Proofs of Legendre's Conjecture and Some Related Conjectures

Wing K. Yu

Abstract

This paper improves my previous proof of the Legendre conjecture by reducing some redundant statements, improving some corollaries, and simplifying two data tables.

Keywords

Legendre conjecture, Oppermann conjecture, Brocard conjecture, Andrica conjecture
prime gaps, prime number distribution

1. Introduction

In this paper, we will use basic algebraic methods to analyze the binomial coefficients $\binom{\lambda n}{n}$, where λ and n are positive integers, to prove the Legendre conjecture, the Oppermann conjecture, the Brocard conjecture, and the Andrica conjecture [1], [2], [3], [4], [5].

We use the following convention and notation:
$\mathbb{N}_0$ = {0, 1, 2, 3, 4,…}, a set of natural numbers including zero.
$\mathbb{Z}$ = {… -4, -3, -2, -1, 0, 1, 2, 3, 4,…}, a set of integers.
$\mathbb{Z}^+$ = $\{n \in \mathbb{N}_0 : \ n \neq 0\}$ = {1, 2, 3, 4, 5,…}, a set of positive integers.
$N, m, n, \lambda, i, j, k \in \mathbb{Z}^+$
$\mathbb{P}$ = { p_1, p_2, p_3, p_4, p_5, …} = {2, 3, 5, 7, 11,…}, a set of prime numbers.
$p\ , p_i\ , p_m\ , p_n\ , p_k \in \mathbb{P}$
$\mathbb{R}$ = { $\mathbb{Z}$, n_j/n_i, $n_k^{n_j/n_i}$, $e, \ldots \mid n_i, n_j, n_k \in \mathbb{Z} : n_i \neq 0\}$, a set of real numbers.
$a, b, u, v, x, e \in \mathbb{R}$
$n\#$ = $\prod_{n\geq p} p$, the primorial function that is the product of all distinct prime numbers p that are less than or equal to n.

$\pi(x)$ = $\sum_{x\geq p} 1$, the prime counting function that counts the number of all distinct prime numbers p less than or equal to number x.

Binomial coefficient: C_n^k = $\binom{n}{k}$ = $\frac{n!}{k!\,(n-k)!}$

log denotes the natural logarithm: $log\ e$ = 1.

Ceiling and floor: ⌈ 3.42 ⌉ = 4, and ⌊ 3.42 ⌋ = 3.

Intervals: (a, b) = { $d \in \mathbb{Z}$: $a < d < b$ }, $(a, b]$ means $a < d \leq b$, and $[a, b]$ means $a \leq d \leq b$.

$\Gamma_{a\geq p>b}\{N\}$ denotes the prime number factorization operator of the integer expression N. It is the product of the prime numbers in the decomposition of N in the range of $a \geq p > b \geq 1$.
$\Gamma_{a\geq p>b}\{N\}$ has some properties:

It is always true that $\Gamma_{a\geq p>b}\{N\} \geq 1$. **(1.1)**

If there is no prime number in N within the range of $a \geq p > b$, then $\Gamma_{a\geq p>b}\{N\} = 1$, or vice versa, if $\Gamma_{a\geq p>b}\{N\} = 1$, then there is no prime number in N within the range of $a \geq p > b$. **(1.2)**

For example, when $\lambda = 5$ and $n = 4$, $\Gamma_{16\geq p>10}\left\{\binom{20}{4}\right\} = 13^0 \cdot 11^0 = 1$. No prime number 13 or 11 is in $\binom{20}{4}$ in the range of $16 \geq p > 10$.

If there is at least one prime number in N in the range of $a \geq p > b$, then $\Gamma_{a\geq p>b}\{N\} > 1$, or vice versa, if $\Gamma_{a\geq p>b}\{N\} > 1$, then there is at least one prime number in N within the range of $a \geq p > b$. **(1.3)**

For example, when $\lambda = 5$ and $n = 4$, $\Gamma_{18\geq p>16}\left\{\binom{20}{4}\right\} = 17 > 1$. Prime number 17 is in $\binom{20}{4}$ within the range of $18 \geq p > 16$.

Let $v_p(n)$ be the $p\text{-}adic\ valuation$ of n, the exponent of the highest power of p that divides n.

We define *R(p)* by the inequalities $p^{R(p)} \leq \lambda n < p^{R(p)+1}$, and determine $v_p(n)$ of $\binom{\lambda n}{n}$.

Because for any real numbers a and *b*, the expression of $\lfloor a+b \rfloor - \lfloor a \rfloor - \lfloor b \rfloor$ is 0 or 1,

$v_p\left(\binom{\lambda n}{n}\right) = v_p((\lambda n)!) - v_p(((\lambda-1)n)!) - v_p(n!) = \sum_{i=1}^{R(p)} \left(\left\lfloor \frac{\lambda n}{p^i} \right\rfloor - \left\lfloor \frac{(\lambda-1)n}{p^i} \right\rfloor - \left\lfloor \frac{n}{p^i} \right\rfloor\right) \leq R(p)$.

Thus, if p divides $\binom{\lambda n}{n}$, then $v_p\left(\binom{\lambda n}{n}\right) \leq R(p) \leq \log_p(\lambda n)$, or $p^{v_p\left(\binom{\lambda n}{n}\right)} \leq p^{R(p)} \leq \lambda n$ **(1.4)**

If $\lambda n \geq p > \lfloor \sqrt{\lambda n} \rfloor$, then $0 \leq v_p\left(\binom{\lambda n}{n}\right) \leq R(p) \leq 1$. **(1.5)**

Among the first six consecutive natural numbers are three prime numbers 2, 3 and 5. Then, each additional six consecutive natural numbers, at most one can add two prime numbers, $p \equiv 1 \text{ (MOD 6)}$ and $p \equiv 5 \text{ (MOD 6)}$. Thus, $\pi(n) \leq \left\lfloor \frac{n}{3} \right\rfloor + 2 \leq \frac{n}{3} + 2$. **(1.6)**

Since $\binom{2n-1}{n}$ is an integer and all the prime numbers in the range of $(n+1) \leq p \leq (2n-1)$ appear in its numerator but not in its denominator, we have

$\frac{(2n-1)\#}{n\#} \leq \binom{2n-1}{n} = \frac{1}{2}\left(\binom{2n-1}{n-1} + \binom{2n-1}{n}\right) < \frac{1}{2}(1+1)^{2n-1} = 2^{2n-2}$.

