An Adaptive Quantum Circuit for Dempster's Rule of Combination

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Abstract

Harnessing the superior computational potential of quantum computing, an Adaptive Quantum Circuit for Dempster's Rule of Combination (AQC-DRC) is proposed to facilitate quantum-level belief and plausibility decision-making based on quantum evidence theory (QET). The AQC-DRC achieves a deterministic realization of DRC, guaranteeing precise fusion outcomes without information loss, while exponentially reducing the computational complexity of evidence combination and markedly improving fusion efficiency. It is founded that the quantum basic probability amplitude (QBPA) in QET can be naturally used to express the quantum amplitude encoding. In addition, the quantum basic probability (QBP) in QET, which forms quantum basic probability distribution (QBPD), can be naturally used to express the quantum measurement outcomes for quantum belief level decision-making. Furthermore, the quantum plausibility (QPl) function in QET also can be naturally used to express the quantum measurement outcomes for quantum plausibility level decision-making. These findings open up new perspectives

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and enhance the physical interpretation of quantum measurement outcomes.
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1. Introduction

As an effective approach for uncertainty reasoning, Dempster–Shafer evidence theory (DSET) [1, 2] offers a powerful framework for representing and managing uncertainty through the basic probability assignment (BPA) function. The Dempster's rule of combination (DRC), a core component of DSET, possesses several desirable properties that make it particularly suitable for multisource information fusion. (1) Commutativity: ensures that the fusion result remains invariant regardless of the order in which evidence is combined; (2) Associativity: provides the system with flexible capabilities for structured and sequential fusion; and (3) Consistency: guarantees that, in the absence of new valid information, the outcome of the evidence combination remains unchanged. These advantages support flexible integration of multisource information, enable recursive and incremental computation, and facilitate the scalability of reasoning systems. However, the computational complexity of DRC increases exponentially with the number of elements in the frame of discernment.

The rapid development of quantum computing offers a new research perspective for addressing the computational complexity challenges in Dempster's rule of combination of Dempster–Shafer evidence theory [1, 2]. Leveraging the principles of quantum parallelism and quantum entanglement, quantum computing provides the potential to significantly accelerate the processing of large-scale uncertain information. In particular, it opens up new possibilities for overcoming the exponential computation complexity issues inherent in classical evidence reasoning frameworks based on DRC, thus providing an innovative approach to efficient information fusion and decision-making.

Harnessing the superior computational potential of quantum computing, an Adaptive Quantum Circuit for Dempster's Rule of Combination (AQC-DRC) is proposed to facilitate quantum-level belief and plausibility decision-making based on quantum evidence theory (QET). The AQC-DRC achieves a deterministic realization of DRC, guaranteeing precise fusion outcomes without information loss, while exponentially reducing the computational complexity of evidence combination and markedly improving fusion efficiency.

In this study, it is founded that the quantum basic probability amplitude (QBPA) in QET [3, 4] can be naturally used to express the quantum amplitude encoding. In addition, the quantum basic probability (QBP) in QET, which forms quantum basic probability distribution (QBPD), can be naturally used to express the quantum measurement outcomes for quantum belief level decision-making. Furthermore, the quantum plausibility (QPl) function in QET also can be naturally used to express the quantum measurement outcomes for quantum plausibility level decision-making. These findings open up new perspectives and enhance the physical interpretation of quantum measurement outcomes.

2. Preliminaries

In this section, we review some basic concepts of Dempster-Shafer evidence theory (DSET) [1, 2] and quantum evidence theory (QET) [3, 4].

2.1. DSET: Dempster–Shafer evidence theory [1, 2]

Definition 1 (Frame of discernment). Let Ω be a frame of discernment (FOD), consisting of a set of mutually exclusive and collectively nonempty events:

$$\Omega = \{h_1, h_2, ..., h_i, ..., h_n\}. \tag{1}$$

Let 2^Ω be the power set of Ω, denoted as:

$$2^\Omega = \{\emptyset, \{h_1\}, \{h_2\}, ..., \{h_n\}, \{h_1, h_2\}, ..., \{h_1, h_2, ..., h_i\}, ..., \Omega\}, \tag{2}$$

where $\emptyset$ is an empty set.

Definition 2 (Hypothesis or proposition). H_j is defined as a hypothesis or proposition when $H_j \subseteq \Omega$.

