 3^{n-1} + 3^{n-2}.2^a + 3^{n-3}2^{a+b} + \ldots\ldots\ldots + 2^{a+b+c+\cdots(n-1)th\ term}}{2^{a+b+c+d+e+\cdots nth\ term}}$$

$\Rightarrow 2^S.16k + 2^s.m = 3^n 16.k + 3^n.m + 3^{n-1} + 3^{n-2}.2^a + 3^{n-3}.2^{a+b} + \ldots\ldots\ldots + 2^{S'}$

$S = a+b+c+ \ldots.. + n^{th}$ term, $S' = a+b+c+ \ldots\ldots + (n-1)^{th}$ term

$$k = \frac{m(3^n - 2^S) + 3^{n-1} + 3^{n-2}.2^a + 3^{n-3}2^{a+b} + \ldots\ldots\ldots + 2^{S'}}{16.(2^S - 3^n)}$$

$$= -\frac{m}{16} + \frac{3^{n-1} + 3^{n-2}.2^a + 3^{n-3}2^{a+b} + \ldots\ldots\ldots + 2^{S'}}{16.(2^S - 3^n)} = \frac{1}{16}\,[\,R - m]$$

Where, $R = \frac{3^{n-1} + 3^{n-2}.2^a + 3^{n-3}2^{a+b} + \ldots\ldots\ldots + 2^{S'}}{(2^S - 3^n)} = \frac{N}{D}$

$N = 3^{n-1} + 3^{n-2}.2^a + 3^{n-3}2^{a+b} + \ldots\ldots\ldots + 2^{S'}$

Now, if all exponents are equal, i.e. a = b = c = …. = p, we get S′ = p + p + … up to q terms = q.p
And, S = p + p + p + …. up to n terms = n.p

$D = \frac{1}{2^S - 3^n} = \frac{1}{2^{n.p} - 3^n}$ and, $N = \sum_{x=0}^{n-1} (3)^{n-1-x}.2^{q.p}$

$N = 3^{n-1} \sum_{q=0}^{n-1} \left(\frac{2^p}{3}\right)^q$ => Represents a GP sum series with common ratio 'r' = $\frac{2^p}{3}$ and first term = $\frac{2^p}{3}$

$$= 3^{n-1} \cdot \frac{r^n - 1}{r - 1} = 3^{n-1} \cdot \left[\frac{\left(\frac{2^p}{3}\right)^n - 1}{\frac{2^p}{3} - 1}\right] = \frac{2^{n.p} - 3^n}{2^p - 3}$$

$$R = \frac{N}{D} = \left[\frac{2^{n.p} - 3^n}{2^p - 3}\right].\left[\frac{1}{2^{n.p} - 3^n}\right] \qquad \Rightarrow \qquad R = \frac{1}{2^p - 3}$$

The exponents are so constrained that they can be equal to 1 or 2 or 3 or ≥ 4

Case I: a = b = c = d = … = 1 $R = \frac{1}{2^1-3} = -1$

Case II: a = b = c = d = … = 2 $R = \frac{1}{2^2-3} = 1$

Case III: a = b = c = d =…. = 3 $R = \frac{1}{2^3-3} = \frac{1}{5}$

Case IV: a = b = c = d =….. = 4 +u $R = \frac{1}{2^{4+u}-3}$; R < 1

Case II is only producing an integer R = 1 which leads to $k = \frac{1}{16}[1 - m]$
m can be equal to all odd integers from 1 to 15 for odd integers presented as 16k +m.
Only m = 1 yields a valid result k = 0
So, by theorem 1, only one looping sequence can exist in which k = 0, m = 1, (a,b,c…) = (2,2,2…)
The starting integer = 16k+m = 16x0 +1 = 1
Deviations: Any of the exponents $a_j > 2$, all others remain same, symmetry in the numerator N disrupted. Let us consider following deviations from the uniformity:

A) **Arbitrary adjustments**: If $a_j = 2 + \partial$ and any other exponent a_i is adjusted such that, $a_i = 2 - \partial$,
$S = 2 + \partial + 2 - \partial + 2 + 2 + \ldots$ (up to n^{th} term) = 2n
$D = 2^{np} - 3^n = 4^n - 3^n$
The denominator remains same but some terms in the telescoping sum changes disrupting the symmetry: $R \neq Z^+$
B) **Increased or decreased exponents**: If at least one of the exponentials increases, $a_i = 2 + u$;
u => positive or negative integer
Let us rewrite the numerator $N = 3^{n-1} + T_1 + T_2 + T_3 + \ldots + 2^{S'}$
After introducing the deviation 'u', every 'T' picks up a factor 2^u :
$N' = 3^{n-1} + T_1.\,2^u + T_2.\,2^u + T_3.\,2^u + \ldots + 2^{S'}.\,2^u = 3^{n-1} + 2^u.(T_1 + T_2 + T_3 + \ldots + 2^{S'})$ (factoring out 2^u)
By definition, $3^{n-1} + T_1 + T_2 + T_3 + \ldots + 2^{S'} = N$
Therefore, $N' = 3^{n-1} + 2^u (N - 3^{n-1}) = 2^u.N + (1 - 2^u).3^{n-1}$
And adding up the deviation, the new denominator, $D' = 2^{2n+u} - 3^n$.
In this deterministic framework u may take the values 1, 2 or any integer ≥ 3.
$N' = 2^u.N - 3^{n-1}$ or, $2^u.N - 3.3^{n-1}$ or, $2^u.N - 7.3^{n-1}$ or, $2^u.N - 15.3^{n-1}$ etc.
Can be written as, $N' = 2^u.N - i.3^{n-1}$ (i € 1, 3, 7, 15 etc.)
$N' = 2^u.N - i.3^{n-1} = 2^u (2^{2n} - 3^n) - i.3^{n-1} = 2^{2n+u} - 3^n + 3^n - i.3^{n-1} = D' + 3^n - i.3^{n-1}$.
$R = \frac{N'}{D'} = \frac{D'+3^n-i\cdot 3^{n-1}}{D'} = 1 + \frac{3^n-i\cdot 3^{n-1}}{D'}$ => cannot be an integer as $D' > 3^n - i\cdot 3^{n-1}$
C) Even if more exponents are altered, $N' = 3^{n-1} + T_1.\,2^u + T_2.\,2^{u+v} + T_3.\,2^{u+v+l} + \ldots + 2^{S'+u+v+l+\ldots}$.
And, $D' = 2^{S+u+v+l+\ldots} - 3^n$.