The proof proceeds by induction on n.
If $n = 3$, then $n\# = 6 < 8 = 2^{2n-3}$.
If $n = 4$, then $n\# = 6 < 32 = 2^{2n-3}$.
If $n = (2m-1)$ is odd and $n \geq 5$, then $m \geq 3$ and then
$n\# = (2m-1)\# < m\# \cdot 2^{2m-2} < 2^{2m-3} \cdot 2^{2m-2} = 2^{4m-5} = 2^{2n-3}$.
If $n = 2m$ is even and $n \geq 6$, then $m \geq 3$ and then

$n\# = (2m)\# = (2m-1)\# < m\# \cdot 2^{2m-2} < 2^{2m-3} \cdot 2^{2m-2} = 2^{4m-5} < 2^{4m-3} = 2^{2n-3}$.

Thus, when $n \geq 3$, $n\# = \prod_{n \geq p} p < 2^{2n-3}$. **(1.7)**

From the prime number decomposition, when $n > \lfloor \sqrt{\lambda n} \rfloor$,

$\binom{\lambda n}{n} = \Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!} \right\} \cdot \Gamma_{n \geq p > \lfloor \sqrt{\lambda n} \rfloor} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!} \right\} \cdot \Gamma_{\lfloor \sqrt{\lambda n} \rfloor \geq p} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!} \right\}$.

When $n \leq \lfloor \sqrt{\lambda n} \rfloor$, $\binom{\lambda n}{n} \leq \Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!} \right\} \cdot \Gamma_{\lfloor \sqrt{\lambda n} \rfloor \geq p} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!} \right\}$.

Thus, $\binom{\lambda n}{n} \leq \Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!} \right\} \cdot \Gamma_{n \geq p > \lfloor \sqrt{\lambda n} \rfloor} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!} \right\} \cdot \Gamma_{\lfloor \sqrt{\lambda n} \rfloor \geq p} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!} \right\}$

$\Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!} \right\} = \Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{((\lambda-1)n)!} \right\}$ since all prime numbers in $n!$ do not appear in the range of $\lambda n \geq p > n$.

Referring to **(1.5)** and **(1.7)**, when $n \geq (\lambda - 2) \geq 13$, then $\lfloor \sqrt{\lambda n} \rfloor \geq 13$, and

$\Gamma_{n \geq p > \lfloor \sqrt{\lambda n} \rfloor} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!} \right\} \leq \frac{\prod_{n \geq p} p}{13\#} < \frac{2^{2n-3}}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13} = \frac{2^{2n-4}}{15015}$.

Referring to **(1.4)** and **(1.6)**, $\Gamma_{\lfloor \sqrt{\lambda n} \rfloor \geq p} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!} \right\} \leq (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}$.

Thus, when $n \geq (\lambda - 2) \geq 13$, $\binom{\lambda n}{n} < \Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{((\lambda-1)n)!} \right\} \cdot \frac{2^{2n-4}}{15015} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}$. **(1.8)**

2. Lemmas

Lemma 1: If $x \geq 3$, then $\frac{2(2x-1)}{x-1} > \left(\frac{x}{x-1}\right)^x$. **(2.1)**

Proof:

When $x \geq 3$, let $f_1(x) = \frac{2(2x-1)}{x-1}$; then, $f_1{}'(x) = \frac{2(x-1)(2x-1)' - 2(2x-1)(x-1)'}{(x-1)^2} = \frac{-2}{(x-1)^2} < 0$.

Thus, $f_1(x)$ is a strictly decreasing function for $x \geq 3$.

Since $f_1(3) = 5$, and $\lim_{x \to \infty} f_1(x) = 4$, for $x \geq 3$, we have $5 \geq f_1(x) = \frac{2(2x-1)}{x-1} \geq 4$.

Let $f_2(x) = \left(\frac{x}{x-1}\right)^x$, then $f_2{}'(x) = \left(\left(\frac{x}{x-1}\right)^x\right)' = \left(\frac{x}{x-1}\right)^x \cdot \left(log \frac{x}{x-1} - \frac{1}{x-1}\right)$ **(2.1.1)**

When $x \geq 3$, $\frac{1}{x-1} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x^5} + \frac{1}{x^6} + \cdots$

Using the formula: $log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$,

$log \frac{x}{x-1} = log \frac{1}{1+\frac{-1}{x}} = -log\left(1 + \frac{-1}{x}\right) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \frac{1}{4x^4} + \frac{1}{5x^5} + \frac{1}{6x^6} + \cdots$

Thus, for $x \geq 3$, $log \frac{x}{x-1} - \frac{1}{x-1} < 0$.

Since $\left(\frac{x}{x-1}\right)^x$ is a positive number for $x \geq 3$, $f_2{}'(x) = \left(\frac{x}{x-1}\right)^x \cdot \left(log \frac{x}{x-1} - \frac{1}{x-1}\right) < 0$.

Thus, $f_2(x)$ is a strictly decreasing function when $x \geq 3$.

Since $f_2(3) = 3.375$ and $\lim_{x\to\infty} f_2(x) = e \approx 2.718$,

when $x \geq 3$, $3.375 \geq f_2(x) = \left(\frac{x}{x-1}\right)^x \geq e$. **(2.1.2)**

Since for $x \geq 3$, $f_1(x)$ has a lower bound of 4 and $f_2(x)$ has an upper bound of 3.375,

$f_1(x) = \frac{2(2x-1)}{x-1} > f_2(x) = \left(\frac{x}{x-1}\right)^x$. ■

Lemma 2: For $n \geq 2$ and $\lambda \geq 3$, $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$. **(2.2)**

Proof:

When $\lambda \geq 3$ and $n = 2$, $\binom{\lambda n}{n} = \binom{2\lambda}{2} = \frac{2\lambda(2\lambda-1)(2\lambda-2)!}{2(2\lambda-2)!} = \lambda(2\lambda-1)$. **(2.2.1)**

$\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}} = \frac{\lambda^{2\lambda-\lambda+1}}{2(\lambda-1)^{2(\lambda-1)-\lambda+1}} = \frac{\lambda(\lambda-1)}{2} \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda}$. **(2.2.2)**

Referring to **(2.1)**, when $x = \lambda \geq 3$, we have $\frac{2(2\lambda-1)}{\lambda-1} > \left(\frac{\lambda}{\lambda-1}\right)^{\lambda}$. **(2.2.3)**

Since $\frac{\lambda(\lambda-1)}{2}$ is a positive number for $\lambda \geq 3$, referring to **(2.2.1)** and **(2.2.2)**, when $\frac{\lambda(\lambda-1)}{2}$ multiplies both sides of **(2.2.3)**, we have

$\left(\frac{\lambda(\lambda-1)}{2}\right)\left(\frac{2(2\lambda-1)}{\lambda-1}\right) = \lambda(2\lambda-1) = \binom{\lambda n}{n} > \left(\frac{\lambda(\lambda-1)}{2}\right)\left(\frac{\lambda}{\lambda-1}\right)^{\lambda} = \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$.