Definition 3 (Basic probability assignment). In FOD Ω, a basic probability assignment (BPA) m, also called a mass function, is defined as a mapping:

$$m : 2^\Omega \to [0, 1], \tag{3}$$

satisfying

$$m(\emptyset) = 0 \quad \text{and} \sum_{H_j \subseteq \Omega} m(H_j) = 1. \tag{4}$$

Definition 4 (Focal element). Let m be a BPA. $\forall H_j \subseteq \Omega$, if $m(H_j) > 0$, H_j is called a focal element in DSET.

Definition 5 (Belief function). Let H_j and H_k be two propositions such that $H_j, H_k \subseteq \Omega$. A belief function Bel, mapping from 2^Ω to $[0,1]$, is defined by

$$\mathrm{Bel}(H_j) = \sum_{H_k \subseteq H_j} m(H_k). \tag{5}$$

Definition 6 (Plausibility function). Let H_j and H_k be two propositions such that $H_j, H_k \subseteq \Omega$. A plausibility function Pl, mapping from 2^Ω to $[0,1]$, is defined by

$$\mathrm{Pl}(H_j) = \sum_{H_k \cap H_j \neq \emptyset} m(H_k) = 1 - Bel(\bar{H}_j), \quad \bar{H}_j = \Omega - H_j. \tag{6}$$

Definition 7 (Dempster's rule of combination). Let m_1 and m_2 be two independent BPAs in FOD Ω with propositions $H_k, H_h \subseteq \Omega$, respectively. Dempster's rule of combination (DRC), represented in the form $m_1 \oplus m_2$, is defined by

$$m_1 \oplus m_2(H_j) = \begin{cases} \dfrac{1}{1-K} \displaystyle\sum_{H_k \cap H_h = H_j} m_1(H_k) m_2(H_h), & H_j \neq \emptyset, \\ 0, & H_j = \emptyset, \end{cases} \tag{7}$$

with

$$K = \sum_{H_k \cap H_h = \emptyset} m_1(H_k) m_2(H_h), \tag{8}$$

where K is the conflict coefficient between m_1 and m_2.

2.2. QET: Quantum evidence theory [3, 4]

Definition 8 (Quantum frame of discernment). Let $|\Phi\rangle$ be a quantum frame of discernment (QFOD), consisting of a set of mutually exclusive and collectively nonempty events, each of which is expressed as an orthonormal basis $|\phi_g\rangle$ in a Hilbert space:

$$|\Phi\rangle = \{|\phi_1\rangle, ..., |\phi_g\rangle, ..., |\phi_n\rangle\}. \tag{9}$$

Let $2^{|\Phi\rangle}$ be the power set of $|\Phi\rangle$, denoted as:

$$2^{|\Phi\rangle} = \{|\emptyset\rangle, \{|\phi_1\rangle\}, \{|\phi_2\rangle\}, ..., \{|\phi_n\rangle\}, \{|\phi_1\phi_2\rangle\}, ..., \{|\phi_1\phi_2...\phi_g\rangle\}, ..., |\Phi\rangle\}, \tag{10}$$

which can be simply represented as:

$$2^{|\Phi\rangle} = \{|\emptyset\rangle, |\phi_1\rangle, |\phi_2\rangle, ..., |\phi_n\rangle, |\phi_{12}\rangle, ..., |\phi_{12...g}\rangle, ..., |\phi_{12...n}\rangle\}, \tag{11}$$

where $\emptyset$ is an empty set.

Definition 9 (Quantum hypothesis or proposition). $|\psi_j\rangle$ is defined as a quantum hypothesis or proposition when $|\psi_j\rangle \subseteq |\Phi\rangle$.

Definition 10 (Quantum basic probability amplitude function). A quantum basic probability amplitude (QBPA) function $\mathbb{Q}_{\mathbb{M}}$ in QFOD $|\Phi\rangle$, also referred to as a quantum mass function (QMF), is defined as a mapping:

$$\mathbb{Q}_{\mathbb{M}} : 2^{|\Phi\rangle} \to \mathbb{C}, \tag{12}$$

satisfying

$$\begin{aligned} &\mathbb{Q}_{\mathbb{M}}(|\emptyset\rangle) = 0, \\ &\mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle) = \varphi(|\psi_j\rangle)e^{i\theta(|\psi_j\rangle)}, \quad |\psi_j\rangle \subseteq |\Phi\rangle, \\ &\sum_{|\psi_j\rangle \subseteq |\Phi\rangle} |\mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle)|^2 = 1, \end{aligned} \tag{13}$$

in which $i = \sqrt{-1}$; $\varphi(|\psi_j\rangle) \in [0, 1]$ represents the modulus of $\mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle)$; $\theta(|\psi_j\rangle)$ denotes a phase term of $\mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle)$; $\mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle)$ denote a quantum basic probability amplitude for $|\psi_j\rangle$; and $|\mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle)|^2 = \varphi^2(|\psi_j\rangle)$ denotes the modulus squared of $\mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle)$.