The term 3^{n-1} remains invariant, while subsequent terms scale by 2^u, 2^{u+v}, $2^{u+v+l+\ldots}$ etc. This disrupts the telescoping sum, as the weights 3^{n-1} and 2^S no longer align. Numerator changes polinomially, sub-exponentially whereas the denominator changes exponentially keeping $R \neq \mathbb{Z}^+$
Therefore, only 1- 4 - 1 i.e. 4 – 2 - 1 loop exists and no odd positive integer >1 reappears. (Proved)

Corollary: If no odd integer>1 reappears in Collatz sequence, no even integers > 4 reappears in Collatz sequence.

Theorem 2: An integer is capable of forming r cycles, it cannot form (r+1) cycles in a modular loop.

Proof: For a sequence with n (n>1) steps, if the modular form becomes $16k_n + m$, from $16k_1 + m$ then, after first cycle:

$$16k_n + m_n = \frac{3^n 16.k + 3^n.m + 3^{n-1} + 3^{n-2}.2^a + 3^{n-3}2^{a+b} + \ldots\ldots + 2^{a+b+c+\cdots(n-1)th\ term}}{2^{a+b+c+d+e+\cdots nth\ term}}$$

$S = a + b + c + \ldots. + n^{th}$ term

$$k_n = \frac{3^n 16.k + (3^n - 2^S).m + 3^{n-1} + 3^{n-2}.2^a + 3^{n-3}2^{a+b} + \ldots\ldots + 2^{a+b+c+\cdots(n-1)th\ term}}{16.2^S}$$

This takes the form of $k_n = \dfrac{3^n 16.k + z}{2^p}$,

Where $z = (3^n - 2^S).m + 3^{n-1} + 3^{n-2}.2^a + 3^{n-3}2^{a+b} + \ldots\ldots\ldots + 2^{a+b+c+\cdots(n-1)th\ term}$

And, $16.2^S = 2^{s+4}$ is written as 2^p .

This means, k is such a unique core integer due to which 16k+m is capable of completing 1 cycle of this loop.

After 2^{nd} cycle, k becomes k_n, therefore, substituting k by k_n,

$$k_{n+1} = \frac{3^n(kn).16.+z}{2^p} = \frac{3^{2n}16^2.k + 3^n.16.z + 2^p.z}{2^{2.p}}$$

Similarly, after r^{th} cycle, $k_{n+r} = \dfrac{3^{r.n}16^r.k + 3^{(r-1).n}.16^{r-1}.z + 3^{(r-2).n}.16^{r-2}.z + \ldots\ldots\ldots 2^{(r-1)p}.z}{2^{r.p}}$

Or, $k_{n+r} = \dfrac{3^{r.n}16^r.k + Z'}{2^{r.p}}$; $z' = 3^{r.n-1}.16^{r-1}.z + 3^{r.n-2}.16^{r-2}.z + \ldots\ldots\ldots.2^{(r-1)p}.z$

$$k = \frac{2^{r.p}(K\_n+r) - Z'}{3^{r.n}16^r}$$ => k is a unique core integer that is capable of completing r cycles.

And, after $(r+1)^{th}$ cycle, $k_{n+r+1} = \dfrac{3^{(r+1).n}16^r.k + 3^{r.n}.16^r.z + 3^{(r-1).n}.16^{r-1}.z + \ldots\ldots\ldots 2^{r.p}.z}{2^{(r+1).p}}$

Or, $k_{n+r+1} = \dfrac{3^{r.n}16^r.k + z''}{2^{r.p}}$; $z'' = 3^{(r+1).n-1}.16^r.z + 3^{r.n}.16^r.z + \ldots\ldots\ldots.2^{r.p}.z$

$$k = \frac{2^{(r+1).p}(K_n+r+1) - z''}{3^{(r+1).n}.16^r}$$ => k is a unique core integer that is capable of completing (r+1) cycles of the same loop.

Evidently, $\dfrac{2^{r.p}(K\_n+r) - Z'}{3^{r.n}16^r} \neq \dfrac{2^{(r+1).p}(K_n+r+1) - z''}{3^{(r+1).n}16^r}$

This shows, the core integer capable of completing r cycles cannot complete (r+1) cycles. Therefore, every loop is bound to a finite number of cycles. (Proved)

Section 4: Tracking of Transformation Paths: With the classification of integers and transformation rules, randomness of the conjecture is replaced by strict mathematical discipline. It is now essential to track all possible transformation paths between the defined types. Depth First Search (DFS) algorithm are the most efficient tool for this purpose. Using a DFS python code, **911 looping sequences** and **692 non-looping** are exhaustively tracked. The DFS code is attached in the appendix section. Out of the

911 looping sequences and **692 non-looping** sequences, it is found that:
1) only **49** found are of increasing or diverging growth tendency,
2) only **22** are found to be increasing/diverging,
Non-looping sequences are less relevant in proving convergence/divergence nature of the conjecture. Diverging looping sequences, if found stable, lead to disproving the conjecture whereas unstable or bounded growth indicates a universal convergence. All increasing looping sequences and some samples of decreasing looping sequences are displayed in the appendix section.