Thus, $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ when $\lambda \geq 3$ and $n = 2$. **(2.2.4)**

By induction on n, when $\lambda \geq 3$, if $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ is true for n, then for $n+1$,

$$\binom{\lambda(n+1)}{n+1} = \binom{\lambda n+\lambda}{n+1} = \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)(\lambda n+1)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)(n+1)} \cdot \binom{\lambda n}{n}$$

$$\binom{\lambda(n+1)}{n+1} > \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)(\lambda n+1)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)(n+1)} \cdot \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$$

$$\binom{\lambda(n+1)}{n+1} > \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)} \cdot \frac{\lambda n+1}{n} \cdot \frac{1}{(n+1)} \cdot \frac{\lambda^{\lambda n-\lambda+1}}{(\lambda-1)^{(\lambda-1)n-\lambda+1}}$$

Notice $\frac{\lambda n+1}{n} > \lambda$, and $\frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)} > \left(\frac{\lambda}{\lambda-1}\right)^{(\lambda-1)}$

because $\frac{\lambda n+\lambda}{\lambda n+\lambda-n-1} = \frac{\lambda}{\lambda-1}$; $\frac{\lambda n+\lambda-1}{\lambda n+\lambda-n-2} > \frac{\lambda}{\lambda-1}$; $\cdots$ $\frac{\lambda n+2}{\lambda n-n+1} > \frac{\lambda}{\lambda-1}$. Thus,

$$\binom{\lambda(n+1)}{n+1} > \frac{\lambda^{\lambda-1}}{(\lambda-1)^{(\lambda-1)}} \cdot \frac{\lambda}{1} \cdot \frac{1}{(n+1)} \cdot \frac{\lambda^{\lambda n-\lambda+1}}{(\lambda-1)^{(\lambda-1)n-\lambda+1}} = \frac{\lambda^{\lambda(n+1)-\lambda+1}}{(n+1)(\lambda-1)^{(\lambda-1)(n+1)-\lambda+1}} \quad \textbf{(2.2.5)}$$

From **(2.2.4)** and **(2.2.5)**, we have for $n \geq 2$ and $\lambda \geq 3$, $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ ■

Lemma 3: When $x \geq 13$ and $f_4(x) = \frac{60060(x+2)^2 \cdot \left(\left(\frac{x+1}{4}\right) \cdot e\right)^{(x-1)}}{(x^2+2x)^{\frac{x+1}{3}+3}}$,

$$f_4{}'(x) = f_4(x) \cdot \left(ln\left(\frac{x+1}{4}\right) - \frac{ln(x)}{3} - \frac{ln(x+2)}{3} - \frac{2}{x+1} - \frac{2}{3(x+2)} - \frac{10}{3x} + \frac{4}{3}\right). \quad \textbf{(2.3)}$$

Proof:

When $x \geq 13$, let $f_4(x) = 60060\,\frac{u(x)}{v(x)}$ where $u(x) = (x+2)^2 \cdot \left(\left(\frac{x+1}{4}\right) \cdot e\right)^{(x-1)}$ and $v(x) = (x^2+2x)^{\frac{x+1}{3}+3}$.

$$u'(x) = \left((x+2)^2 \cdot \left(\left(\frac{x+1}{4}\right) \cdot e\right)^{(x-1)}\right)'$$

$$= 2(x+2) \cdot \left(\left(\frac{x+1}{4}\right) \cdot e\right)^{(x-1)} + \left(\left(\left(\frac{x+1}{4}\right) \cdot e\right)^{(x-1)}\right)' \cdot (x+2)^2$$

$$= 2(x+2) \cdot \left(\left(\frac{x+1}{4}\right) \cdot e\right)^{(x-1)} + (x+2)^2 \cdot \left(\left(\frac{x+1}{4}\right) \cdot e\right)^{(x-1)} \cdot \left((x-1) \cdot log\left(\frac{x+1}{4} \cdot e\right)\right)'$$

$$= \frac{2}{x+2} \cdot u(x) + u(x) \cdot \left(log\left(\frac{x+1}{4}\right) + 1 + \frac{x-1}{x+1}\right)$$

$$= \left(\frac{2}{x+2} - \frac{2}{x+1} + log\left(\frac{x+1}{4}\right) + 2\right) \cdot u(x)$$

$$v'(x) = \left((x^2+2x)^{\frac{x+1}{3}+3}\right)' = \left(x(x+2)\right)^{\frac{x+1}{3}+3} \cdot \left(\left(\frac{x+1}{3}+3\right) \cdot log(x^2+2x)\right)'$$

$$= v(x) \cdot \left(\left(\frac{x+1}{3}+3\right) \cdot log(x^2+2x)\right)' = v(x) \cdot \left(\frac{1}{3}log(x) + \frac{1}{3}log(x+2) + \left(\frac{x+1}{3}+3\right) \cdot \left(\frac{1}{x+2} + \frac{1}{x}\right)\right)$$

$$= v(x) \cdot \left(\frac{1}{3}log(x) + \frac{1}{3}log(x+2) + \frac{x+1}{3x} + \frac{x+1}{3(x+2)} + \frac{3}{x+2} + \frac{3}{x}\right)$$

$$= v(x) \cdot \left(\frac{1}{3}log(x) + \frac{1}{3}log(x+2) + \frac{2}{3} + \frac{8}{3(x+2)} + \frac{10}{3x}\right)$$

$$f_4{}'(x) = 60060\,\frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{\left(v(x)\right)^2}$$

$$= 60060\,\frac{u(x)}{v(x)} \cdot \left(\frac{2}{x+2} - \frac{2}{x+1} + log\left(\frac{x+1}{4}\right) + 2 - \frac{1}{3}log(x) - \frac{1}{3}log(x+2) - \frac{2}{3} - \frac{8}{3(x+2)} - \frac{10}{3x}\right)$$

$$= f_4(x) \cdot \left(log\left(\frac{x+1}{4}\right) - \frac{log(x)}{3} - \frac{log(x+2)}{3} - \frac{2}{x+1} - \frac{2}{3(x+2)} - \frac{10}{3x} + \frac{4}{3}\right) \quad \blacksquare$$

