Proofs of Legendre Conjecture and Some Related Conjectures

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Abstract

This paper is an improvement on my previous work and proves the Legendre, Oppermann, Brocard, and Andrica conjectures using basic analytical methods.

Keywords

Legendre conjecture, Oppermann conjecture, Brocard conjecture, Andrica conjecture
prime gaps, prime number distribution

1. Introduction

In this paper, we will use basic algebraic methods to analyze the binomial coefficients $\binom{\lambda n}{n}$, where λ and n are positive integers, to prove the Legendre conjecture, the Oppermann conjecture, the Brocard conjecture, and the Andrica conjecture [1], [2], [3], [4], [5].

Definition: $\Gamma_{a\geq p>b}\left\{\binom{\lambda n}{n}\right\}$ denotes the prime number factorization operator of the integer expression $\binom{\lambda n}{n}$. It is the product of the prime numbers in the decomposition of $\binom{\lambda n}{n}$ in the range of $a \geq p > b$. In this operator, p is a prime number, a and b are real numbers, and $a \geq p > b \geq 1$.

It has some properties:

It is always true that $\Gamma_{a\geq p>b}\left\{\binom{\lambda n}{n}\right\} \geq 1$. **(1.1)**

If there is no prime number in $\binom{\lambda n}{n}$ within the range of $a \geq p > b$, then $\Gamma_{a\geq p>b}\left\{\binom{\lambda n}{n}\right\} = 1$, or vice versa, if $\Gamma_{a\geq p>b}\left\{\binom{\lambda n}{n}\right\} = 1$, then there is no prime number in $\binom{\lambda n}{n}$ within the range of $a \geq p > b$. **(1.2)**

For example, when $\lambda = 5$ and $n = 4$, $\Gamma_{16\geq p>10}\left\{\binom{20}{4}\right\} = 13^0 \cdot 11^0 = 1$. No prime number 13 or 11 is in $\binom{20}{4}$ in the range of $16 \geq p > 10$.

If there is at least one prime number in $\binom{\lambda n}{n}$ in the range of $a \geq p > b$, then $\Gamma_{a\geq p>b}\left\{\binom{\lambda n}{n}\right\} > 1$, or vice versa, if $\Gamma_{a\geq p>b}\left\{\binom{\lambda n}{n}\right\} > 1$, then there is at least one prime number in $\binom{\lambda n}{n}$ within the range of $a \geq p > b$. **(1.3)**

For example, when $\lambda = 5$ and $n = 4$, $\Gamma_{18\geq p>16}\left\{\binom{20}{4}\right\} = 17 > 1$. Prime number 17 is in $\binom{20}{4}$ within the range of $18 \geq p > 16$.

Let $v_p(n)$ be the $p\text{-}adic\ valuation$ of n, the exponent of the highest power of p that divides n.

We define *R(p)* by the inequalities $p^{R(p)} \le \lambda n < p^{R(p)+1}$, and determine $v_p(n)$ of $\binom{\lambda n}{n}$.

$v_p\left(\binom{\lambda n}{n}\right) = v_p((\lambda n)!) - v_p(((\lambda-1)n)!) - v_p(n!) = \sum_{i=1}^{R(p)}\left(\left\lfloor\frac{\lambda n}{p^i}\right\rfloor - \left\lfloor\frac{(\lambda-1)n}{p^i}\right\rfloor - \left\lfloor\frac{n}{p^i}\right\rfloor\right) \le R(p)$

because for any real numbers a and b, the expression of $\lfloor a+b\rfloor - \lfloor a\rfloor - \lfloor b\rfloor$ is 0 or 1.

Thus, if p divides $\binom{\lambda n}{n}$, then $v_p\left(\binom{\lambda n}{n}\right) \le R(p) \le \log_p(\lambda n)$, or $p^{v_p\left(\binom{\lambda n}{n}\right)} \le p^{R(p)} \le \lambda n$ **(1.4)**

If $\lambda n \ge p > \lfloor\sqrt{\lambda n}\rfloor$, then $0 \le v_p\left(\binom{\lambda n}{n}\right) \le R(p) \le 1.$ **(1.5)**

Let $\pi(n)$ be the number of distinct prime numbers less than or equal to n. Among the first six consecutive natural numbers are three prime numbers 2, 3 and 5. Then, for each additional six consecutive natural numbers, at most one can add two prime numbers, $p \equiv 1 \ (\mathrm{MOD}\ 6)$ and $p \equiv 5\ (\mathrm{MOD}\ 6)$. Thus, $\pi(n) \le \left\lfloor\frac{n}{3}\right\rfloor + 2 \le \frac{n}{3} + 2.$ **(1.6)**

Let the primordial function $n\# = \prod_{n\ge p} p$ where the product is taken over all distinct prime numbers p less than or equal to the integer n.