Lemma 3: Growth tendency of looping and non-looping sequences is determined by comparing accumulation of power of 3 in numerator with accumulation of power of 2 in denominator.
Proof: an odd integer x, after n transformations forms odd integer =

$$x \rightarrow \frac{3x+1}{2^a} \rightarrow \frac{3^2x+3+2^a}{2^{a+b}} \ldots \rightarrow \frac{3^n+3^{n-1}+3^{n-2}.2^a+3^{n-3}.2^{a+b}+\cdots+2^{a+b+c+\cdots(n-1)th\ term}}{2^{a+b+c+\cdots nt\ \ term}} = \frac{3^n.x+T}{2^p}$$

For large value of x, T is negligible. Therefore, if $3^n > 2^p$ then the sequence is increasing and if $3^n < 2^p$, it is decreasing.
Threshold ratio: The ratio, that decides whether a sequence is increasing or decreasing is the ratio of p to n such that $p > n.\log_2 3$
$\log_2 3 \approx 1.585$, hence, $p > 1.585.n$
For example, n = 3, threshold $p \geq (1.585 \times 3) \approx 5$ will hold $2^p > 3^n$. This concept aligns with **Terence Tao**'s probabilistic observation of threshold ratio $\frac{number\ of\ even\ steps}{number\ of\ odd\ steps} \approx \log_2 3 \approx 1.585$

Demonstrative Examples:
A) Looping sequence: **Type 2** to **Type 7** to **Type 8** to **Type 4** to **Type 2**:
Step I: Starting Type 2 = 16k + 3 → 24k + 5 (3x + 1, followed by division by 2).
Step II: 24k + 5 (Type 7, according to sequence) → 9k + 2 (3x + 1, followed by division by 8).
Step III: 9k + 2 (Type 8) → $\frac{27k+7}{2}$ (3x + 1, followed by division by 2).
Step IV: $\frac{27k+7}{2}$ (Type 4) → $\frac{81k+2}{4}$ (3x + 1, followed by division by 2)
The filial modulo class of the loop = Type 2 (say, 16k′ +3) = $\frac{81k+}{4}$.
k′ = $\frac{81k+}{64}$ => This equation is satisfied by the unique k = 37 + 64.n yielding k′ = 47 + 81.n
The loop will continue for a single cycle if initiated by the core integer k = 37 + 64.n only.
Growth tendency = (numerator's power of 3)/(denominator's power of 2) = $3^4/ 2^6 > 1$ = increasing.

B) Non-looping sequence: **Type 1** to **Type 3** to **Type 2** to **Type 7** to **Type 5** to **Type 6**:
Step I: Starting Type 1 =16k+ 1 → 12k + 1 (3x + 1 operation followed by division by 4).
Step II: 12k + 1 (Type 3, according to sequence) → $\frac{9k+1}{4}$ (3x + 1, followed by division by 16).
Step III: $\frac{9k+1}{4}$ (Type 2, according to sequence) → $\frac{27k+}{8}$ (3x + 1, followed by division by 2).
Step IV: $\frac{27k+7}{8}$ (Type 7, according to sequence) → $\frac{81k+2}{64}$ (3x + 1, followed by division by 8).
Step V: $\frac{81k+}{64}$ (Type 5, according to sequence) → $\frac{243k+151}{256}$ (3x + 1, followed by division by 4)
The last modulo class Type 6 (say, 16k′+11) = $\frac{243k+151}{256}$

=> $k' = \frac{243k-2665}{4096}$: The equation is satisfied by k = 3635 + 4096.n

yielding $k' = 215 + 243.n$ ($n \in Z^+$).

This path is initiated only by core integer = 3635 + 4096.n type only.

Growth tendency = (power of 3 in numerator)/(power of 2 in denominator) = $3^5/ 2^{12} <1$ = decreasing.
Similarly, Type 1- 3 – 2 – 7 – 5 – 3 – 4 – 6 path initiated by core integer, k = 2611 + 16384.n only.

Table 14: Class-wise Analysis of Non-Looping Sequences:

Types	Total Non-recurrent Sequences	Decreasing	Increasing
Type1 (16k+1)	115	112	03
Type 2 (16k+3)	91	88	03
Type 3 (16k+5)	116	116	00
Type 4 (16k+7)	72	70	02
Type 5 (16k+9)	75	71	04
Type 6 (16k+11)	74	72	02
Type 7 (16k+13)	114	112	02
Type 8 (16k+15)	35	29	06
All Types	692	670	22

Section 5: Samples of Looping sequences:

Case 1: **Type 4** to **type 2** to **type 7** to **type 8** to **type 4:** Let the parent integer be 16k+7.

1^{st} Cycle: 16k+7 → 24k+ 11 (by 3x+1 operation, followed by division by 2 as divisibility of type4 is 2)

24k+11 → 36k+17 (by 3x+1 operation, followed by division by 2 as divisibility of type2 is 2).

36k+17 → $\frac{27k+1}{2}$ (by 3x+1 operation, followed by division by 2 as divisibility of type7 is 8).

$\frac{27k}{2}$ → $\frac{81k+4}{4}$ (by 3x+1 operation, followed by division by 2 as divisibility of type8 is 2).

2^{nd} Cycle: $\frac{81k+41}{4} \rightarrow \frac{243k+127}{8} \rightarrow \frac{729k+389}{16} \rightarrow \frac{2187k+1183}{128} \rightarrow \frac{6561k+367}{256}$.

3^{rd} Cycle: $\frac{6561k+367}{256} \rightarrow \frac{19683k+1128}{512} \rightarrow \frac{59049k+3437}{1024} \rightarrow \frac{177147k+1041}{8192} \rightarrow \frac{531441k+320621}{16384}$.