Lemma 4: When $x \geq 13$ and $f_5(x) = log\left(\frac{x+1}{4}\right) - \frac{log(x)}{3} - \frac{log(x+2)}{3} - \frac{2}{x+1} - \frac{2}{3(x+2)} - \frac{10}{3x} + \frac{4}{3}$,

$$f_5{}'(x) = \frac{x^2+2x-2}{3x(x+1)(x+2)} + \frac{2}{(x+1)^2} + \frac{2}{3(x+2)^2} + \frac{10}{3x^2} > 0. \quad \textbf{(2.4)}$$

Proof:

When $x \geq 13$ and $f_5(x) = log\left(\frac{x+1}{4}\right) - \frac{log(x)}{3} - \frac{log(x+2)}{3} - \frac{2}{x+1} - \frac{2}{3(x+2)} - \frac{10}{3x} + \frac{4}{3}$,

$$f_5{}'(x) = \frac{1}{x+1} - \frac{1}{3x} - \frac{1}{3(x+2)} + \frac{2}{(x+1)^2} + \frac{2}{3(x+2)^2} + \frac{10}{3x^2}$$

$$= \frac{3x^2+6x}{3x(x+1)(x+2)} - \frac{x^2+3x+2}{3x(x+1)(x+2)} - \frac{x^2+x}{3x(x+1)(x+2)} + \frac{2}{(x+1)^2} + \frac{2}{3(x+2)^2} + \frac{10}{3x^2}$$

$$= \frac{x^2+2x-2}{3x(x+1)(x+2)} + \frac{2}{(x+1)^2} + \frac{2}{3(x+2)^2} + \frac{10}{3x^2} > 0 \quad \blacksquare$$

Lemma 5: When $x \geq 13$ and $f_7(x) = log\left(\frac{x+1}{4}\right) + 1 - \frac{\sqrt{x+2}}{6\sqrt{x}} \cdot (log(x+2) + log(x) + 2) - \frac{3}{x}$,

$$f_7{}'(x) = \left(\frac{1}{x+1} - \frac{1}{3\sqrt{x(x+2)}}\right) + \frac{log(x+2)+log(x)}{6x\sqrt{x(x+2)}} + \frac{3}{x^2} > 0. \quad \textbf{(2.5)}$$

Proof:

When $x \geq 13$ and $f_7(x) = log\left(\frac{x+1}{4}\right) + 1 - \frac{\sqrt{x+2}}{6\sqrt{x}} \cdot (log(x+2) + log(x) + 2) - \frac{3}{x}$,

$f_7{}'(x) = \frac{1}{x+1} - \frac{\sqrt{x+2}}{6\sqrt{x}} \cdot \left(\frac{1}{x+2} + \frac{1}{x}\right) + \frac{log(x+2)+log(x)+2}{6x\sqrt{x(x+2)}} + \frac{3}{x^2}$

$= \left(\frac{1}{x+1} - \frac{1}{3\sqrt{x(x+2)}}\right) + \frac{log(x+2)+log(x)}{6x\sqrt{x(x+2)}} + \frac{3}{x^2}$

Because $\frac{1}{x+1} - \frac{1}{3\sqrt{x(x+2)}} = \frac{3\sqrt{x(x+2)}-(x+1)}{3(x+1)\sqrt{x(x+2)}} \cdot \frac{3\sqrt{x(x+2)}+(x+1)}{3\sqrt{x(x+2)}+(x+1)} = \frac{8x^2+16x-1}{9(x+1)\sqrt{x(x+2)}\left(\sqrt{x(x+2)}+(x+1)\right)} > 0$,

$\frac{log(x+2)+log(x)}{6x\sqrt{x(x+2)}} > 0$, and $\frac{3}{x^2} > 0$,

$f_7{}'(x) = \left(\frac{1}{x+1} - \frac{1}{3\sqrt{x(x+2)}}\right) + \frac{log(x+2)+log(x)}{6x\sqrt{x(x+2)}} + \frac{3}{x^2} > 0$ when $x \geq 13$. ■

Lemma 6: When $m \geq \lambda - 2 \geq 13$, $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)m}{i} \geq p > \frac{\lambda m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right) = 1.$ **(2.6)**

Proof:

When $m \geq \lambda - 2 \geq 13$, in the products of $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)m}{i} \geq p > \frac{\lambda m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)$, there are intervals of $\left(\frac{\lambda m}{i+1}, \frac{(\lambda-1)m}{i}\right]$ where i = 1 to λ −2. If there is no prime number in any interval $\left(\frac{\lambda m}{i+1}, \frac{(\lambda-1)m}{i}\right]$, then referring to **(1.2)**, $\Gamma_{\frac{(\lambda-1)m}{i} \geq p > \frac{\lambda m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$ = 1; otherwise all the results are as follows.

In $\Gamma_{\frac{(\lambda-1)m}{1} \geq p > \frac{\lambda m}{2}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$, every distinct prime number with $\frac{(\lambda-1)m}{1} \geq p > \frac{\lambda m}{2}$ in the numerator $(\lambda m)!$ also exactly appears in the denominator $((\lambda-1)m)!$. They cancel each other out in $\frac{(\lambda m)!}{((\lambda-1)m)!}$. Referring to **(1.2)**, $\Gamma_{\frac{(\lambda-1)m}{1} \geq p > \frac{\lambda m}{2}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$ = 1.

In $\Gamma_{\frac{(\lambda-1)m}{2} \geq p > \frac{\lambda m}{3}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$, every distinct prime number that satisfies $\frac{(\lambda-1)m}{2} \geq p > \frac{\lambda m}{3}$ in the numerator $(\lambda m)!$ is of the form of $2p \cdot p$ = $(2)! \cdot p^2$. The same is true for the denominator $((\lambda-1)m)!$. They cancel each other out in $\frac{(\lambda m)!}{((\lambda-1)m)!}$, then $\Gamma_{\frac{(\lambda-1)m}{2} \geq p > \frac{\lambda m}{3}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$ = 1.

In $\Gamma_{\frac{(\lambda-1)m}{3} \geq p > \frac{\lambda m}{4}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$, every distinct prime number that satisfies $\frac{(\lambda-1)m}{3} \geq p > \frac{\lambda m}{4}$ in the numerator $(\lambda m)!$ is of the form of $3p \cdot 2p \cdot p$ = $(3)! \cdot p^3$. The same is true for the denominator $((\lambda-1)m)!$. They cancel each other out in $\frac{(\lambda m)!}{((\lambda-1)m)!}$, then $\Gamma_{\frac{(\lambda-1)m}{3} \geq p > \frac{\lambda m}{4}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$ = 1.

