A Method of Finding All Existing Starting Numbers For Finite Arbitrarily Long Collatz Trajectories That Obey Any Behavior/Dynamics of Our Choosing.

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Abstract

We provide a general analytic formula to construct all existing starting odd numbers that obey our desired finite arbitrarily long Collatz trajectory, meaning that these starting odd numbers obey our pre-designated maximum factors of 2 at each iteration of the reduced Collatz map. We also provide another general analytic formula for finding the resulting odd numbers after N iterations of the reduced Collatz map. These formulas shed light on the structure of Collatz trajectories and other properties. We can also use this information to find in finite steps all existing Collatz trajectories that become 1 after any finite N iterations. We also will see that the "location" of all of the 1's in Collatz Conjecture can be found by solving a special case of the discrete log problem.

1 Introduction

Collatz Conjecture is the hypothesis that for any positive integer n, by repeated applications of the Collatz map (shown below), this sequence of numbers will eventually reach 1.

Collatz map:

$$n \longrightarrow \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases}$$

In this paper we will provide some new theorems on finite arbitrarily long Collatz trajectories.

We will provide a general analytic formula for finding all starting odd numbers for any desired Collatz trajectory dynamic.

We then show for any desired dynamics of a finite arbitrarily long collatz trajectory, there exist infinite many starting numbers that satisfy this particular dynamic. Also that the difference in value between all such subsequent starting numbers for any given desired dynamic is a constant.

We also show how to construct all existing Collatz trajectories that result in 1 after N iterations.

2 Definitions

In this paper we will only talk about the reduced Collatz map which we will just simply refer to as "Collatz map" (shown below) . The reduced Collatz map simplifies the Collatz map by combining all the consecutive 'n is even' steps into a single step. Therefore this map takes a positive odd number to another positive odd number by the following operations:

Also in this paper, we will only discuss the cases where $N \geq 2$. For the case N=1 it is a simple exercise but will not be discussed here.

Definition: Collatz map

$2n + 1 \longrightarrow \frac{(2n+1)3+1}{2^k}$, where k denotes the maximum number of factors of 2 that $(2n + 1)3 + 1$ contains. Hence the results is another positive odd number.

Definition: Iteration N

Iteration N means that a starting odd number $2n_0 + 1$ (which is also synonymous with "iteration 0's odd number") has had the Collatz map (reduced) applied to it N times with the resulting odd number now belonging to iteration N and is now labeled $2n_N + 1$. Starting odd numbers are labeled as $2n_0 + 1$ to designate that it belongs to iteration 0 by definition i.e. has not had the collatz map applied to it yet. The k in this map is also labeled as k_1 to denote it belongs also to iteration 1. This is further clarified in the below Collatz map:

$2n_0 + 1 \longrightarrow \frac{(2n_0+1)3+1}{2^{k_1}}$ which is now equal to $2n_1 + 1$, which is an odd number in iteration 1 for some natural number n_1.

Definition: We will also use the following nomenclature interchangeably: $V_N, 2n_N + 1$, value at iteration N.

Definition: Reverse Collatz Map: Is simply the reverse process of the Collatz Map (i.e. reduced Collatz Map) i.e.

$2n_1 + 1 \longrightarrow \frac{(2n_1+1)2^{k_1}-1}{3}$, which now goes back to $2n_0 + 1$, which is an odd number in iteration 0 for some natural number n_0.

Definition: Collatz Trajectory:

A sequence of odd numbers as the result of repeated applications of the Collatz map.

3 Proposition 1:

$$2n_N+1 = \frac{\ddots\frac{\frac{\frac{(2n_0+1)\cdot 3+1}{2^{k_1}}\cdot 3+1}{2^{k_2}}\cdot 3+1}{2^{k_3}}}{2^{k_N}} = \frac{3^N n_0 + 2\cdot 3^{N-1} + 3^{N-2}2^{k_1-1} + 3^{N-3}2^{k_1+k_2-1} + 3^{N-4}2^{k_1+k_2+k_3-1} + 3^{N-5}2^{k_1+k_2+k_3+k_4-1} + ... + 3^{N-N}2^{k_1+k_2+k_3+...+k_{N-1}-1}}{2^{k_1+k_2+k_3+...+k_N-1}}$$