Since $\binom{2n-1}{n}$ is an integer and all the primes in the range of $(n+1) \le p \le (2n-1)$ appear in its numerator but not in its denominator, we have

$\frac{(2n-1)\#}{n\#} \le \binom{2n-1}{n} = \frac{1}{2}\left(\binom{2n-1}{n-1} + \binom{2n-1}{n}\right) < \frac{1}{2}(1+1)^{2n-1} = 2^{2n-2}$.

The proof proceeds by induction on n.

If $n = 3$, then $n\# = 6 < 8 = 2^{2n-3}$.

If $n = 4$, then $n\# = 6 < 32 = 2^{2n-3}$.

If $n = (2m-1)$ is odd and $n \ge 5$, then $m \ge 3$ and then

$n\# = (2m-1)\# < m\# \cdot 2^{2m-2} < 2^{2m-3} \cdot 2^{2m-2} = 2^{4m-5} = 2^{2n-3}$.

If $n = 2m$ is even and $n \ge 6$, then $m \ge 3$ and then

$n\# = (2m)\# = (2m-1)\# < m\# \cdot 2^{2m-2} < 2^{2m-3} \cdot 2^{2m-2} = 2^{4m-5} < 2^{4m-3} = 2^{2n-3}$.

Thus, when $n \ge 3$, $n\# = \prod_{n\ge p} p < 2^{2n-3}$. **(1.7)**

From the prime number decomposition, when $n > \lfloor\sqrt{\lambda n}\rfloor$,

$\binom{\lambda n}{n} = \Gamma_{\lambda n\ge p>n}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} \cdot \Gamma_{n\ge p>\lfloor\sqrt{\lambda n}\rfloor}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} \cdot \Gamma_{\lfloor\sqrt{\lambda n}\rfloor\ge p}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\}$.

When $n \le \lfloor\sqrt{\lambda n}\rfloor$, $\binom{\lambda n}{n} \le \Gamma_{\lambda n\ge p>n}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} \cdot \Gamma_{\lfloor\sqrt{\lambda n}\rfloor\ge p}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\}$.

Thus, $\binom{\lambda n}{n} \le \Gamma_{\lambda n\ge p>n}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} \cdot \Gamma_{n\ge p>\lfloor\sqrt{\lambda n}\rfloor}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} \cdot \Gamma_{\lfloor\sqrt{\lambda n}\rfloor\ge p}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\}$

$\Gamma_{\lambda n\ge p>n}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} = \Gamma_{\lambda n\ge p>n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}$ since all prime numbers in $n!$ do not appear in the range of $\lambda n \ge p > n$.

Referring to **(1.5)** and **(1.7)**, when $n \ge (\lambda - 2) \ge 13$, then $\lfloor\sqrt{\lambda n}\rfloor \ge 13$.

$\Gamma_{n \geq p > \lfloor\sqrt{\lambda n}\rfloor}\left\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\right\} \leq \frac{\prod_{n \geq p} p}{13\#} \leq \frac{2^{2n-3}}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13} = \frac{2^{2n-4}}{15015}$.

Referring to **(1.4)** and **(1.6)**, $\Gamma_{\lfloor\sqrt{\lambda n}\rfloor \geq p}\left\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\right\} \leq (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}$.

Thus, when $n \geq (\lambda - 2) \geq 13$, $\binom{\lambda n}{n} < \Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} \cdot \frac{2^{2n-4}}{15015} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}$. **(1.8)**

2. Lemmas

Lemma 1: If a real number $x \geq 3$, then $\frac{2(2x-1)}{x-1} > \left(\frac{x}{x-1}\right)^x$. **(2.1)**

Proof:

Let $f_1(x) = \frac{2(2x-1)}{x-1}$; then, $f_1{}'(x) = \frac{2(x-1)(2x-1)' - 2(2x-1)(x-1)'}{(x-1)^2} = \frac{-2}{(x-1)^2} < 0$.

Thus, $f_1(x)$ is a strictly decreasing function for $x > 1$.

Since $f_1(3) = 5$, and $\lim_{x\to\infty} f_1(x) = 4$, for $x \geq 3$, we have $5 \geq f_1(x) = \frac{2(2x-1)}{x-1} \geq 4$.

Let $f_2(x) = \left(\frac{x}{x-1}\right)^x$, then $f_2{}'(x) = \left(\left(\frac{x}{x-1}\right)^x\right)' = \left(\frac{x}{x-1}\right)^x \cdot \left(ln\frac{x}{x-1} - \frac{1}{x-1}\right)$ **(2.1.1)**

When $x \geq 3$, $\frac{1}{x-1} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x^5} + \frac{1}{x^6} + \cdots$

Using the formula: $ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$,

$ln\frac{x}{x-1} = ln\frac{1}{1+\frac{-1}{x}} = -ln\left(1 + \frac{-1}{x}\right) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \frac{1}{4x^4} + \frac{1}{5x^5} + \frac{1}{6x^6} + \cdots$

Thus, for $x \geq 3$, $ln\frac{x}{x-1} - \frac{1}{x-1} < 0$.