…

In $\Gamma_{\frac{(\lambda-1)m}{\lambda-2} \geq p > \frac{\lambda m}{\lambda-1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$, every distinct prime number that satisfies $\frac{(\lambda-1)m}{\lambda-2} \geq p > \frac{\lambda m}{\lambda-1}$ in the numerator $(\lambda m)!$ is of the form of $(\lambda-2) \cdots 3p \cdot 2p \cdot p$ = $(\lambda-2)! \cdot p^{\lambda-2}$. The same is true for the denominator $((\lambda-1)m)!$.

They cancel each other in $\frac{(\lambda m)!}{((\lambda-1)m)!}$, then $\Gamma_{\frac{(\lambda-1)m}{\lambda-2}\geq p>\frac{\lambda m}{\lambda-1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$ = 1.

Thus, When $m \geq \lambda - 2 \geq 13$, $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)m}{i}\geq p>\frac{\lambda m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right) = 1$. ■

3. A Prime Number between $(\lambda - 1)n$ and λn when $n \geq (\lambda - 2) \geq 13$

Proposition:

For $n \geq \lambda - 2 \geq 13$, there is at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **(3.1)**

Proof:

Applying **(2.2)** to **(1.8)**, when $n \geq (\lambda - 2) \geq 13$,

$$\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}} < \binom{\lambda n}{n} < \Gamma_{\lambda n\geq p>n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} \cdot \frac{2^{2n-4}}{15015} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}.$$

Because $(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2} > 0$ and $\frac{2^{2n-4}}{15015} > 0$,

$$\Gamma_{\lambda n\geq p>n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > \frac{\lambda^{\lambda n-\lambda+1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}\cdot\frac{2^{2n-4}}{15015}\cdot n(\lambda-1)^{(\lambda-1)n-\lambda+1}} = \frac{60060\lambda^2\cdot\left(\left(\frac{\lambda-1}{4}\right)\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} > 0.$$

Referring to **(2.1.2)**, when $\lambda \geq 3$, $\left(\frac{\lambda}{\lambda-1}\right)^{\lambda} \geq e$. Thus, when $n \geq (\lambda - 2) \geq 13$,

$$\Gamma_{\lambda n\geq p>n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > \frac{60060\lambda^2\cdot\left(\left(\frac{\lambda-1}{4}\right)\cdot e\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} = f_3(n,\lambda) > 0. \quad \textbf{(3.2)}$$

Let $x \geq y - 2 \geq 13$, then, $f_3(x,y) = \frac{60060y^2\cdot\left(\left(\frac{y-1}{4}\right)\cdot e\right)^{(x-1)}}{(xy)^{\frac{\sqrt{xy}}{3}+3}}$ is a continuous function because both the numerator and the denominator are positive real numbers. **(3.3)**

When $x = y - 2 \geq 13$,

$$f_3(x,y) = \frac{60060y^2\cdot\left(\left(\frac{y-1}{4}\right)\cdot e\right)^{(x-1)}}{(xy)^{\frac{\sqrt{xy}}{3}+3}} = \frac{60060(x+2)^2\cdot\left(\left(\frac{x+1}{4}\right)\cdot e\right)^{(x-1)}}{(x^2+2x)^{\frac{\sqrt{x(x+2)}}{3}+3}}$$

$$> f_4(x) = \frac{60060(x+2)^2\cdot\left(\left(\frac{x+1}{4}\right)\cdot e\right)^{(x-1)}}{(x^2+2x)^{\frac{x+1}{3}+3}} > 0. \quad \textbf{(3.4)}$$

Referring to **(2.3)**,

$$f_4{}'(x) = f_4(x)\cdot\left(\log\left(\frac{x+1}{4}\right) - \frac{\log(x)}{3} - \frac{\log(x+2)}{3} - \frac{2}{x+1} - \frac{2}{3(x+2)} - \frac{10}{3x} + \frac{4}{3}\right) = f_4(x)\cdot f_5(x)$$

where $f_5(x) = \log\left(\frac{x+1}{4}\right) - \frac{\log(x)}{3} - \frac{\log(x+2)}{3} - \frac{2}{x+1} - \frac{2}{3(x+2)} - \frac{10}{3x} + \frac{4}{3}$

Referring to **(2.4)**, $f_5{}'(x) = \frac{x^2+2x-2}{3x(x+1)(x+2)} + \frac{2}{(x+1)^2} + \frac{2}{3(x+2)^2} + \frac{10}{3x^2} > 0$ when $x \geq 13$.

Thus, $f_5(x)$ is a strictly increasing function for $x \geq 13$.

When $x = 13$, $f_5(x) = log\left(\frac{13+1}{4}\right) - \frac{log(13)}{3} - \frac{log(15)}{3} - \frac{2}{13+1} - \frac{2}{3(13+2)} - \frac{10}{39} + \frac{4}{3} \approx 0.385 > 0$.

Because when $x \geq 13$, $f_5(x) > 0$, then $f_4{}'(x) = f_4(x) \cdot f_5(x) > 0$.

Thus, $f_4(x)$ is a strictly increasing function when $x \geq 13$.

Referring to **(3.4)**, as long as $x = (y-2) \geq 13$, because $f_3(x,y) > f_4(x)$, $f_3(x,y)$ is an increasing function respect to both x and y.

Thus, when $x = (y-2) \geq 13$, $f_3(x+1, y+1) > f_3(x,y)$. **(3.5)**

And, when $n = (\lambda - 2) \geq 13$, $f_3(n+1, \lambda+1) \geq f_3(n, \lambda)$. **(3.6)**

Referring to **(3.3)**, when $x \geq y - 2 \geq 13$,

$$\frac{\partial f_3(x,y)}{\partial x} = f_3(x,y) \cdot \left(log\left(\frac{y-1}{4}\right) + 1 - \frac{\sqrt{y}}{6\sqrt{x}} \cdot log(xy) - \frac{\sqrt{y}}{3\sqrt{x}} - \frac{3}{x}\right) = f_3(x,y) \cdot f_6(x,y)$$

where $f_6(x,y) = log\left(\frac{y-1}{4}\right) + 1 - \frac{\sqrt{y}}{6\sqrt{x}} \cdot log(xy) - \frac{\sqrt{y}}{3\sqrt{x}} - \frac{3}{x}$. **(3.7)**

Let $x = y - 2 \geq 13$,

$$f_6(x,y) = f_7(x) = log\left(\frac{x+1}{4}\right) + 1 - \frac{\sqrt{x+2}}{6\sqrt{x}} \cdot (log(x+2) + log(x) + 2) - \frac{3}{x}.$$

Referring to **(2.5)**, $f_7{}'(x) = \left(\frac{1}{x+1} - \frac{1}{3\sqrt{x(x+2)}}\right) + \frac{log(x+2)+log(x)}{6x\sqrt{x(x+2)}} + \frac{3}{x^2} > 0$ when $x \geq 13$.