The last term of the loop represents filial Type 4 integer and may be represented as 16m +7

Therefore, $16k'+7 = \frac{531441k+320621}{16384}$

$k' = \frac{531441k+20599}{262144}$ This equation is satisfied by the general expression k = 246723 + 262144.n which yields $k' = 500179 + 531441.n$ ($n \in Z^+$)

The parent integer should be 16k+7 = 3947575 +2^{22}.n
The calculations show that the said looping sequence initiated by **3947575** +2^{22}.n type of integer will continue for 3 full cycles and will reach to 16m+7 = **8002871** + 8503056.n. Let us verify with n = 0 :

1^{st} Cycle: 3947575(type4) → 5921363(type2) → 8882045(type7) → 3330767(type8)
→ 4996151(type4) → 7494227(type2) → 11241341(type7) → 4215503(type8) → 6323255(type4)

2^{nd} Cycle: 6323255(type4)→ 9484883(type2) → 14227325(type7) → 5335247(type8) → **8002871(=**

16 x 500179 +7 => type4)

And then, the 3rd cycle: 8002871(type4) → 12004307 (type2) → 18006461(type7) → 6752423 (type4)
So, **the loop terminates after 3rd cycle due to type mismatch**.
Case 2: **Type 2** – **type 7** – **type 8** – **type 4** – **type 6** – **type 1** – **type 5** – **type 2**:
Parent integer = 16k+3

1st Cycle: 16k+3 → 24k+5 → 9k+2 → $\frac{27k+}{2}$ → $\frac{81k+23}{4}$ → $\frac{243k+}{8}$ → $\frac{729k+227}{32}$ → $\frac{2187k+713}{128}$

2nd Cycle: $\frac{2187k+713}{128}$ → $\frac{6561k+2267}{256}$ → $\frac{19683k+705}{2048}$ → $\frac{59049k+232}{4096}$ → $\frac{177147k+73753}{8192}$ →

$\frac{531441k+22}{16384}$ → $\frac{1594323k+704}{65536}$ → $\frac{4782969k+217}{262144}$

After 2nd cycle, the filial integer, say, 16m+3 = $\frac{4782969k+217}{262144}$ => m = $\frac{4782969k+1393}{262144x\ 16}$
The equation is satisfied by k = 232346 + 2^{22}.n yielding m = 2649556 + 3^{14}.n

Therefore, this loop with two cycles is initiated by the parent integer = 16k+7 = 37175379 + 2^{26}.n and the filial integer will be 16m+7 = 42392899 + 76527504.n

Demonstration (with n = 0):
1st Cycle: 37175379 (type2) → 55763069 (type7) → 20911151 (type8) → 31366727 (type4) → 47050091(type6) → 70575137 (type1) → 52931353 (type5) → 39698515 (type2)
2nd Cycle: 39698515 (type2) → 59547773 (type7) → 22330415(type8) → 33495623(type4) → 50243435 (type6) → 75365153 (type 1) → 56523865 (type5) → **42392899** (type2)
And then, 3rd cycle: 42392899 (type2) → 63589349 (type3) : 2-7-8-4-6-1-5-2 sequence terminates.

Case 3: **Type 6** – **Type 5** –**Type 6** loop: Parent integer = 16k+11
1st Cycle: $16k_1+11$ → 24k + 17 → $18k_1 + 13$;
$16k_2 + 11 = 18k_1 +13$
$16k_2 = 18k_1 + 2$
$k_2 = \frac{18k+2}{16} = \frac{9k+1}{8}$
2nd Cycle: $k_3 = \frac{9k2+}{8} = \frac{9\{(9k+1)/8\}+1}{8} = \frac{81k+}{64}$
3rd Cycle: $k_4 = \frac{9k3+1}{8} = \frac{9\{(81k+\quad)/64\}+1}{8} = \frac{729k+217}{512}$
4th Cycle: $k_5 = \frac{9k4+1}{8} = \frac{9\{(729k+217)/512\}+1}{8} = \frac{6561k+246}{4096}$
=> k5 = $\frac{6561k+246}{4096}$; the equation is satisfied by k = 4095 + 2^{12}.n yielding $k_5 = 6560 + 3^8$.n

Therefore, the loop initiates with parent integer 16k+11 = **65531** + 2^{16}.n and terminates after 4 cycles with filial integer 16m + 11 = **104971**+ 104976.n
Demonstration with n = 0
1st Cycle: **65531** (type6) → 98297 (type5) → 73723 (type6)

2nd Cycle: 73723 (type6) → 110585 (type5) → 82939 (type6)

3rd Cycle: 82939 (type6) → 124409 (type5) → 93307 (type6)

4th Cycle: 93307 (type6) → 139961 (type5) → **104971** (type6)

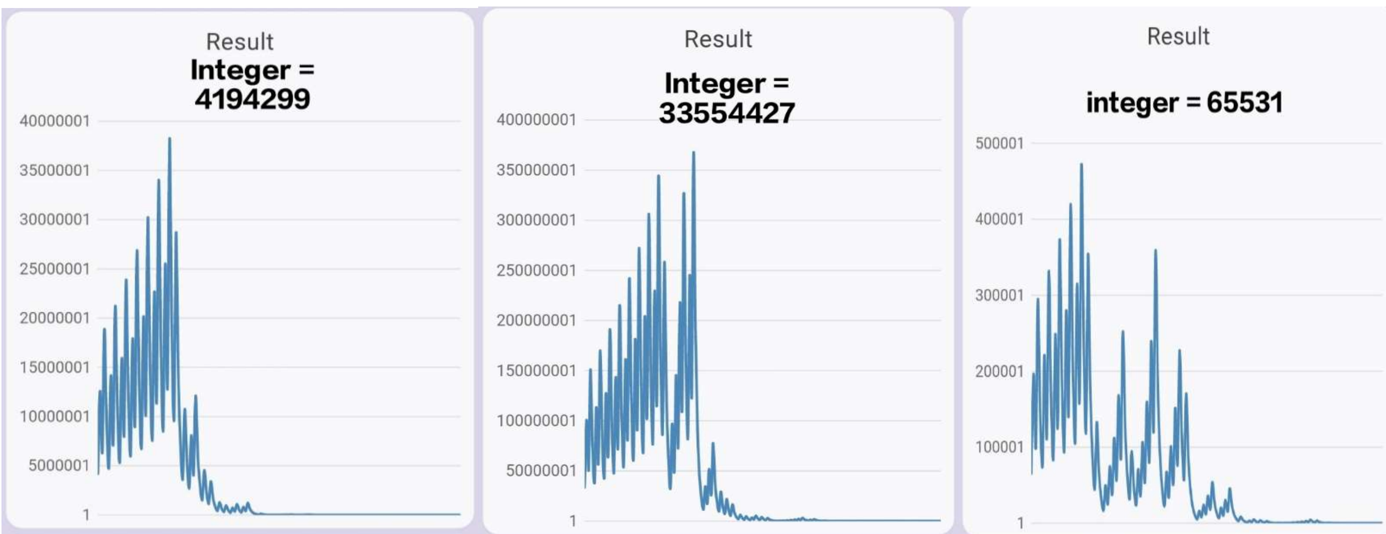
And then: (type6) → 157457 (= 9841 x 16 +1 =>**type1**): The loop terminates as 6-5-6 sequence breaks at this point.