Since $\left(\frac{x}{x-1}\right)^x$ is a positive number for $x \geq 3$, $f_2{}'(x) = \left(\frac{x}{x-1}\right)^x \cdot \left(ln\frac{x}{x-1} - \frac{1}{x-1}\right) < 0$.

Thus $f_2(x)$ is a strictly decreasing function when $x \geq 3$.

Since $f_2(3) = 3.375$ and $\lim_{x\to\infty} f_2(x) = e \approx 2.718$,

when $x \geq 3$, $3.375 \geq f_2(x) = \left(\frac{x}{x-1}\right)^x \geq e$. **(2.1.2)**

Since for $x \geq 3$, $f_1(x)$ has a lower bound of 4 and $f_2(x)$ has an upper bound of 3.375,

$f_1(x) = \frac{2(2x-1)}{x-1} > f_2(x) = \left(\frac{x}{x-1}\right)^x$ is proven. **(2.1.3)**

Lemma 2: For $n \geq 2$ and $\lambda \geq 3$, $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n - \lambda + 1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$. **(2.2)**

Proof:

When $\lambda \geq 3$ and $n = 2$, $\binom{\lambda n}{n} = \binom{2\lambda}{2} = \frac{2\lambda(2\lambda-1)(2\lambda-2)!}{2(2\lambda-2)!} = \lambda(2\lambda - 1)$. **(2.2.1)**

$\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}} = \frac{\lambda^{2\lambda-\lambda+1}}{2(\lambda-1)^{2(\lambda-1)-\lambda+1}} = \frac{\lambda(\lambda-1)}{2} \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda}$. **(2.2.2)**

Referring to **(2.1)**, when $x = \lambda \geq 3$, we have $\frac{2(2\lambda-1)}{\lambda-1} > \left(\frac{\lambda}{\lambda-1}\right)^{\lambda}$. **(2.2.3)**

Since $\frac{\lambda(\lambda-1)}{2}$ is a positive number for $\lambda \geq 3$, referring to **(2.2.1)** and **(2.2.2)**, when $\frac{\lambda(\lambda-1)}{2}$ multiplies both sides of **(2.2.3)**, we have

$$\left(\frac{\lambda(\lambda-1)}{2}\right)\left(\frac{2(2\lambda-1)}{\lambda-1}\right) = \lambda(2\lambda-1) = \binom{\lambda n}{n} > \left(\frac{\lambda(\lambda-1)}{2}\right)\left(\frac{\lambda}{\lambda-1}\right)^{\lambda} = \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}.$$

Thus, $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ when $\lambda \geq 3$ and $n = 2$. **(2.2.4)**

By induction on n, when $\lambda \geq 3$, if $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ is true for n, then for $n+1$,

$$\binom{\lambda(n+1)}{n+1} = \binom{\lambda n+\lambda}{n+1} = \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)(\lambda n+1)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)(n+1)} \cdot \binom{\lambda n}{n}$$

$$\binom{\lambda(n+1)}{n+1} > \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)(\lambda n+1)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)(n+1)} \cdot \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$$

$$\binom{\lambda(n+1)}{n+1} > \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)} \cdot \frac{\lambda n+1}{n} \cdot \frac{1}{(n+1)} \cdot \frac{\lambda^{\lambda n-\lambda+1}}{(\lambda-1)^{(\lambda-1)n-\lambda+1}}$$

Notice $\frac{\lambda n+1}{n} > \lambda$, and $\frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)} > \left(\frac{\lambda}{\lambda-1}\right)^{(\lambda-1)}$

because $\frac{\lambda n+\lambda}{\lambda n+\lambda-n-1} = \frac{\lambda}{\lambda-1}$; $\frac{\lambda n+\lambda-1}{\lambda n+\lambda-n-2} > \frac{\lambda}{\lambda-1}$; $\cdots$ $\frac{\lambda n+2}{\lambda n-n+1} > \frac{\lambda}{\lambda-1}$. Thus,

$$\binom{\lambda(n+1)}{n+1} > \frac{\lambda^{\lambda-1}}{(\lambda-1)^{(\lambda-1)}} \cdot \frac{\lambda}{1} \cdot \frac{1}{(n+1)} \cdot \frac{\lambda^{\lambda n-\lambda+1}}{(\lambda-1)^{(\lambda-1)n-\lambda+1}} = \frac{\lambda^{\lambda(n+1)-\lambda+1}}{(n+1)(\lambda-1)^{(\lambda-1)(n+1)-\lambda+1}} \quad \textbf{(2.2.5)}$$

From **(2.2.4)** and **(2.2.5)**, we have for $n \geq 2$ and $\lambda \geq 3$, $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$

Thus, **Lemma 2** is proven.