Definition 11 (Quantum focal element). Let $\mathbb{Q}_{\mathbb{M}}$ be a QBPA. $\forall|\psi_j\rangle \subseteq |\Phi\rangle$, if $|\mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle)|^2$ or $\varphi(|\psi_j\rangle) > 0$, $|\psi_j\rangle$ is called a quantum focal element in QET.

Definition 12 (Quantum basic probability distribution). The quantum basic probability distribution (QBPD) of $\mathbb{Q}_{\mathbb{M}}$, is defined as:

$$\mathrm{M} : 2^{|\Phi\rangle} \rightarrow [0, 1], \tag{14}$$

and satisfies:

$$\begin{aligned} &\mathrm{M}(|\emptyset\rangle) = 0, \\ &\mathrm{M}(|\psi_j\rangle) = |\mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle)|^2, \quad |\psi_j\rangle \subseteq |\Phi\rangle, \\ &\sum_{|\psi_j\rangle \subseteq |\Phi\rangle} \mathrm{M}(|\psi_j\rangle) = 1, \end{aligned} \tag{15}$$

where $|\mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle)|^2 = \mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle)\widehat{\mathbb{Q}}_{\mathbb{M}}(|\psi_j\rangle) = \varphi^2(|\psi_j\rangle) = x_j^2 + y_j^2$, in which $\widehat{\mathbb{Q}}_{\mathbb{M}}(|\psi_j\rangle)$ is the complex conjugate of $\mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle)$, e.g., $\widehat{\mathbb{Q}}_{\mathbb{M}}(|\psi_j\rangle) = x_j - y_j i$.

Definition 13 (Quantum basic probability). In QET, $\mathrm{M}(|\psi_j\rangle)$ ($|\psi_j\rangle \subseteq |\Phi\rangle$) is called quantum basic probability (QBP), which represents the degree of belief or support to $|\psi_j\rangle$.

Definition 14 (Quantum belief function). Let $\mathbb{Q}_{\mathbb{M}}$ be a QBPA with proposition $|\psi_j\rangle \subseteq |\Phi\rangle$. A quantum belief function QBel in QET, mapping from $2^{|\Phi\rangle}$ to $[0, 1]$, is defined by:

$$\mathrm{QBel}(|\psi_j\rangle) = \sum_{|\psi_p\rangle \subseteq |\psi_j\rangle} \left|\mathbb{Q}_{\mathbb{M}}(|\psi_p\rangle)\right|^2, \quad |\psi_j\rangle \subseteq |\Phi\rangle. \tag{16}$$

According to Eq. (15), Eq. (16) can also be represented as:

$$\mathrm{QBel}(|\psi_j\rangle) = \sum_{|\psi_p\rangle \subseteq |\psi_j\rangle} \varphi^2(|\psi_j\rangle) = \sum_{|\psi_p\rangle \subseteq |\psi_j\rangle} \mathrm{M}(|\psi_p\rangle), \quad |\psi_j\rangle \subseteq |\Phi\rangle. \tag{17}$$

Therefore, when $\mathrm{M} = m$, Eq. (17) becomes:

$$\mathrm{QBel}(|\psi_j\rangle) = \sum_{|\psi_p\rangle \subseteq |\psi_j\rangle} m(|\psi_p\rangle), \quad |\psi_j\rangle \subseteq |\Phi\rangle, \tag{18}$$

which is consistent with the classical Bel in DSET.

Definition 15 (Quantum plausibility function). Let $\mathbb{Q}_{\mathbb{M}}$ be a QBPA with proposition $|\psi_j\rangle \subseteq |\Phi\rangle$. A quantum plausibility (QPl) function in QET, mapping from $2^{|\Phi\rangle}$ to $[0,1]$, is defined by:

$$\mathrm{QPl}(|\psi_j\rangle) = \sum_{|\psi_p\rangle \cap |\psi_j\rangle \neq \emptyset} \left| \mathbb{Q}_{\mathbb{M}}(|\psi_p\rangle) \right|^2, \quad |\psi_j\rangle \subseteq |\Phi\rangle. \tag{19}$$