Thus, when $x \geq 13$, $f_7(x)$ is a strictly increasing function.

When $x = (y-2) \geq 13$, because $f_6(x,y) = f_7(x)$, $f_6(x,y)$ is an increasing function respect to both x and y.

Thus, when $x = (y-2) \geq 13$, $f_6(x+1, y+1) > f_6(x,y)$. **(3.8)**

Referring to **(3.7)**, when $x \geq (y-2) \geq 13$,

$$\frac{\partial f_6(x,y)}{\partial x} = \frac{\partial}{\partial x}\left(log\left(\frac{y-1}{4}\right) + 1 - \frac{\sqrt{y}}{6\sqrt{x}} \cdot log(xy) - \frac{\sqrt{y}}{3\sqrt{x}} - \frac{3}{x}\right)$$

$$= \frac{\sqrt{y}}{12x\sqrt{x}} \cdot log(xy) - \frac{\sqrt{y}}{6x\sqrt{x}} + \frac{\sqrt{y}}{6x\sqrt{x}} + \frac{3}{x^2} = \frac{\sqrt{y}}{12x\sqrt{x}} \cdot log(xy) + \frac{3}{x^2} > 0.$$

Thus, when $x \geq (y-2) \geq 13$, $f_6(x,y)$ is an increasing function respect to x. **(3.9)**

When $x = (y-2) = 13$, $f_6(x,y) = log\left(\frac{15-1}{4}\right) + 1 - \frac{\sqrt{15} \cdot log(195)}{6\sqrt{13}} - \frac{\sqrt{15}}{3\sqrt{13}} - \frac{3}{13} \approx 0.720 > 0$.

Referring to **(3.8)**, when $x = (y-2) \geq 13$, $f_6(x,y) > 0$.

Referring to **(3.9)**, when $x \geq (y-2) \geq 13$, $f_6(x,y) > 0$.

Referring to **(3.7)**, when $x \geq (y-2) \geq 13$, since $f_3(x,y) > 0$ and $f_6(x,y) > 0$, then $\frac{\partial f_3(x,y)}{\partial x} > 0$, and $f_3(x,y)$ is an increasing function respect to x.

Thus, when $x \geq (y-2) \geq 13$, $f_3(x+1, y) > f_3(x,y)$. **(3.10)**

And, when $n \geq (\lambda - 2) \geq 13,\ f_3(n+1,\lambda) \geq f_3(n,\lambda)$. **(3.11)**

When $n = (\lambda - 2) = 13$,

$$f_3(n,\lambda) = \frac{60060\lambda^2 \cdot \left(\left(\frac{\lambda-1}{4}\right)\cdot e\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} = \frac{60060 \cdot 15^2 \cdot \left(\left(\frac{15-1}{4}\right)\cdot e\right)^{(13-1)}}{(15\cdot 13)^{\frac{\sqrt{15\cdot 13}}{3}+3}} \approx \frac{7.432\text{E+18}}{3.386\text{E+17}} > 1.$$

Referring to **(3.6)**, when $n = (\lambda - 2) \geq 13,\ \ f_3(n,\lambda) > 1$.

Referring to **(3.11)**, when $n \geq (\lambda - 2) \geq 13,\ \ f_3(n,\lambda) > 1$.

Thus, from **(3.2)**, when $n \geq \lambda - 2 \geq 13,\ \ \Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > f_3(n,\lambda) > 1$. **(3.12)**

Let $m \geq n$. When $m \geq n \geq \lambda - 2 \geq 13$, $\Gamma_{\lambda m \geq p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} > f_3(m\,,\lambda) > 1$. **(3.13)**

From **(1.8)**, when $n \geq (\lambda - 2) \geq 13$, if there is a prime number p in $\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}$, then $p \geq n+1 = \sqrt{(n+2)n+1} > \sqrt{\lambda n}$. From **(1.5)**, $0 \leq v_p\left(\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}\right) \leq R(p) \leq 1$.

Thus, when $m \geq n \geq \lambda - 2 \geq 13$, every distinct prime number in $\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}$ and also in $\Gamma_{\lambda m \geq p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$ has a power of 0 or 1.

$$\Gamma_{\lambda m \geq p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$$

$$= \Gamma_{\lambda m \geq p > (\lambda-1)m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} \cdot \prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)m}{i} \geq p > \frac{\lambda m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} \cdot \Gamma_{\frac{\lambda m}{i+1} \geq p > \frac{(\lambda-1)m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right).$$

Referring to **(2.6)**, $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)m}{i} \geq p > \frac{\lambda m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right) = 1$.

Thus, $\Gamma_{\lambda m \geq p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} = \Gamma_{\lambda m \geq p > (\lambda-1)m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} \cdot \prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{\lambda m}{i+1} \geq p > \frac{(\lambda-1)m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)$

$\Gamma_{\lambda m \geq p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} = \prod_{i=1}^{\lambda-1}\left(\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)$. **(3.14)**

$\prod_{i=1}^{\lambda-1}\left(\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)$ is the product of (λ –1) sectors from i = 1 to i = ($\lambda - 1$).

Each of these sectors is the prime number factorization of $\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$ where $\frac{(\lambda m)!}{((\lambda-1)m)!}$ is the product of the consecutive integers in the interval of ($\frac{(\lambda-1)m}{i}$, $\frac{\lambda m}{i}$] .

From **(3.13)** and **(3.14)**, when $m \geq n \geq \lambda - 2 \geq 13,\ \ \prod_{i=1}^{\lambda-1}\left(\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right) > 1$.

Referring to **(1.1)**, $\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} \geq 1$. Thus, when $m \geq n \geq \lambda - 2 \geq 13$, at least one of the sectors in $\prod_{i=1}^{\lambda-1}\left(\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)$ is greater than one.

Let $\Gamma_{\frac{\lambda m}{j}\geq p>\frac{(\lambda-1)m}{j}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}>1$ be such a sector with $(\lambda-1)\geq j\geq 1$, and let $m=nj$.