3. A Prime Number between $(\lambda - 1)n$ and λn when $n \geq (\lambda - 2) \geq 13$

Proposition:

For $n \geq \lambda - 2 \geq 13$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **(3.1)**

Proof:

Applying **(2.2)** to **(1.8)**, when $n \geq (\lambda - 2) \geq 13$,

$$\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}} < \binom{\lambda n}{n} < \Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} \cdot \frac{2^{2n-4}}{15015} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}.$$

Because $(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2} > 0$ and $\frac{2^{2n-4}}{15015} > 0$,

$$\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > \frac{\lambda^{\lambda n-\lambda+1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}\cdot\frac{2^{2n-4}}{15015}\cdot n(\lambda-1)^{(\lambda-1)n-\lambda+1}} = \frac{60060\lambda^2\cdot\left(\left(\frac{\lambda-1}{4}\right)\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} > 0\ .$$

Referring to **(2.1.2)**, when $\lambda \geq 3$, $\left(\frac{\lambda}{\lambda-1}\right)^{\lambda} \geq e$. Thus, when $n \geq (\lambda - 2) \geq 13$,

$$\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > \frac{60060\lambda^2\cdot\left(\left(\frac{\lambda-1}{4}\right)\cdot e\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} = f_3(n,\lambda) > 0\ . \qquad \textbf{(3.2)}$$

Let $x \geq 13$ and $y \geq 15$, where x and y are both real numbers.

When $x = y - 2$,

$$f_3(x,y) = \frac{60060y^2\cdot\left(\left(\frac{y-1}{4}\right)\cdot e\right)^{(x-1)}}{(yx)^{\frac{\sqrt{yx}}{3}+3}} = \frac{60060(x+2)^2\cdot\left(\left(\frac{x+1}{4}\right)\cdot e\right)^{(x-1)}}{((x+2)\cdot x)^{\frac{\sqrt{x\cdot(x+2)}}{3}+3}}$$

$$> f_4(x) = \frac{60060(x+2)^2\cdot\left(\left(\frac{x+1}{4}\right)\cdot e\right)^{(x-1)}}{((x+2)\cdot x)^{\frac{x+1}{3}+3}} > 0\ . \qquad \textbf{(3.3)}$$

$$f_4{}'(x) = f_4(x)\cdot\left(\frac{2}{x+2} + ln\left(\frac{x+1}{4}\right) + \frac{4}{3} - \frac{2}{x+1} - \frac{1}{3}ln\big((x+2)\cdot x\big) - \frac{10}{3x} - \frac{8}{3(x+2)}\right) = f_4(x)\cdot f_5(x)$$

where $f_5(x) = \frac{2}{x+2} + ln\left(\frac{x+1}{4}\right) + \frac{4}{3} - \frac{2}{x+1} - \frac{1}{3}ln\big((x+2)\cdot x\big) - \frac{10}{3x} - \frac{8}{3(x+2)}$

$$f_5{}'(x) = \frac{-2}{(x+2)^2} + \frac{1}{x+1} + \frac{2}{(x+1)^2} - \frac{1}{3x} - \frac{1}{3(x+2)} + \frac{10}{3x^2} + \frac{8}{3(x+2)^2}$$

$$= \frac{-2x^2-4x-2}{(x+1)^2\cdot(x+2)^2} + \frac{2x^2+8x+8}{(x+1)^2\cdot(x+2)^2} + \frac{3x^2+6x}{3x(x+1)(x+2)} - \frac{x^2+3x+2}{3x(x+1)(x+2)} - \frac{x^2+x}{3x(x+1)(x+2)} + \frac{10}{3x^2} + \frac{8}{3(x+2)^2}$$

$f_5{}'(x) = \frac{4x+6}{(x+1)^2\cdot(x+2)^2} + \frac{x^2+2x-2}{3x(x+1)(x+2)} + \frac{10}{3x^2} + \frac{8}{3(x+2)^2} > 0$ when $x \geq 3$.

Thus, $f_5(x)$ is a strictly increasing function for $x \geq 3$.

When $x = 13$, $f_5(x) = \frac{2}{13+2} + ln\left(\frac{13+1}{4}\right) + \frac{4}{3} - \frac{2}{13+1} - \frac{ln(15)}{3} - \frac{ln(13)}{3} - \frac{10}{39} - \frac{8}{45} \approx 0.384 > 0.$

Thus, for $x \geq 13$, $f_5(x) > 0$. Then, $f_4{}'(x) = f_4(x)\cdot f_5(x) > 0$, and $f_4(x)$ is a strictly increasing function for $x \geq 13$.

Referring to **(3.3)**, as long as $x = (y-2) \geq 13$, $f_3(x,y)$ is an increasing function respect to both x and y, because $f_3(x,y) > f_4(x)$.