According to Eq. (15), Eq. (19) can also be represented as:

$$\mathrm{QPl}(|\psi_j\rangle) = \sum_{|\psi_p\rangle \cap |\psi_j\rangle \neq \emptyset} \varphi^2(|\psi_j\rangle) = \sum_{|\psi_p\rangle \cap |\psi_j\rangle \neq \emptyset} \mathrm{M}(|\psi_p\rangle), \quad |\psi_j\rangle \subseteq |\Phi\rangle. \tag{20}$$

Therefore, when $\mathrm{M} = m$, Eq. (20) becomes:

$$\mathrm{QPl}(|\psi_j\rangle) = \sum_{|\psi_p\rangle \cap |\psi_j\rangle \neq \emptyset} m(|\psi_p\rangle), \quad |\psi_j\rangle \subseteq |\Phi\rangle, \tag{21}$$

which is consistent with the classical Pl in DSET.

3. AQC-DRC: Adaptive Quantum Circuit for Dempster's Rule of Combination

The AQC-DRC consists of the following three components: 1) quantum amplitude encoding for BPA, 2) construction of the adaptive quantum circuit for DRC, and 3) measurement in the adaptive quantum circuit for decision-making.

3.1. Quantum amplitude encoding for BPA

In this section, QBPA in QET is expressed for quantum amplitude encoding. In this context, a BPA is encoded as a superposition over an n-qubit quantum state.

Definition 16 (QBPA expression for quantum amplitude encoding). Let $\mathbb{Q}_{\mathbb{M}}$ be a QBPA on the QFOD $|\Phi\rangle = \{|\phi_1\rangle, \ldots, |\phi_i\rangle, \ldots, |\phi_n\rangle\}$ with quantum proposition $|\psi_j\rangle \subseteq |\Phi\rangle$. The QBPA expression for quantum amplitude encoding, also called a quantum superposition state of the QBPA, is defined as:

$$\begin{aligned} |\mathbb{Q}_{\mathbb{M}}\rangle &= \sum_{|\psi_j\rangle \subseteq |\Phi\rangle} \mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle)|\psi_j\rangle \\ &= \sum_{|\psi_j\rangle \subseteq |\Phi\rangle} \varphi(|\psi_j\rangle) e^{i\theta(|\psi_j\rangle)} |\psi_j\rangle, \\ \sum_{|\psi_j\rangle \subseteq |\Phi\rangle} &|\mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle)|^2 = \varphi^2(|\psi_j\rangle) = 1, \end{aligned} \tag{22}$$

where

$$|\psi_j\rangle = \bigotimes_{i=1}^{n} |\delta_{ji}\rangle = |\delta_{jn}\rangle \cdots |\delta_{ji}\rangle \cdots |\delta_{j2}\rangle |\delta_{j1}\rangle, \tag{23}$$

and

$$\delta_{ji} = \begin{cases} 1, & \text{if } |\phi_i\rangle \in |\psi_j\rangle, \\ 0, & \text{if } |\phi_i\rangle \notin |\psi_j\rangle. \end{cases} \tag{24}$$

When $\theta(|\psi_j\rangle) = 0$, the QBPA expression for quantum amplitude encoding can be represented as:

$$\begin{aligned} |\mathbb{Q}_{\mathbb{M}}\rangle &= \sum_{|\psi_j\rangle \subseteq |\Phi\rangle} \varphi(|\psi_j\rangle)|\psi_j\rangle, \\ \sum_{|\psi_j\rangle \subseteq |\Phi\rangle} &\varphi^2(|\psi_j\rangle) = 1. \end{aligned} \tag{25}$$

Definition 17 (Quantum amplitude encoding of BPA). Let m_h be a BPA on the FOD $\Phi = \{\phi_1, \ldots, \phi_i, \ldots, \phi_n\}$ with proposition $\psi_j \subseteq \Phi$. Considering QBPA expression for quantum amplitude encoding, a BPA m_h is encoded into the amplitudes of an n-qubit state as:

$$
\begin{aligned}
|\mathbb{Q}_{\mathbb{M}_h}\rangle = U_E(m_h)|0\rangle^{\otimes n} &= \sum_{|\psi_j\rangle \subseteq |\Phi\rangle} \mathbb{Q}_{\mathbb{M}_h}(|\psi_j\rangle)|\psi_j\rangle \\
&= \sum_{|\psi_j\rangle \subseteq |\Phi\rangle} \varphi_h(|\psi_j\rangle) e^{i\theta_h(|\psi_j\rangle)}|\psi_j\rangle,
\end{aligned}
\tag{26}
$$

satisfying

$$
\varphi_h(|\psi_j\rangle) e^{i\theta_h(|\psi_j\rangle)} = \sqrt{m_h(\psi_j)} e^{i0} = \sqrt{m_h(\psi_j)}, \quad \text{and} \quad \sum_{|\psi_j\rangle \subseteq |\Phi\rangle} \left| \sqrt{m_h(\psi_j)} \right|^2 = 1,
\tag{27}
$$

where $|\psi_j\rangle$ is defined in Definition 16, and U_E denotes a state preparation oracle or operator.