Thus, when $m=nj\geq n\geq\lambda-2\geq 13$,

$$\Gamma_{\lambda m\geq p>m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$$

$$\geq\Gamma_{\frac{\lambda m}{j}\geq p>\frac{(\lambda-1)m}{j}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}=\Gamma_{\frac{\lambda nj}{j}\geq p>\frac{(\lambda-1)nj}{j}}\left\{\frac{(\lambda nj)!}{((\lambda-1)nj)!}\right\}=\Gamma_{\lambda n\geq p>(\lambda-1)n}\left\{\frac{(\lambda nj)!}{((\lambda-1)nj)!}\right\}>1. \qquad \textbf{(3.15)}$$

$$\frac{(\lambda nj)!}{((\lambda-1)nj)!}=\frac{(\boldsymbol{\lambda nj})\cdot(\lambda nj-1)\cdots(\boldsymbol{\lambda nj-j})\cdots(\boldsymbol{\lambda nj-2j})\cdots(\boldsymbol{\lambda nj-(n-1)j})\cdots(\lambda nj-nj+1)\cdot((\lambda-1)nj)!}{((\lambda-1)nj)!}$$

$$\frac{(\lambda nj)!}{((\lambda-1)nj)!}=\frac{\boldsymbol{j}\cdot(\boldsymbol{\lambda n})\cdot(\lambda nj-1)\cdots\boldsymbol{j}\cdot(\boldsymbol{\lambda n-1})\cdots\boldsymbol{j}\cdot(\boldsymbol{\lambda n-2})\cdots\boldsymbol{j}\cdot(\boldsymbol{\lambda n-n+1})\cdots(\lambda nj-nj+1)\cdot((\lambda-1)nj)!}{((\lambda-1)nj)!}$$

Thus, $\frac{(\lambda nj)!}{((\lambda-1)nj)!}$ contains all the factors of (λn), $(\lambda n-1)$, $(\lambda n-2)$,… $(\lambda n-n+1)$ in $\frac{(\lambda n)!}{((\lambda-1)n)!}$.

These factors make up all the consecutive integers in the range of $\lambda n\geq p>(\lambda-1)n$ in $\frac{(\lambda n)!}{((\lambda-1)n)!}$.

Referring to the definition, all prime numbers of $\frac{(\lambda nj)!}{((\lambda-1)nj)!}$ in the ranges of $p>\lambda n$ and in $(\lambda-1)n>p$ do not contribute to $\Gamma_{\lambda n\geq p>(\lambda-1)n}\left\{\frac{(\lambda nj)!}{((\lambda-1)nj)!}\right\}$, nor does j for $(\lambda-1)\geq j\geq 1$.

Only the prime numbers of $\Gamma_{\lambda n\geq p>(\lambda-1)n}\left\{\frac{(\lambda nj)!}{((\lambda-1)nj)!}\right\}$ within the range of $\lambda n\geq p>(\lambda-1)n$ present in $\Gamma_{\lambda n\geq p>(\lambda-1)n}\left\{\frac{(\lambda nj)!}{((\lambda-1)nj)!}\right\}$. Because $\frac{(\lambda n)!}{((\lambda-1)n)!}$ is the product of all the consecutive integers in this range, $\Gamma_{\lambda n\geq p>(\lambda-1)n}\left\{\frac{(\lambda nj)!}{((\lambda-1)nj)!}\right\}=\Gamma_{\lambda n\geq p>(\lambda-1)n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}$. **(3.16)**

Referring to **(3.15)** and **(3.16)**, when $m=nj\geq n\geq\lambda-2\geq 13$,

$$\Gamma_{\lambda m\geq p>m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\geq\Gamma_{\lambda n\geq p>(\lambda-1)n}\left\{\frac{(\lambda nj)!}{((\lambda-1)nj)!}\right\}=\Gamma_{\lambda n\geq p>(\lambda-1)n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}>1.$$

Thus, when $n\geq\lambda-2\geq 13$, $\Gamma_{\lambda n\geq p>(\lambda-1)n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}>1$. Referring to **(1.3)**, there exists at least a prime number p such that $(\lambda-1)n<p\leq\lambda n$.

Thus, **Proposition (3.1)** is proven. It becomes a theorem: **Theorem (3.1)**. ■

4. Proof of Legendre's Conjecture

Legendre's conjecture states that there is a prime number between n^2 and $(n+1)^2$ for every positive integer n. **(4.1)**

Proof:

Referring to **Theorem (3.1)**, for integers $j \geq k-2 \geq 13$, there exists at least a prime number p such that $j(k-1) < p \leq jk$. **(4.2)**

When $k = j+1 \geq 15$, then $j = k-1 \geq 14$.

Applying $k = j+1$ into **(4.2)**, then $j^2 < p \leq j(j+1) < (j+1)^2$.

Let $n = j \geq 14$, then we have $n^2 < p < (n+1)^2$. **(4.3)**

For $1 \leq n \leq 13$, we have a table, **Table 1**, that shows Legendre's conjecture valid. **(4.4)**

Table 1: For $1 \leq n \leq 13$, there is a prime number between n^2 and $(n+1)^2$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13
n^2	1	4	9	16	25	36	49	64	81	100	121	144	169
p	3	5	11	19	29	41	53	67	83	103	127	149	173
$(n+1)^2$	4	9	16	25	36	49	64	81	100	121	144	169	196

Combining **(4.3)** and **(4.4)**, we have proven Legendre's conjecture. ■

Extension of Legendre's conjecture

There are at least two prime numbers, p_n and p_m , between j^2 and $(j+1)^2$ for every positive integer j such that $j^2 < p_n \leq j(j+1)$ and $j(j+1) < p_m < (j+1)^2$ where p_n is the n^{th} prime number, p_m is the m^{th} prime number, and $m \geq n+1$. **(4.5)**

Proof:

Referring to **Theorem (3.1)**, for integers $j \geq k-2 \geq 13$, there exists at least a prime number p such that $j(k-1) < p \leq jk$.

When $k-1 = j \geq 14$, then $j(k-1) = j^2 < p_n \leq jk = j(j+1)$. Thus, there is at least a prime number p_n such that $j^2 < p_n \leq j(j+1)$ when $j = k-1 \geq 14$.

When $j = k-2 \geq 14$, then $k = j+2$.

$j(k-1) = j(j+1) < p_m \leq jk = j(j+2) < (j+1)^2$. Thus, there is at least another prime number p_m such that $j(j+1) < p_m < (j+1)^2$ when $j = k-2 \geq 14$.