Thus, when $x = (y-2) \geq 13$, $f_3(x+1, y+1) > f_3(x,y)$. **(3.4)**

$$\frac{\partial f_3(x,y)}{\partial x} = f_3(x,y)\cdot\left(ln\left(\frac{y-1}{4}\right) + 1 - \frac{\sqrt{y}}{6\sqrt{x}}\cdot ln(yx) - \frac{\sqrt{y}}{3\sqrt{x}} - \frac{3}{x}\right) = f_3(x,y)\cdot f_6(x,y) \qquad \textbf{(3.5)}$$

where $f_6(x,y) = ln\left(\frac{y-1}{4}\right) + 1 - \frac{\sqrt{y}}{6\sqrt{x}}\cdot ln(yx) - \frac{\sqrt{y}}{3\sqrt{x}} - \frac{3}{x}$

When $x = y - 2$, then $f_6(x,y) = f_7(x) = ln\left(\frac{x+1}{4}\right) + 1 - \frac{\sqrt{x+2}}{6\sqrt{x}}\cdot\big(ln(x+2) + ln(x) + 2\big) - \frac{3}{x}$

$f_7{}'(x) = \frac{1}{x+1} - \frac{\sqrt{x+2}}{6\sqrt{x}} \cdot \left(\frac{1}{x+2} + \frac{1}{x}\right) + \frac{ln(x+2)+ln(x)+2}{6x\sqrt{x(x+2)}} + \frac{3}{x^2} = \left(\frac{1}{x+1} - \frac{1}{3\sqrt{x(x+2)}}\right) + \frac{ln(x+2)+ln(x)}{6x\sqrt{x(x+2)}} + \frac{3}{x^2}.$

$\frac{1}{x+1} - \frac{1}{3\sqrt{x(x+2)}} = \frac{3\sqrt{x(x+2)}-(x+1)}{3(x+1)\sqrt{x(x+2)}} \cdot \frac{3\sqrt{x(x+2)}+(x+1)}{3\sqrt{x(x+2)}+(x+1)} = \frac{8x^2+16x-1}{9(x+1)\sqrt{x(x+2)}\left(\sqrt{x(x+2)}+(x+1)\right)} > 0$ when $x > 1$.

When $x \geq 3$, $f_7{}'(x) = \left(\frac{1}{x+1} - \frac{1}{3\sqrt{x(x+2)}}\right) + \frac{ln(x+2)+ln(x)}{6x\sqrt{x(x+2)}} + \frac{3}{x^2} > 0$, then $f_7(x)$ is a strictly increasing function.

When $x = (y-2) \geq 3$, because $f_6(x,y) = f_7(x)$, $f_6(x,y)$ is an increasing function respect to both x and y. **(3.6)**

$\frac{\partial f_6(x,y)}{\partial x} = \frac{\sqrt{y}}{12x\sqrt{x}} \cdot ln(yx) + \frac{3}{x^2} > 0$ when $x \geq 3$ and $y \geq 3$.

Thus, when $x = (y-2) \geq 3$, $f_6(x,y)$ is an increasing function respect to x. **(3.7)**

When $x = (y-2) = 13$, $f_6(x,y) = ln\left(\frac{15-1}{4}\right) + 1 - \frac{\sqrt{15}}{6\sqrt{13}} \cdot ln(195) - \frac{\sqrt{15}}{3\sqrt{13}} - \frac{3}{13} \approx 0.720 > 0.$

Referring to **(3.6)**, when $x = (y-2) \geq 13$, $f_6(x,y) > 0$.

Referring to **(3.7)**, when $x \geq (y-2) \geq 13$, $f_6(x,y) > 0$.

Referring to **(3.5)**, when $x \geq (y-2) \geq 13$, since $f_3(x,y) > 0$ and $f_6(x,y) > 0$, $\frac{\partial f_3(x,y)}{\partial x} > 0$, and $f_3(x,y)$ is an increasing function respect to x.

Thus, when $x \geq (y-2) = 13$, $f_3(x+1,y) > f_3(x,y)$. **(3.8)**

When $x = (y-2) = 13$,

$$f_3(x,y) = \frac{60060y^2 \cdot \left(\left(\frac{y-1}{4}\right) \cdot e\right)^{(x-1)}}{(xy)^{\frac{\sqrt{xy}}{3}+3}} = \frac{60060 \cdot 15^2 \cdot \left(\left(\frac{15-1}{4}\right) \cdot e\right)^{(13-1)}}{(15 \cdot 13)^{\frac{\sqrt{15 \cdot 13}}{3}+3}} \approx \frac{7.432\mathrm{E}{+}18}{3.386\mathrm{E}{+}17} > 1.$$

Referring to **(3.4)**, when $x = (y-2) \geq 13$, $f_3(x,y) > 1$.

Referring to **(3.8)**, when $x \geq (y-2) \geq 13$, $f_3(x,y) > 1$.

Let $x = n$ and $y = \lambda$, then when $n \geq (\lambda - 2) \geq 13$, $f_3(n,\lambda) > 1$. **(3.9)**

Referring to **(3.2)**, when $n \geq \lambda - 2 \geq 13$, $\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > f_3(n,\lambda) > 1$.