3.2. Construction of an adaptive quantum circuit for DRC

The encoded quantum states of BPAs $\{|\mathbb{Q}_{\mathbb{M}_1}\rangle, \ldots, |\mathbb{Q}_{\mathbb{M}_h}\rangle, \ldots, |\mathbb{Q}_{\mathbb{M}_k}\rangle\}$ will be combined by a series of specific quantum operators, which can be categorized as one types of U_C designed through Toffoli gates. Then, after implementing U_C, we obtain:

$$
\begin{aligned}
\rho_{\mathbb{Q}_{\mathbb{M}_{21}}} &= \sum_{|\psi_t\rangle \subseteq |\Phi\rangle} \sum_{\cap |\psi_j\rangle = |\psi_t\rangle} \prod_{1 \le h \le 2} \left| \varphi_h(|\psi_j\rangle) e^{i\theta_h(|\psi_j\rangle)} \right|^2 |\psi_t\rangle\langle\psi_t| \\
&= \sum_{|\psi_t\rangle \subseteq |\Phi\rangle} \sum_{\cap |\psi_j\rangle = |\psi_t\rangle} \prod_{1 \le h \le 2} \left| \sqrt{m_h(\psi_j)} \right|^2 |\psi_t\rangle\langle\psi_t|.
\end{aligned}
\tag{28}
$$

The output in terms of $\rho_{\mathbb{Q}_{\mathbb{M}_{h\ldots1}}}$ after implementing U_C based on Toffoli gates are delivered to $|0\rangle^{\otimes n}$-qubit. Then, after implementing U_C, we obtain:

$$\begin{aligned}\rho_{\mathbb{Q}_{\mathbb{M}_{h\ldots1}}} &= \sum_{|\psi_t\rangle\subseteq|\Phi\rangle}\ \sum_{\cap|\psi_j\rangle=|\psi_t\rangle}\ \prod_{1\leq h\leq k}\left|\varphi_h(|\psi_j\rangle)e^{i\theta_h(|\psi_j\rangle)}\right|^2 |\psi_t\rangle\langle\psi_t| \\ &= \sum_{|\psi_t\rangle\subseteq|\Phi\rangle}\ \sum_{\cap|\psi_j\rangle=|\psi_t\rangle}\ \prod_{1\leq h\leq k}\left|\sqrt{m_h(\psi_j)}\right|^2 |\psi_t\rangle\langle\psi_t|.\end{aligned} \tag{29}$$

3.3. Measurement in the adaptive quantum circuit for decision-making

We define two types of measurement operators in terms of the quantum belief level and the plausibility level decision-making for different application requirements.

3.3.1. Quantum measurement for quantum belief level decision-making

Definition 18 (Measurement operator for quantum belief level). The measurement operator U_M^{QB} is defined for the quantum belief level decision-making as:

$$U_M^{QB} = \{\mathcal{M}_{|\psi_t\rangle}||\psi_t\rangle \subseteq \Phi\}, \tag{30}$$

and

$$\mathcal{M}_{|\psi_t\rangle} = |\psi_t\rangle\langle\psi_t|, \tag{31}$$

where $|\psi_t\rangle$ is defined in Definition 16.

Definition 19 (QBP expression for quantum measurement outcomes). Let $\rho_{\mathbb{Q}_{\mathbb{M}_{k\ldots1}}}$ be a density operator with regards to the trace of the output of U_C. Let $U_M^{QB} = \{\mathcal{M}_{|\psi_t\rangle} = |\psi_t\rangle\langle\psi_t|||\psi_t\rangle \subseteq \Phi\}$ be a set of measurement operators. The quantum basic probability (QBP) expression for quantum measurement outcomes is defined as:

$$\mathrm{M}(|\psi_t\rangle) = \frac{\mathrm{Tr}\left(\mathcal{M}^\dagger_{|\psi_t\rangle}\mathcal{M}_{|\psi_t\rangle}\cdot\rho_{\mathbb{Q}_{\mathbb{M}_{k\ldots1}}}\right)}{\sum\limits_{\substack{|\psi_v\rangle\subseteq|\Phi\rangle\\|\psi_v\rangle\neq\emptyset}}\mathrm{Tr}\left(\mathcal{M}^\dagger_{|\psi_v\rangle}\mathcal{M}_{|\psi_v\rangle}\cdot\rho_{\mathbb{Q}_{\mathbb{M}_{k\ldots1}}}\right)}, \tag{32}$$

which forms a quantum basic probability assignment (QBPA) M.