Thus, when $j \geq 14$, there are at least two prime numbers p_n and p_m between j^2 and $(j+1)^2$ such that $j^2 < p_n \leq j(j+1) < p_m < (j+1)^2$ where $m \geq n+1$ for $p_m > p_n$. **(4.6)**

For $1 \leq j \leq 13$, we have a table, **Table 2**, that shows that **(4.5)** is valid. **(4.7)**

Table 2: For $1 \leq j \leq 18$, there are 2 primes such that $j^2 < p_n \leq j(j+1) < p_m < (j+1)^2$.

j	1	2	3	4	5	6	7	8	9	10	11	12	13
j^2	1	4	9	16	25	36	49	64	81	100	121	144	169
p_n	2	5	11	19	29	41	53	67	83	103	127	149	173
$j(j+1)$	2	6	12	20	30	42	56	72	90	110	132	156	182
p_m	3	7	13	23	31	43	59	73	97	113	137	163	191
$(j+1)^2$	4	9	16	25	36	49	64	81	100	121	144	169	196

Combining **(4.6)** and **(4.7)**, we have proven **(4.5)**. It becomes a theorem: **Theorem (4.5)**. ■

5. The Proofs of Three Related Conjectures

Oppermann's conjecture states that for every integer $x > 1$, there is at least one prime number between $x(x-1)$ and x^2, and at least another prime number between x^2 and $x(x+1)$. **(5.1)**

Proof:

Theorem (4.5) states that there are at least two prime numbers, p_n and p_m , between j^2 and $(j+1)^2$ for every positive integer j such that $j^2 < p_n \leq j(j+1) < p_m < (j+1)^2$ where $m \geq n+1$ for $p_m > p_n$.

$j(j+1)$ is a composite number except $j = 1$. Since $j^2 < p_n \leq j(j+1)$ is valid for every positive integer j, when we replace j with $j+1$, we have $(j+1)^2 < p_k < (j+1)(j+2)$. Thus, we have $j(j+1) < p_m < (j+1)^2 < p_k < (j+1)(j+2)$. **(5.2)**

When $x > 1$, then $(x-1) \geq 1$. Substituting j with $(x-1)$ in **(5.2)**, we have
$x(x-1) < p_m < x^2 < p_k < x(x+1)$ **(5.3)**

Thus, we have proven Oppermann's conjecture. ■

Brocard's conjecture states that there are at least 4 prime numbers between $(p_n)^2$ and $(p_{n+1})^2$, where p_n is the n^{th} prime number, for every $n > 1$. **(5.4)**

Proof:

Theorem (4.5) states that there are at least two prime numbers, p_n and p_m , between j^2 and $(j+1)^2$ such that $j^2 < p_n \leq j(j+1)$ and $j(j+1) < p_m < (j+1)^2$ for every positive integer j , where $m \geq n+1$ for $p_m > p_n$. When $j > 1$, $j(j+1)$ is a composite number. Then **Theorem (4.5)** can be written as $j^2 < p_n < j(j+1)$ and $j(j+1) < p_m < (j+1)^2$.

In the prime number series: $p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ p_4 = 7,\ p_5 = 11, \ldots$ Except p_1, all prime numbers are odd numbers. Their intervals are 2 or more. Thus, when $n > 1$, $(p_{n+1} - p_n) \geq 2$. Thus, we have $p_n < (p_n + 1) < (p_n + 2) \leq p_{n+1}$ when $n > 1$. **(5.5)**

Applying **Theorem (4.5)** to **(5.5)**, when $n > 1$, we have at least two prime numbers p_{m1} , and p_{m2} in between $(p_n)^2$ and $(p_n+1)^2$ such that $(p_n)^2 < p_{m1} < p_n(p_n+1) < p_{m2} < (p_n+1)^2$, and at least two more prime numbers p_{m3} , p_{m4} in between $(p_n+1)^2$ and $(p_n+2)^2$ such that $(p_n+1)^2 < p_{m3} < (p_n+1)(p_n+2) < p_{m4} < (p_n+2)^2 \leq (p_{n+1})^2$.

Thus, there are at least 4 prime numbers between $(p_n)^2$ and $(p_{n+1})^2$ for $n > 1$ such that
$(p_n)^2 < p_{m1} < p_n(p_n+1) < p_{m2} < (p_n+1)^2 < p_{m3} < (p_n+1)(p_n+2) < p_{m4} < (p_{n+1})^2$ **(5.6)**

Thus, Brocard's conjecture is proven. ■

Andrica's conjecture is named after Dorin Andrica. It is a conjecture regarding the gaps between prime numbers. The conjecture states that the inequality $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ holds for all n where p_n is the n^{th} prime number. If $g_n = p_{n+1} - p_n$ denotes the n^{th} prime gap, then Andrica's conjecture can also be rewritten as $g_n < 2\sqrt{p_n} + 1$. **(5.7)**

Proof:

From **Theorem (4.5)**, for every positive integer j, there are at least two prime numbers p_n and p_m between j^2 and $(j+1)^2$ such that $j^2 < p_n \leq j(j+1) < p_m < (j+1)^2$ where $m \geq n+1$ for $p_m > p_n$. Since $m \geq n+1$, we have $p_m \geq p_{n+1}$.

Thus, we have $j^2 < p_n$ **(5.8)**

and $p_{n+1} \leq p_m < (j+1)^2$. **(5.9)**

Since j, p_n , p_{n+1}, and $(j+1)$ are positive integers,

$j < \sqrt{p_n}$ **(5.10)**

and $\sqrt{p_{n+1}} < j + 1$. **(5.11)**

Applying **(5.10)** to **(5.11)**, we have $\sqrt{p_{n+1}} < \sqrt{p_n} + 1$. **(5.12)**

Thus, $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ holds for all n since in **Theorem (4.5)**, j holds for all positive integers.

Using the prime gap to prove this conjecture, from **(5.8)** and **(5.9)**, we have

$g_n = p_{n+1} - p_n < (j+1)^2 - j^2 = 2j + 1$. From **(5.10)**, $j < \sqrt{p_n}$.

Thus, $g_n = p_{n+1} - p_n < 2\sqrt{p_n} + 1$. **(5.13)**

Thus, Andrica's conjecture is proven. ■

6. References

[1] W. Yu, (2023) The Proofs of Legendre's Conjecture and Three Related Conjectures. *Journal of Applied Mathematics and Physics*, **11**, 1319-1336.

[2] Mathword, https://mathworld.wolfram.com/LegendresConjecture.

[3] Oppermann, L. (1882), *Om vor Kundskab om Primtallenes Mængde given Grændser,* Oversigt Oversigt over Det Kongelige Danske Videnskabernes Selskabs Forhandlinger og Dets Medlemmers Arbejder: 169–179

[4] Wikenigma, https://wikenigma.org.uk/content/mathematics/brocards_conjecture

[5] Andrica, D. (1986). "Note on a conjecture in prime number theory". *Studia Univ. Babes–Bolyai Math*. **31** (4): 44–48. ISSN 0252-1938. Zbl 0623.10030.