Let integer $m \geq n$. When $m \geq n \geq \lambda - 2 \geq 13$, $\Gamma_{\lambda m \geq p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} > f_3(m,\lambda) > 1$. **(3.10)**

Referring to **(1.8)**, when $n \geq (\lambda - 2) \geq 13$, if there is a prime number p in $\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}$, then $p \geq n+1 = \sqrt{(n+2)n+1} > \sqrt{\lambda n}$. From **(1.5)**, $0 \leq v_p\left(\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}\right) \leq R(p) \leq 1$.

Thus, when $m \geq n \geq \lambda - 2 \geq 13$, every distinct prime number in $\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}$ and in $\Gamma_{\lambda m \geq p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$ has a power of 0 or 1.

$\Gamma_{\lambda m \geq p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} =$

$= \Gamma_{\lambda m \geq p > (\lambda-1)m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} \cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{(\lambda-1)m}{i} \geq p > \frac{\lambda m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} \cdot \Gamma_{\frac{\lambda m}{i+1} \geq p > \frac{(\lambda-1)m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)$

In $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{(\lambda-1)m}{i} \geq p > \frac{\lambda m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)$, for every distinct prime number p in these ranges, the numerator $(\lambda m)!$ has the product of $p \cdot 2p \cdot 3p \cdots ip = (i)! \cdot p^i$. The denominator $((\lambda - 1)m)!$ also has the same product of $(i)! \cdot p^i$. Thus, they cancel each other in $\frac{(\lambda m)!}{((\lambda-1)m)!}$.

Referring to **(1.2)**, $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{(\lambda-1)m}{i} \geq p > \frac{\lambda m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right) = 1.$

Thus, $\Gamma_{\lambda m \geq p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} = \Gamma_{\lambda m \geq p > (\lambda-1)m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} \cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda m}{i+1} \geq p > \frac{(\lambda-1)m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)$

$\Gamma_{\lambda m \geq p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} = \prod_{i=1}^{i=\lambda-1}\left(\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right).$ **(3.11)**

$\prod_{i=1}^{i=\lambda-1}\left(\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)$ is the product of (λ –1) sectors from i = 1 to i = (λ – 1).

Each of these sectors is the prime number factorization of the product of the consecutive integers between $\frac{(\lambda -1)m}{i}$ and $\frac{\lambda m}{i}$.

From **(3.10)** and **(3.11)**, when $m \geq n \geq \lambda - 2 \geq 13$, $\prod_{i=1}^{i=\lambda-1}\left(\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right) > 1.$

Referring to **(1.1)**, $\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} \geq 1$. Thus, when $m \geq n \geq \lambda - 2 \geq 13$, at least one of the sectors in $\prod_{i=1}^{i=\lambda-1}\left(\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)$ is greater than one.

Let $\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} > 1$ be such a sector and let $m = ni$ where $(\lambda - 1) \geq i \geq 1$.

Thus, when $m = ni \geq n \geq \lambda - 2 \geq 13$,

$$\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} = \Gamma_{\frac{\lambda ni}{i} \geq p > \frac{(\lambda-1)ni}{i}}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\} = \Gamma_{\lambda n \geq p > (\lambda-1)n}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\} > 1. \quad \textbf{(3.12)}$$

$$\frac{(\lambda ni)!}{((\lambda-1)ni)!} = \frac{(\boldsymbol{\lambda ni})\cdot(\lambda ni-1)\cdots(\boldsymbol{\lambda ni-i})\cdots(\boldsymbol{\lambda ni-2i})\cdots(\boldsymbol{\lambda ni-(n-1)i})\cdots(\lambda ni-ni+1)\cdot((\lambda-1)ni)!}{((\lambda-1)ni)!}$$

$$\frac{(\lambda ni)!}{((\lambda-1)ni)!} = \frac{\boldsymbol{i}\cdot(\boldsymbol{\lambda n})\cdot(\lambda ni-1)\cdots\boldsymbol{i}\cdot(\boldsymbol{\lambda n-1})\cdots\boldsymbol{i}\cdot(\boldsymbol{\lambda n-2})\cdots\boldsymbol{i}\cdot(\boldsymbol{\lambda n-n+1})\cdots(\lambda ni-ni+1)\cdot((\lambda-1)ni)!}{((\lambda-1)ni)!}$$

Thus, $\frac{(\lambda ni)!}{((\lambda-1)ni)!}$ contains all the factors of (λn), $(\lambda n - 1)$, $(\lambda n - 2)$,... $(\lambda n - n + 1)$ in $\frac{(\lambda n)!}{((\lambda-1)n)!}$.

These factors make up all the consecutive integers in the range of $\lambda n \geq p > (\lambda - 1)n$ in $\frac{(\lambda n)!}{((\lambda-1)n)!}$. Thus, $\frac{(\lambda ni)!}{((\lambda-1)ni)!}$ contains $\frac{(\lambda n)!}{((\lambda-1)n)!}$.