After implementing the measurement operator U_M^{QB}, for $|\psi_t\rangle \subseteq \Phi, |\psi_t\rangle \neq \emptyset$, we obtain the quantum basic probability (QBP) for each $|\psi_t\rangle$:

$$\mathrm{M}(|\psi_t\rangle) = \frac{\mathrm{Tr}\left(\mathcal{M}^{\dagger}_{|\psi_t\rangle}\mathcal{M}_{|\psi_t\rangle} \cdot \rho_{\mathbb{Q}_{\mathbb{M}_{k\ldots1}}}\right)}{\sum\limits_{\substack{|\psi_v\rangle \subseteq |\Phi\rangle \\ |\psi_v\rangle \neq \emptyset}} \mathrm{Tr}\left(\mathcal{M}^{\dagger}_{|\psi_v\rangle}\mathcal{M}_{|\psi_v\rangle} \cdot \rho_{\mathbb{Q}_{\mathbb{M}_{k\ldots1}}}\right)} = \frac{\sum\limits_{\cap|\psi_j\rangle = |\psi_t\rangle} \prod\limits_{1\leq h\leq k} \left|\varphi_h(|\psi_j\rangle)e^{i\theta_h(|\psi_j\rangle)}\right|^2}{\sum\limits_{\substack{|\psi_v\rangle \subseteq \Phi \\ |\psi_v\rangle \neq \emptyset}} \sum\limits_{\cap|\psi_j\rangle = |\psi_v\rangle} \prod\limits_{1\leq h\leq k} \left|\varphi_h(|\psi_j\rangle)e^{i\theta_h(|\psi_j\rangle)}\right|^2}. \tag{33}$$

Because $\left|\varphi_h(|\psi_j\rangle)e^{i\theta_h(|\psi_j\rangle)}\right|^2 = \left|\sqrt{m_h(\psi_j)}\right|^2$ and $m(\psi_t) = \mathrm{M}(|\psi_t\rangle)$, the combined BAA can be generated:

$$m(\psi_t) = \mathrm{M}(|\psi_t\rangle) = \frac{\sum\limits_{\cap|\psi_j\rangle = |\psi_t\rangle} \prod\limits_{1\leq h\leq k} \left|\sqrt{m_h(\psi_j)}\right|^2}{\sum\limits_{\substack{|\psi_v\rangle \subseteq \Phi \\ |\psi_v\rangle \neq \emptyset}} \sum\limits_{\cap|\psi_j\rangle = |\psi_v\rangle} \prod\limits_{1\leq h\leq k} \left|\sqrt{m_h(\psi_j)}\right|^2}. \tag{34}$$

For $|\psi_t\rangle = |\emptyset\rangle$, we have

$$\begin{aligned} K = \mathrm{Tr}\left(\mathcal{M}^{\dagger}_{|\emptyset\rangle}\mathcal{M}_{|\emptyset\rangle} \cdot \rho_{\mathbb{Q}_{\mathbb{M}_{k\ldots1}}}\right) &= \sum_{\cap|\psi_j\rangle = |\emptyset\rangle} \prod_{1\leq h\leq k} \left|\varphi_h(|\psi_j\rangle)e^{i\theta_h(|\psi_j\rangle)}\right|^2 \\ &= \sum_{\cap|\psi_j\rangle = |\emptyset\rangle} \prod_{1\leq h\leq k} \left|\sqrt{m_h(\psi_j)}\right|^2. \end{aligned} \tag{35}$$

Then, for $|\psi_t\rangle \subseteq \Phi, |\psi_t\rangle \neq \emptyset$, we also have

$$m(\psi_t) = \frac{\mathrm{Tr}\left(\mathcal{M}^{\dagger}_{|\psi_t\rangle}\mathcal{M}_{|\psi_t\rangle} \cdot \rho_{\mathbb{Q}_{\mathbb{M}_{k\ldots1}}}\right)}{1 - \mathrm{Tr}\left(\mathcal{M}^{\dagger}_{|\emptyset\rangle}\mathcal{M}_{|\emptyset\rangle} \cdot \rho_{\mathbb{Q}_{\mathbb{M}_{k\ldots1}}}\right)} = \frac{\sum\limits_{\cap|\psi_j\rangle = |\psi_t\rangle} \prod\limits_{1\leq h\leq k} \left|\sqrt{m_h(\psi_j)}\right|^2}{1-K}. \tag{36}$$