5th Cycle: $k_6 = \frac{9k5+1}{8} = \frac{9\{(6561k+2465)/4096+}{8}$

$k_6 = \frac{3^{10}k+26281}{2^{15}}$ is satisfied by $k = 32767 + 2^{15}.n$

The starting integer which is capable of forming 5 cycles is 16x32767 + 11 = 524283

Similarly, integers having core integers of the series $262143 + 2^{18}.n$ is capable of forming 6 cycles, $2097151 + 2^{21}.n$ of 7 cycles, $16777215 + 2^{24}.n$ of 8 cycles etc. Loop will eventually terminate after the defined number of cycles in each case. Visuals of the cycles are presented in the following:



Section 6: Infinite Oscillation: In this conjecture, some hypothetical integers are believed to yield infinitely alternative odd-even parity patterns upon 3x+1 operation followed by division by 2. To dissolve this hypothesis, we shall demonstrate Type 8 integers having odd core value (k) those can only transform to Type 8 on repeated iterations.

We take an extreme $(1111....)_2$ parity pattern, i.e. 16k + 15, where, $k = 2^x - 1$.

$16k + 15 = 16(2^x - 1) + 15 = 2^{x+4} - 16 + 15 = 2^{x+4} - 1 = 2^n - 1$ type integer.

By 3x + 1 operation, $2^n - 1 \rightarrow 3.2^n - 2 \rightarrow 3^n - 1$: This can be shown by stepwise synthetic deformation:

$N_1 = 3N_0 + 1 = 4 + 3\,(2 + 2^2 + 2^3 + \ldots + 2^n)$ (Performing 3x + 1 operation)

Or, $N_2 = 2 + 3\,(1 + 2 + 2^2 + 2^3 + \ldots\ldots + 2^{n-1}) = 5 + (2 + 2^2 + 2^3 + \ldots\ldots + 2^{n-1})$ – this is evidently odd term.

Or, $N_3 = 3N_2 + 1 = 16 + 3^2\,(2 + 2^2 + 2^3 + \ldots\ldots + 2^{n-1})$

Or, $N_4 = 8 + 3^2\,(1 + 2 + 2^2 + 2^3 + \ldots\ldots + 2^{n-2}) = 17 + 3^2(2 + 2^2 + 2^3 + \ldots\ldots + 2^{n-2})$ – Odd term

Or, $N_5 = 3N_4 + 1 = 52 + 3^3(2 + 2^2 + 2^3 + \ldots\ldots + 2^{n-2})$

Or, $N_6 = = 26 + 3^3(1 + 2 + 2^2 + 2^3 + \ldots\ldots + 2^{n-3}) = 53 + 3^3(2 + 2^2 + 2^3 + \ldots\ldots + 2^{n-3})$ – Odd term

Or, $N_7 = 3N_6 + 1 = 160 + 3^4(2 + 2^2 + 2^3 + \ldots\ldots + 2^{n-3})$

Or, $N_8 = 80 + 3^4(1 + 2 + 2^2 + 2^3 + \ldots + 2^{n-4}) = 161 + 3^4(2 + 2^2 + 2^3 + \ldots + 2^{n-4})$ – Odd term.

A closer look at the integers generated as odd terms reveals that, they bear a common form of $2.3^{x+1} - 1$ like, $2.3^2 - 1 = 17$, $2.3^3 - 1 = 53$, $2.3^4 - 1 = 161$ …..

It can be concluded, all $2^n - 1$(Type 8) integers transform into $3^n - 1$ representing some other modulo residual form than Type 8. Following table depicts some aligned results:

Table 15:

n	$2^{n+1} - 1$	$(3^{n+1} - 1)/2^x$	Type of $(3^n - 1)/2^x$	Number of steps taken to generate
4	31	121	Type 1	10+1
5	63	91	Type 6	12 +3
6	127	1093	Type 3	14+1
7	255	205	Type 7	16+ 5
8	511	9841	Type1	18 +1
9	1023	7381	Type 3	20+3
16	393214	64570081	Type 1	34+1
21	4194303	1961316225	Type 1	44+4

Evidently, boundary integers having $(11111\ldots1)_2$ binary pattern, generate alternative even-odd integers unless the power of 2 exhausted. Thereafter these integers transform into $3^n - 1$ type with disrupted symmetry of binary pattern. $3^n - 1$ integers converge rather easily than Type8.

Section 7: Some Tangible Results: Enormous Numbers With Shortest Path of Convergence:

The shortest route to convergence is widely discussed $\frac{2^{2n}-1}{3}$ ($n \in Z^+$) Some examples: 5, 21, 341 …

These are all 16k+5 i.e Type 3 integers. As per transformation rules, shortest routes to Type 3 are:

1) **Type1** to **Type 3**,
2) Type 2 to Type 3,
3) Type 3 to Type 3,
4) Type 4 to Type 2 to Type 3,
5) Type 5 to Type 2 to Type 3,
6) Type 6 to Type 1 to Type 3,
7) Type 7 to Type 3,
8) Type 8 to Type 4 to Type 2 to Type 3.

We'll form mathematical equations for all transformations. To establish the principle, lets demonstrate the last and the longest route:

Step I: Type8: 16k+5 → 24k+23 (3x+, followed by division by 2) => This is Type 4

Step II: 24k+23 → 36k + 35 => This is Type 2

Step III: 36k + 35 → 54k + 53 => This is Type 3

To adopt the shortest path, Type 3 must be = $\frac{2^{2n}-1}{3}$ = 54k + 53

Solving, 16k + 15 = $\frac{2^{2n+3}-65}{81}$ in the equation, n has a periodicity = 14 + 27m (m ∈Z^+) and after a few iterations, it reaches 2^{2n}. 2^{2n} takes 2n more steps to reach 1.