Referring to the definition, all prime numbers in $\frac{(\lambda ni)!}{((\lambda-1)ni)!}$ in the ranges of $\lambda ni \geq p > \lambda n$ and

$(\lambda - 1)n > p$ do not contribute to $\Gamma_{\lambda n \geq p > (\lambda-1)n}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\}$, nor does i for $(\lambda - 1) \geq i \geq 1$. Only the prime numbers in the prime factorization of $\frac{(\lambda ni)!}{((\lambda-1)ni)!}$ in the range of $\lambda n \geq p > (\lambda - 1)n$ present in $\Gamma_{\lambda n \geq p > (\lambda-1)n}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\}$. Since $\frac{(\lambda n)!}{((\lambda-1)n)!}$ is the product of all the consecutive integers in this range, $\Gamma_{\lambda n \geq p > (\lambda-1)n}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\} = \Gamma_{\lambda n \geq p > (\lambda-1)n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}$.

Referring to **(3.12)**, when $m = ni \geq n \geq \lambda - 2 \geq 13$, $\Gamma_{\lambda n \geq p > (\lambda-1)n}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\} > 1$. Thus, when $n \geq \lambda - 2 \geq 13$, $\Gamma_{\lambda n \geq p > (\lambda-1)n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > 1$. Referring to **(1.3)**, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

Thus, **Proposition (3.1)** is proven. It becomes a theorem: **Theorem (3.1)**.

4. Proof of Legendre's Conjecture

Legendre's conjecture states that there is a prime number between n^2 and $(n+1)^2$ for every positive integer n. **(4.1)**

Proof:

Referring to **Theorem (3.1)**, for integers $j \geq k - 2 \geq 13$, there exists at least a prime number p such that $j(k - 1) < p \leq jk$. **(4.2)**

When $k = j + 1 \geq 15$, then $j = k - 1 \geq 14$.

Applying $k = j + 1$ into **(4.2)**, then $j^2 < p \leq j(j+1) < (j+1)^2$.

Let $n = j \geq 14$, then we have $n^2 < p < (n+1)^2$. **(4.3)**

For $1 \leq n \leq 13$, we have a table, **Table 1**, that shows Legendre's conjecture valid. **(4.4)**

Table 1: For $1 \leq n \leq 13$, there is a prime number between n^2 and $(n+1)^2$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13
n^2	1	4	9	16	25	36	49	64	81	100	121	144	169
p	3	5	11	19	29	41	53	67	83	103	127	149	173
$(n+1)^2$	4	9	16	25	36	49	64	81	100	121	144	169	196

Combining **(4.3)** and **(4.4)**, we have proven Legendre's conjecture.

Extension of Legendre's conjecture

There are at least two prime numbers, p_n and p_m, between j^2 and $(j+1)^2$ for every positive integer j such that $j^2 < p_n \leq j(j+1)$ and $j(j+1) < p_m < (j+1)^2$ where p_n is the n^{th} prime number, p_m is the m^{th} prime number, and $m \geq n + 1$. **(4.5)**

Proof:

Referring to **Theorem (3.1)**, for integers $j \geq k - 2 \geq 13$, there exists at least a prime number p such that $j(k-1) < p \leq jk$.

When $k-1=j\geq 14$, then $j(k-1)=j^2<p_n\leq jk=j(j+1)$. Thus, there is at least a prime number p_n such that $j^2<p_n\leq j(j+1)$ when $j=k-1\geq 14$.
When $j=k-2\geq 14$, then $k=j+2$.
$j(k-1)=j(j+1)<p_m\leq jk=j(j+2)<(j+1)^2$. Thus, there is at least another prime number p_m such that $j(j+1)<p_m<(j+1)^2$ when $j=k-2\geq 14$.
Thus, when $j\geq 14$, there are at least two prime numbers p_n and p_m between j^2 and $(j+1)^2$ such that $j^2<p_n\leq j(j+1)<p_m<(j+1)^2$ where $m\geq n+1$ for $p_m>p_n$. **(4.6)**

For $1\leq j\leq 13$, we have a table, **Table 2**, that shows that **(4.5)** is valid. **(4.7)**

Table 2: For $1\leq j\leq 18$, there are 2 primes such that $j^2<p_n\leq j(j+1)<p_m<(j+1)^2$.

j	1	2	3	4	5	6	7	8	9	10	11	12	13
j^2	1	4	9	16	25	36	49	64	81	100	121	144	169
p_n	2	5	11	19	29	41	53	67	83	103	127	149	173
$j(j+1)$	2	6	12	20	30	42	56	72	90	110	132	156	182
p_m	3	7	13	23	31	43	59	73	97	113	137	163	191
$(j+1)^2$	4	9	16	25	36	49	64	81	100	121	144	169	196

Combining **(4.6)** and **(4.7)**, we have proven **(4.5)**. It becomes a theorem: **Theorem (4.5)**.