When implementing AQC-DRC based on the quantum measurement for quantum belief level decision-making, denoted as AQC-DRC$_{\mathrm{QB}}$, a decision can be made as follow:

$$\delta = |\psi_t\rangle, \quad \text{and} \quad w = \arg\max_t\{\mathcal{M}(|\psi_t\rangle)\} = \arg\max_t\{m(\psi_t)\}. \tag{37}$$

3.3.2. Quantum measurement for quantum plausibility level decision-making

On the basis of the density matrix of $\rho_{\mathbb{Q}_{\mathbb{M}_{h\ldots1}}}$, we obtain:

$$\begin{aligned}\rho_{\mathbb{Q}^w_{\mathbb{M}_{h\ldots1}}} &= \sum_{|\phi_w\rangle\in|\psi_t\rangle} \sum_{\cap|\psi_j\rangle=|\psi_t\rangle} \prod_{1\le h\le k} \left|\varphi_h(|\psi_j\rangle)e^{i\theta_h(|\psi_j\rangle)}\right|^2 |1\rangle\langle1|+ \\ &\quad \left(1-\sum_{|\phi_w\rangle\in|\psi_t\rangle} \sum_{\cap|\psi_j\rangle=|\psi_t\rangle} \prod_{1\le h\le k} \left|\varphi_h(|\psi_j\rangle)e^{i\theta_h(|\psi_j\rangle)}\right|^2\right)|0\rangle\langle0| \\ &= \sum_{|\phi_w\rangle\in|\psi_t\rangle} \sum_{\cap|\psi_j\rangle=|\psi_t\rangle} \prod_{1\le h\le k} \left|\sqrt{m_h(\psi_j)}\right|^2 |1\rangle\langle1|+ \\ &\quad \left(1-\sum_{|\phi_w\rangle\in|\psi_t\rangle} \sum_{\cap|\psi_j\rangle=|\psi_t\rangle} \prod_{1\le h\le k} \left|\sqrt{m_h(\psi_j)}\right|^2\right)|0\rangle\langle0|.\end{aligned} \tag{38}$$

Definition 20 (Measurement operator for quantum plausibility level). The measurement operator $U^{QPl}_{\mathcal{M}}$ is defined for quantum plausibility level decision-making as follows:

$$U^{QPl}_{\mathcal{M}} = \{\mathcal{M}_{|u\rangle}|u\in\{0,1\}\}, \tag{39}$$

and

$$\mathcal{M}_{|u\rangle} = |u\rangle\langle u|. \tag{40}$$

Definition 21 (QPl expression for quantum measurement outcomes). Let $\rho^w_{\mathbb{Q}_{\mathbb{M}_{k\ldots1}}}$ be the density operator of the w-th qubit in terms of the output of U^C. Let $U^{QPl}_{\mathcal{M}} = \{\mathcal{M}_{|u\rangle} = |u\rangle\langle u||u\in\{0,1\}\}$ and $U^{QB}_{M} = \{\mathcal{M}_{|\psi_t\rangle} = |\psi_t\rangle\langle\psi_t|||\psi_t\rangle\subseteq\Phi\}$ be a set of measurement operators. The quantum plausibility (QPl) expression for quantum measurement outcomes is defined as:

$$\mathrm{QPl}(|\psi_w\rangle) = \frac{\mathrm{Tr}\left(\mathcal{M}^\dagger_{|1\rangle}\mathcal{M}_{|1\rangle}\cdot\rho^w_{\mathbb{Q}_{\mathbb{M}_{k\ldots1}}}\right)}{1-\mathrm{Tr}\left(\mathcal{M}^\dagger_{|\emptyset\rangle}\mathcal{M}_{|\emptyset\rangle}\cdot\rho_{\mathbb{Q}_{\mathbb{M}_{k\ldots1}}}\right)}. \tag{41}$$