All integers in this scope are necessarily Type8 and follow the shortest convergence route to unity.

Let's verify:

For m =0, n = 14, $\frac{2^{2n+3}-65}{81}$ gives = 26512143 = 16 x 1657008 +15 - a Type 8 integer.

Convergence of 26512143 will have 3 more odd integers (Type4, Type2 and Type3) and 3 more odd integer in between before reaching 2^{28}.

For m =1, n = 41 $\frac{2^{2n+3}-65}{81}$ gives = **477600323798372019637007** - a Type8 integer.

Convergence of this huge number will generate three more odd integers (Type 4, Type 2 and Type 3) and two more even integers in between before reaching $\mathbf{2^{82}}$.

Step I: 477600323798372019637007→ 1432800971395116058911022 (even) – By 3x+1

Step II: 1432800971395116058911022 → **716400485697558029455511**(odd: Type 4) division by 2.

Step III: 716400485697558029455511→ **2149201457092674088366534** (even).

Step IV: 2149201457092674088366534 → **1074600728546337044183267** (odd: Type 2).

Step V: 1074600728546337044183267 → **3223802185639011132549802** (even).

Step VI: 3223802185639011132549802→ **1611901092819505566274901**(odd: Type 3)

Step VII: 1611901092819505566274901 → **4835703278458516698824707** (even) = $\mathbf{2^{82}}$

These 24 digit odd numbers of quintillion magnitude can be predicted without any computing machine's validation. Next number in this series comes up with a dimension of $\mathbf{2^{139}}$ and behaves in the same way. Sets of such enormous integers belonging to other modulo classes are listed in the following:

1) Type1 = $\frac{2^{2n+2}-7}{9}$ n = 3m +1 (m ∈ Z^+)

2) Type 2 = $\frac{2^{2n+1}-5}{9}$ n = 3m +2 (m ∈ Z^+)

3) Type 3 = $\frac{2^{2n}-1}{3}$ n = 3m +2 (m ∈ Z^+)

4) Type 4 = $\frac{2^{2n+2}-19}{27}$ n = 9m +5 (m ∈ Z^+)

5) Type 5 = $\frac{2^{2n+3}-29}{27}$ n = 9m +8 (m ∈ Z^+)

6) Type 6 = $\frac{2^{2n+3}-23}{27}$ n = 9m +4 (m ∊ Z^+)

7) Type 7 = $\frac{2^{2n+3}-11}{9}$ n = 3m +2 (m ∊ Z^+)

Conclusion: In addition to computational methods, modulo- residual classes are also useful in validating integers of enormous size.

Section 8: Demonstration of Real Some Looping sequences:

Example 1: 27(16 x 1 + 11= >Type 6) → 41 → 31 → 47 → 71 → 107 → 161 → 121 → 91 → 137 → 103 → 155 → 233 → 175 → 263 → 395 → 593 → 445 → 167 → 251 → 377 → 283 → 425 → 319 → 479 → 719 → 1079 → 1619 → 2429 → 911 → 1367 → 2051 → 3077 → 577 → 433 → 325 → 61 → 23 → 35 → 53 → 5 → 1

Types of the above integers in the same sequence: Type 6 → Type 5 → Type 8 → Type 8 → Type 4 → Type 6 → Type 1 → Type 5 → Type 6 → Type 5 → Type 4 → Type 6 → Type 5 → Type 8 → Type 4 → Type 6 → Type 1 → Type 7 → Type 4 → Type 6 → Type 5 → Type 6 → Type 5 → Type 8 → Type 8 → Type 8 → Type 4 → Type 2 →> Type 7 → Type 8 → Type 4 → Type 2 → Type 3 → Type 1 → Type 1 → Type 3 → Type 7 → Type 4 → Type 2 →> Type 3 → Type 3 → Type 1
An illustration, how 27, a Type 6 integer has followed loops like 5-6-5, 8-8, 8-4-2-7-8. All loops being unstable, it has converged to unity.

Example 2: 431 (16 x 16 + 15 => Type 8) → 647 (16 x 40 + 7 => Type 4) → 971 (16 x 60 + 11 => Type 6) → 1457 (16 x 91 + 1 => Type 1) → 1093 (16 x 68 + 5 => Type 3) → 205 (16 x 12 + 13 => Type 7) → 77 (16 x 4 +13 => Type 7) → 29 (16 x 1 + 13 => Type 7) → 11 (16 x 0 + 11 => Type 6) → 17 (16 x 1 +1 => Type 1) → 13 (16 x 0 +13 => Type 7) → 5 (16 x 0 + 5 => Type 3) → 1 (Type 1)
Looping sequences: (Type 7 – Type 6 – Type 1 – Type 7) and (Type 1 – Type 7 – Type 3 – Type 1) and (Type 7 – Type 7 – Type 7).

Section 9: Convergence Argument:

A Collatz sequence can logically have only three outcomes:
1) An unbound growth,
2) An infinite loop with sustaining magnitude, and,
3) Convergence to unity.

Theorem 1 and lemma 3 effectively rules out existence of non-trivial integer loops while theorem 2 eliminates all possibilities of unbound growth due to occurrence of increasing modular loops.

Applicability of Pigeonhole Principle:
1) All odd integers are confined within the finite framework of 8 modulo residual classes. This ensures all odd integers in the sequence, after transformations, must repeatedly fall into one of these modular classes and revisit previous modular classes. Type 1, 3, 5, and 7 (divisibility = 2^2, 2^4, 2^2 and 2^3 respectively) contribute to a net reduction in magnitude due to their divisibility properties. This reduction outweighs the cumulative growth induced by all Types.
3) No integer can reappear in a Collatz sequence by theorem 1 and lemma 3.

