Artificial Prime Numbers in the Fibonacci Sequence: A Structural Approach

José Acevedo Jiménez
Santiago, Dominican Republic

Abstract: In this article, it is proven that for every $n \geq 3$, there exists at least one artificial prime number q such that $F_n < q < F_{2n}$, where F_k denotes the k-th number in the Fibonacci sequence. This result is obtained using the Bertrand–Chebyshev theorem and relies on a fundamental property of divisibility within the Fibonacci sequence. Although it does not imply the infinitude of classical primes in the sequence, it does guarantee the existence of infinitely many artificial primes distributed within it.

Keywords: Artificial primes, Fibonacci numbers, Number theory, Divisibility, Prime numbers, Bertrand–Chebyshev theorem.

1. Introduction

The notion of a prime number has been a cornerstone in the development of number theory. In this research, we study an extension of the concept known as relative primality or artificial primality, defined with respect to the structure of a specific set of integer–in this case, the Fibonacci sequence.
A number $q \in S$, with $S \subset \mathbb{N}$, if called an artificial prime number if there exists no $d \in S$, with $d \neq q$, such that $d \mid q$. This definition depends solely on the set S, allowing us to study primality from an internal perspective within the set.

In this work, we analyze the existence of artificial primes within the set $S = \{F_k : k \geq 3\}$, that is, Fibonacci numbers greater than 1. The main result establishes that for every $n \geq 3$, there is at least one artificial prime between F_n and F_{2n}. This observation implies that there are infinitely many artificial prime numbers in the Fibonacci sequence.

2. Preliminaries

2.1. Fibonacci Sequence

The Fibonacci sequence $\{F_n\}$ is defined by the recurrence:

$F_1 = F_2 = 1,\ F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

2.2. Divisibility in Fibonacci

A fundamental property is:

$$F_m \mid F_n \iff m \mid n.$$

It is also known that:

$$\gcd(F_m, F_n) = F_{\gcd(m,n).}$$

These properties allow us to study divisibility between terms by observing only their indices.

2.3. Artificial Prime Numbers

Given a set $S \subset \mathbb{N}$, we say that a number $q \in S$ is an artificial prime number if:

$$\forall d \in S,\ d \neq q \Rightarrow d \nmid q.$$

In this case of the Fibonacci sequence, we define the set:

$$S = \{F_k : k \geq 3\}.$$

3. Main Theorem

For every integer $n \geq 3$, there exists at least one artificial prime

$F_p \in \{F_k : n < k < 2n\}$.

Proof.

By the Bertrand-Chebyshev theorem, for every integer $n \geq 1$, there exists at least one prime number p such that:

$$n < p \leq 2n.$$

Let p be such a prime. Consider the Fibonacci number F_p. We observe that:

- The index p is prime.
- By the divisibility property in Fibonacci numbers:

$$F_d \mid F_p \iff d \mid p.$$

Since p is prime, the only divisor of p in $\mathbb{N}$ are 1 and p itself.

But since we are working in the set $S = \{F_k : k \geq 3\}$, the index 1 is not included in S.

Therefore, no other term $F_d \in S$, whit $d \neq p$, divides F_p, in other words, F_p has no divisors in S except itself.

$$\Rightarrow F_p \text{ is an artificial prime in } S.$$

Moreover, since $p \in (n, 2n]$, we have $F_p \in (P_n, P_{2n}]$, because the Fibonacci sequence is strictly increasing.

Therefore, there is always at least one artificial prime number between F_n and F_{2n}.

4. Consequences

An immediate consequence of the theorem is that there are infinitely many artificial primes in the Fibonacci sequence. This does not imply that there are infinitely many classical primes among the Fibonacci numbers, which remains an open problem.

Conclusion

This theorem reveals a structure within the Fibonacci sequence: although the distribution of classical primes in it is sparse and not fully understood, the distribution of artificial primes shows a guaranteed regularity. This suggests that the notion of relative primality may offer new pathways to investigate internal properties of recursive numerical sequences.

References

[1] Koshy, Thomas. (2001). *Fibonacci and Lucas Numbers with Applications*. Wiley-Interscience.

[2] Hardy, G. H., & Wright, E. M. (2008). *An Introduction to the Theory of Numbers* (6th ed.). Oxford University Press.