After implementing the measurement operators, the QPl of each element $|\phi_w\rangle$ $(1 \leq w \leq n)$ in FOD are generated directly as follows:

$$\mathrm{QPl}(|\phi_w\rangle) = \frac{\mathrm{Tr}\left(\mathcal{M}^{\dagger}_{|1\rangle}\mathcal{M}_{|1\rangle} \cdot \rho^{w}_{\mathbb{Q}_{\mathbb{M}_{k\ldots1}}}\right)}{1 - \mathrm{Tr}\left(\mathcal{M}^{\dagger}_{|\emptyset\rangle}\mathcal{M}_{|\emptyset\rangle} \cdot \rho_{\mathbb{Q}_{\mathbb{M}_{k\ldots1}}}\right)} = \frac{\sum\limits_{|\phi_w\rangle\in|\psi_t\rangle}\ \sum\limits_{\cap|\psi_j\rangle=|\psi_t\rangle}\ \prod\limits_{1\leq h\leq k}\left|\varphi_h(|\psi_j\rangle)e^{i\theta_h(|\psi_j\rangle)}\right|^2}{1-K_G}. \tag{42}$$

Because $\left|\varphi_h(|\psi_j\rangle)e^{i\theta_h(|\psi_j\rangle)}\right|^2 = \left|\sqrt{m_h(\psi_j)}\right|^2$ and $\mathrm{Pl}(\psi_w) = \mathrm{QPl}(|\phi_w\rangle)$, the Pl for $|\phi_w\rangle$ can be generated:

$$\mathrm{Pl}(\phi_w) = \mathrm{QPl}(|\phi_w\rangle) = \frac{\sum\limits_{|\phi_w\rangle\in|\psi_t\rangle}\ \sum\limits_{\cap|\psi_j\rangle=|\psi_t\rangle}\ \prod\limits_{1\leq h\leq k}\left|\sqrt{m_h(\psi_j)}\right|^2}{1-K_G}. \tag{43}$$

When implementing AQC-DRC based on the quantum plausibility level decision-making, denoted as AQC-DRC$_{\mathrm{QPl}}$, a decision can be made as follow:

$$\delta = |\phi_w\rangle, \quad \text{and} \quad w = \arg\max_w\{\mathrm{QPl}(|\phi_w\rangle)\} = \arg\max_w\{\mathrm{Pl}(\phi_w)\}. \tag{44}$$

4. Computational complexity analysis

Assume that there are n elements in the frame of discernment (FOD) and k pieces of evidence, with a total of N focal elements.

For quantum belief-level decision-making, the time complexity of the classical DRC is $O(kN2^{2n})$. In contrast, with sufficient auxiliary qubits, the time complexity of AQC-DRC, in terms of both the circuit depth and the normalization process, denoted as AQC-DRC$_{\mathrm{QB}}$, is $O(kn + N)$. Through comparative analysis, the time complexity of AQC-DRC$_{\mathrm{QB}}$ achieves an exponential reduction compared to that of the classical DRC. Moreover, the space complexity, corresponding to the number of qubits required by AQC-DRC$_{\mathrm{QB}}$, is

$O(kn)$, which increases linearly with the number of elements n in the FOD and the number of pieces of evidence k.

For quantum plausibility-level decision-making, the time complexity of AQC-DRC, in terms of both the circuit depth and the normalization process, denoted as $\text{AQC-DRC}_{\text{QPl}}$, is $O(kn^2)$. Similarly, through comparative analysis, the time complexity of $\text{AQC-DRC}_{\text{QPl}}$ demonstrates an exponential reduction compared to that of the classical DRC. The space complexity, representing the number of qubits required for quantum plausibility-level decision-making in $\text{AQC-DRC}_{\text{QPl}}$, remains $O(kn)$, which also grows linearly with the number of elements n in the FOD and the number of pieces of evidence k.

5. Conclusion

In this paper, we propose an adaptive quantum circuit for Dempster's rule of combination (AQC-DRC) to support quantum-level belief and plausibility decision-making within the framework of quantum evidence theory (QET). The AQC-DRC enables deterministic computation of evidence combination rules, thereby ensuring high precision in fusion outcomes without information loss. Moreover, it achieves an exponential reduction in computational complexity, making it a promising approach for real-time quantum multi-source information fusion. The architecture of the proposed AQC-DRC is conceptually straightforward and highly scalable, which facilitates its practical implementation.

It is observed that the quantum basic probability amplitude (QBPA) in QET can naturally express the quantum amplitude encoding. The quantum basic probability (QBP)in QET, forming the quantum basic probability distribution (QBPD), can directly express quantum measurement outcomes for belief-level decision-making, while the quantum plausibility (QPl) function in QET can also naturally represent the quantum measurement outcomes for plausibility-level decision-making. These insights not only broaden the understanding of QET, but also provide a more intuitive physical interpretation of quantum measurement outcomes.

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