4) The pigeonhole principle guarantees that sequences confined to finite constraints must reduce. As reduction is inevitable in Collatz conjecture, convergence to 1 is the only outcome.

Section 10: Research Outcomes:

1) **Universal classification of integers**: 16k + m : no integer left out of this classification.

2) **Resolution of apparent chaos**: The long-standing perception that Collatz conjecture is chaotic and cannot be predicted is hereby resolved. With the transformation rules, it is possible to predict convergence path of any integer.

3) **Non-trivial loop redefinition**: Perceived existence of non-trivial loop is hereby affirmed and redefined: a non-trivial loop is a cyclic path of modulo residual patterns in Collatz sequence, not an infinite cycle with recurrence of individual integers. This non-trivial loop of modulo residual classes actually reinforce the statement of the conjecture rather than setting counterexamples.

4) **Universal and inevitable convergence**: As there is no unbound growth and a huge majority of converging loops, a universal convergence is established by ***pigeonhole principle***.

Appendix Section:

Appendix: A: DFS (Python) code:

Code begins:

```
def find_cycles(graph, start_node, current_node, visited, path, results):
    # Add current node to the path and mark as visited
    path.append(current_node)
    visited.add(current_node)
    # Check if we looped back to the start node
    if current_node == start_node and len(path) > 1:
        results.append(list(path))
    else:
        # Traverse each neighbor
        for neighbor in graph[current_node][0]:
            if neighbor not in visited or neighbor == start_node:
                find_cycles(graph, start_node, neighbor, visited.copy(), path[:], results)
    # Remove current node from path after recursion
    path.pop()
count_total = 0
count_convergent = 0
count_divergent = 0
save_str = ""
def print_and_save(string):
    global save_str
    save_str += string + "\n"
    print(string)
def analyze_graph(graph):
    global count_total, count_convergent, count_divergent
    for node in graph:
        results = []
        find_cycles(graph, node, node, set(), [], results)
        # Print table header
        print(f"\nNode {node} Cycles:")
        print(f"{'Path':<50} | {'Sum of Change Factors':<40} | {'Status'}")
        print()

        # Print each cycle in table format
        for path in results:
            # Calculate the sum of change factors for the path
            change_sum = [graph[node][1] for node in path][:-1]
            status = "Convergent" if sum(change_sum) <= 0 else "Divergent"

            path_str = " -> ".join(map(str, path))
            print_and_save(f"{path_str:<50} | {str(change_sum):<40} | {status}")
            # Count totals
            count_total += 1
            count_convergent += 1 if status == "Convergent" else 0
            count_divergent += 1 if status == "Divergent" else 0
        print()
# Run analysis
analyze_graph(transition_table_with_worst_case)
```

Code end

Appendix: B: All Divergent loops:

SL	Loop Sequence	Parent	Filial type	≈ Δ(% magnitude)	Nature of loop
1	1 -> 5 -> 4 -> 6 -> 1	x	(81x+143)/64	50.00	Divergent
2	4 -> 6 -> 1 -> 5 -> 4	x	(81x+73)/64	32.00	Divergent
3	5 -> 6 -> 5	x	(9x+7)/8	16.00	Divergent
4	6 -> 1 -> 5 -> 4 -> 6	x	(81x+101)/64	31.48	Divergent
5	6 -> 5 -> 6	x	(9x+5)/8	14.81	Divergent
6	8 -> 8	x	(3x+1)/2	51.00	Divergent
7	1 -> 5 -> 8 -> 4 -> 2 -> 7 -> 6 -> 1	x	(2187x+5845)/2048	12.24	Divergent
8	1 -> 5 -> 8 -> 4 -> 6 -> 1	x	(243x+493)128	100.00	Divergent
9	2 -> 7 -> 6 -> 1 -> 5 -> 8 -> 4 -> 2	x	(2187x+7087)/2048	13.73	Divergent
10	2 -> 7 -> 8 -> 4 -> 2	x	(81x+125)/64	30.40	Divergent
11	2 -> 7 -> 8 -> 4 -> 6 -> 1 -> 5 -> 2	x	(2187x+4847)/2048	19.26	Divergent
12	4 -> 2 -> 7 -> 6 -> 1 -> 5 -> 8 -> 4	x	(2187x+4371)/2048	16.96	Divergent
13	4 -> 2 -> 7 -> 8 -> 4	x	((81x+89)/64	32.61	Divergent
14	4 -> 6 -> 1 -> 5 -> 2 -> 7 -> 8 -> 4	x	(2187x+3955)/2048	15.21	Divergent
15	4 -> 6 -> 1 -> 5 -> 8 -> 4	x	(243x+283)/128	100.00	Divergent
16	4 -> 6 -> 5 -> 4	x	(27x+19)/16	74.00	Divergent
17	5 -> 4 -> 6 -> 1 -> 5	x	(81x+103)/64	33.00	Divergent
18	5 -> 4 -> 6 -> 5	x	(27x+29)/16	76.00	Divergent
19	6 -> 1 -> 5 -> 2 -> 7 -> 8 -> 4 -> 6	x	(2187x+5836)/2048	17.00	Divergent
20	6 -> 1 -> 5 -> 8 -> 4 -> 2 -> 7 -> 6	x	(2187x+3916)/2048	14.00	Divergent
21	6 -> 1 -> 5 -> 8 -> 4 -> 6	x	(243x+364)/128	100.00	Divergent
22	6 -> 5 -> 4 -> 6	x	(27x+23)/16	74.10	Divergent
23	7 -> 6 -> 1 -> 5 -> 8 -> 4 -> 2 -> 7	x	(2187x+10561)/2048	25.00	Divergent
24	7 -> 8 -> 4 -> 2 -> 7	x	(81x+179)/64	36.00	Divergent
25	7 -> 8 -> 4 -> 6 -> 1 -> 5 -> 2 -> 7	x	(2187x+7201)/2048	11.00	Divergent
26	8 -> 4 -> 2 -> 7 -> 6 -> 1 -> 5 -> 8	x	(2187x+3227)/2048	12.00	Divergent
27	8 -> 4 -> 2 -> 7 -> 8	x	(81x+65)/64	30.00	Divergent
28	8 -> 4 -> 6 -> 1 -> 5 -> 2 -> 7 -> 8	x	(2187x+2683)/2048	11.00	Divergent
29	8 -> 4 -> 6 -> 1 -> 5 -> 8	x	(243x+128)/128	96.00	Divergent
30	2 -> 7 -> 6 -> 5 -> 8 -> 4 -> 2	x	(729x+1765)/512	60.52	Divergent
31	2 -> 7 -> 8 -> 4 -> 6 -> 5 -> 2	x	(729x+1765)/512	57.65	Divergent
32	4 -> 2 -> 7 -> 6 -> 5 -> 8 -> 4	x	(729x+1249)/1024	53.00	Divergent
33	4 -> 6 -> 5 -> 2 -> 7 -> 8 -> 4	x	(729x+1009)/512	51.00	Divergent
34	4 -> 6 -> 5 -> 8 -> 4	x	(81x+73)/32	163.00	Divergent
35	5 -> 2 -> 7 -> 8 -> 4 -> 6 -> 1 -> 5	x	(2187x+6509)/2048	19.50	Divergent
36	5 -> 2 -> 7 -> 8 -> 4 -> 6 -> 5	x	(729x+1999)/512	58.00	Divergent
37	5 -> 8 -> 4 -> 2 -> 7 -> 6 -> 1 -> 5	x	(2187x+4349)/2048	15.30	Divergent
38	5 -> 8 -> 4 -> 2 -> 7 -> 6 -> 5	x	(729x+1279)/512	52.40	Divergent
39	5 -> 8 -> 4 -> 6 -> 1 -> 5	x	(243x+341)/128	100.00	Divergent
40	5 -> 8 -> 4 -> 6 -> 5	x	(81x+103)/32	166.00	Divergent
41	6 -> 5 -> 2 -> 7 -> 8 -> 4 -> 6	x	(729x+1405)/512	52.50	Divergent
42	6 -> 5 -> 8 -> 4 -> 2 -> 7 -> 6	x	(729x+925)/512	50.00	Divergent
43	6 -> 5 -> 8 -> 4 -> 6	x	(81x+85)/32	163.00	Divergent
44	7 -> 6 -> 5 -> 8 -> 4 -> 2 -> 7	x	(729x+2539)/512	61.00	Divergent
45	7 -> 8 -> 4 -> 6 -> 5 -> 2 -> 7	x	(729x+2059)/512	56.25	Divergent
46	8 -> 4 -> 2 -> 7 -> 6 -> 5 -> 8	x	(729x+1164)/512	50.00	Divergent
47	8 -> 4 -> 6 -> 5 -> 2 -> 7 -> 8	x	(729x+745)/512	50.00	Divergent
48	8 -> 4 -> 6 -> 5 -> 8	x	(81x+65)/32	60.00	Divergent
49	1 ->5 -> 2 -> 7 -> 8 -> 4 -> 6 -> 1	x	(2187x+8725)/2048	31	Divergent