5. The Proofs of Three Related Conjectures

Oppermann's conjecture states that for every integer $x>1$, there is at least one prime number between $x(x-1)$ and x^2, and at least another prime number between x^2 and $x(x+1)$. **(5.1)**

Proof:
Theorem (4.5) states that there are at least two prime numbers, p_n and p_m , between j^2 and $(j+1)^2$ for every positive integer j such that $j^2<p_n\leq j(j+1)<p_m<(j+1)^2$ where $m\geq n+1$ for $p_m>p_n$.
$j(j+1)$ is a composite number except $j=1$. Since $j^2<p_n\leq j(j+1)$ is valid for every positive integer j, when we replace j with $j+1$, we have $(j+1)^2<p_v<(j+1)(j+2)$.
Thus, we have $j(j+1)<p_m<(j+1)^2<p_v<(j+1)(j+2)$. **(5.2)**
When $x>1$, then $(x-1)\geq 1$. Substituting j with $(x-1)$ in **(5.2)**, we have
$x(x-1)<p_m<x^2<p_v<x(x+1)$ **(5.3)**
Thus, we have proven Oppermann's conjecture.

Brocard's conjecture states that there are at least 4 prime numbers between $(p_n)^2$ and $(p_{n+1})^2$, where p_n is the n^{th} prime number, for every $n>1$. **(5.4)**

Proof:

Theorem (4.5) states that there are at least two prime numbers, p_n and p_m , between j^2 and $(j+1)^2$ such that $j^2 < p_n \leq j(j+1)$ and $j(j+1) < p_m < (j+1)^2$ for every positive integer j , where $m \geq n+1$ for $p_m > p_n$. When $j > 1$, $j(j+1)$ is a composite number. Then **Theorem (4.5)** can be written as $j^2 < p_n < j(j+1)$ and $j(j+1) < p_m < (j+1)^2$.

In the prime number series: $p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ p_4 = 7,\ p_5 = 11, \ldots$ Except p_1, all prime numbers are odd numbers. Their intervals are 2 or more. Thus, when $n > 1$, $(p_{n+1} - p_n) \geq 2$. Thus, we have $p_n < (p_n + 1) < (p_n + 2) \leq p_{n+1}$ when $n > 1$. **(5.5)**

Applying **Theorem (4.5)** to **(5.5)**, when $n > 1$, we have at least two prime numbers p_{m1} , and p_{m2} in between $(p_n)^2$ and $(p_n+1)^2$ such that $(p_n)^2 < p_{m1} < p_n(p_n+1) < p_{m2} < (p_n+1)^2$, and at least two more prime numbers p_{m3} , p_{m4} in between $(p_n+1)^2$ and $(p_n+2)^2$ such that $(p_n+1)^2 < p_{m3} < (p_n+1)(p_n+2) < p_{m4} < (p_n+2)^2 \leq (p_{n+1})^2$.
Thus, there are at least 4 prime numbers between $(p_n)^2$ and $(p_{n+1})^2$ for $n > 1$ such that $(p_n)^2 < p_{m1} < p_n(p_n+1) < p_{m2} < (p_n+1)^2 < p_{m3} < (p_n+1)(p_n+2) < p_{m4} < (p_{n+1})^2$ **(5.6)**

Thus, Brocard's conjecture is proven.

Andrica's conjecture is named after Dorin Andrica. It is a conjecture regarding the gaps between prime numbers. The conjecture states that the inequality $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ holds for all n where p_n is the n^{th} prime number. If $g_n = p_{n+1} - p_n$ denotes the n^{th} prime gap, then Andrica's conjecture can also be rewritten as $g_n < 2\sqrt{p_n} + 1$. **(5.7)**

Proof:
From **Theorem (4.5)**, for every positive integer j, there are at least two prime numbers p_n and p_m between j^2 and $(j+1)^2$ such that $j^2 < p_n \leq j(j+1) < p_m < (j+1)^2$ where $m \geq n+1$ for $p_m > p_n$. Since $m \geq n+1$, we have $p_m \geq p_{n+1}$.
Thus, we have $j^2 < p_n$ **(5.8)**
and $p_{n+1} \leq p_m < (j+1)^2$. **(5.9)**
Since j, p_n , p_{n+1}, and $(j+1)$ are positive integers,
$j < \sqrt{p_n}$ **(5.10)**
and $\sqrt{p_{n+1}} < j + 1$. **(5.11)**

Applying **(5.10)** to **(5.11)**, we have $\sqrt{p_{n+1}} < \sqrt{p_n} + 1$. **(5.12)**

Thus, $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ holds for all n since in **Theorem (4.5)**, j holds for all positive integers. Using the prime gap to prove this conjecture, from **(5.8)** and **(5.9)**, we have $g_n = p_{n+1} - p_n < (j+1)^2 - j^2 = 2j + 1$. From **(5.10)**, $j < \sqrt{p_n}$.
Thus, $g_n = p_{n+1} - p_n < 2\sqrt{p_n} + 1$. **(5.13)**

Thus, Andrica's conjecture is proven.

6. References

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