Appendix: D: Samples of Convergent Loops:

Serial	Loop	Power of 3	Power of 2	Tendency
1	1 -> 3 -> 7 -> 1	3	9	Convergent
2	2 -> 7 -> 3 -> 1 -> 5 -> 2	5	12	Convergent
3	7 -> 3 -> 1 -> 5 -> 2 -> 7	5	12	Convergent
4	7 -> 3 -> 1 -> 7	3	9	Convergent
5	1 -> 3 -> 2 -> 7 -> 1	4	10	Convergent
6	1 -> 3 -> 5 -> 2 -> 7 -> 1	5	10	Convergent
7	1 -> 3 -> 7 -> 5 -> 6 -> 1	5	12	Convergent
8	1 -> 3 -> 7 -> 6 -> 1	4	10	Convergent
9	1 -> 5 -> 2 -> 3 -> 7 -> 1	5	12	Convergent
10	1 -> 5 -> 2 -> 7 -> 3 -> 1	5	12	Convergent
11	1 -> 5 -> 2 -> 7 -> 3 -> 6 -> 1	6	12	Convergent
12	1 -> 7 -> 3 -> 1	3	9	Convergent
13	1 -> 7 -> 3 -> 5 -> 6 -> 1	5	12	Convergent
14	1 -> 7 -> 3 -> 6 -> 1	4	10	Convergent
15	2 -> 3 -> 7 -> 1 -> 5 -> 2	5	12	Convergent
16	2 -> 7 -> 1 -> 3 -> 5 -> 2	5	12	Convergent
17	2 -> 7 -> 3 -> 1 -> 5 -> 4 -> 2	6	13	Convergent
18	2 -> 7 -> 3 -> 5 -> 2	4	10	Convergent
19	2 -> 7 -> 3 -> 6 -> 1 -> 5 -> 2	6	13	Convergent
20	3 -> 1 -> 5 -> 2 -> 7 -> 3	5	12	Convergent
21	3 -> 1 -> 7 -> 3	3	9	Convergent
22	3 -> 7 -> 1 -> 3	3	9	Convergent
23	3 -> 7 -> 1 -> 5 -> 2 -> 3	5	12	Convergent
24	3 -> 7 -> 3	2	7	Convergent
25	4 -> 2 -> 7 -> 3 -> 1 -> 5 -> 4	6	13	Convergent
26	5 -> 2 -> 7 -> 3 -> 1 -> 5	5	12	Convergent
27	5 -> 6 -> 1 -> 3 -> 7 -> 5	5	12	Convergent
28	6 -> 1 -> 3 -> 7 -> 5 -> 6	5	12	Convergent
29	6 -> 1 -> 3 -> 7 -> 6	4	10	Convergent
30	6 -> 1 -> 7 -> 3 -> 5 -> 6	5	12	Convergent
31	7 -> 1 -> 3 -> 2 -> 7	4	13	Convergent
32	7 -> 1 -> 3 -> 5 -> 2 -> 7	5	12	Convergent
33	7 -> 1 -> 3 -> 7	3	9	Convergent