, for N $\geq$ 2, $k_i \geq 1$

Proof by Induction:

Base case: N=2: $2n_2+1 = \frac{\left(\frac{(2n_0+1)\cdot 3+1}{2^{k_1}}\right)\cdot 3+1}{2^{k_2}} = \frac{\left(\frac{6n_0+4}{2^{k_1}}\right)\cdot 3+1}{2^{k_2}} = \frac{\frac{18n_0+12}{2^{k_1}}+1}{2^{k_2}} = \frac{\frac{18n_0+12}{2^{k_1}}+\frac{2^{k1}}{2^{k1}}}{2^{k_2}} = \frac{\frac{18n_0+12+2^{k_1}}{2^{k_1}}}{2^{k_2}} = \frac{18n_0+12+2^{k_1}}{2^{k_1+k_2}} = \frac{9n_0+6+2^{k_1-1}}{2^{k_1+k_2-1}} = \frac{3^2n_0+2\cdot 3^{2-1}+3^{2-2}\cdot 2^{k_1-1}}{2^{k_1+k_2-1}}$

QED

Induction Step:

Assume Nth case is true, then show that if follows that the N+1th case is also true:

$$2n_N+1 = \frac{\frac{\frac{\frac{\frac{(2n_0+1)\cdot 3+1}{2^{k_1}}\cdot 3+1}{2^{k_2}}\cdot 3+1}{2^{k_3}}}{\ddots}}{2^{k_N}} = \frac{3^N n_0 + 2\cdot 3^{N-1} + 3^{N-2}2^{k_1-1} + 3^{N-3}2^{k_1+k_2-1} + 3^{N-4}2^{k_1+k_2+k_3-1} + 3^{N-5}2^{k_1+k_2+k_3+k_4-1} + ... + 3^{N-N}2^{k_1+k_2+k_3+...+k_{N-1}-1}}{2^{k_1+k_2+k_3+...+k_N-1}}$$

$$2n_{N+1}+1 = \frac{\left(\frac{\frac{\frac{\frac{\frac{(2n_0+1)\cdot 3+1}{2^{k_1}}\cdot 3+1}{2^{k_2}}\cdot 3+1}{2^{k_3}}}{\ddots}}{2^{k_N}}\right)3+1}{2^{k_{N+1}}} = \frac{\left(\frac{3^N n_0+2\cdot 3^{N-1}+3^{N-2}2^{k_1-1}+3^{N-3}2^{k_1+k_2-1}+3^{N-4}2^{k_1+k_2+k_3-1}+3^{N-5}2^{k_1+k_2+k_3+k_4-1}+...+3^{N-N}2^{k_1+k_2+k_3+...+k_{N-1}-1}}{2^{k_1+k_2+k_3+...+k_N-1}}\right)\cdot 3+1}{2^{k_{N+1}}}$$

$$= \frac{\left(\frac{3^{N+1}n_0+2\cdot 3^N+3^{N-1}2^{k_1-1}+3^{N-2}2^{k_1+k_2-1}+3^{N-3}2^{k_1+k_2+k_3-1}+3^{N-4}2^{k_1+k_2+k_3+k_4-1}+...+3^{N-N+1}2^{k_1+k_2+k_3+...+k_{N-1}-1}}{2^{k_1+k_2+k_3+...+k_N-1}}\right)+1}{2^{k_{N+1}}}$$

$$= \frac{\left(\frac{3^{N+1}n_0+2\cdot 3^N+3^{N-1}2^{k_1-1}+3^{N-2}2^{k_1+k_2-1}+3^{N-3}2^{k_1+k_2+k_3-1}+3^{N-4}2^{k_1+k_2+k_3+k_4-1}+...+3^{N-N+1}2^{k_1+k_2+k_3+...+k_{N-1}-1}}{2^{k_1+k_2+k_3+...+k_N-1}}\right)+\frac{2^{k_1+k_2+k_3+...+k_N-1}}{2^{k_1+k_2+k_3+...+k_N-1}}}{2^{k_{N+1}}}$$

$$= \frac{\left(\frac{3^{N+1}n_0+2\cdot 3^N+3^{N-1}2^{k_1-1}+3^{N-2}2^{k_1+k_2-1}+3^{N-3}2^{k_1+k_2+k_3-1}+3^{N-4}2^{k_1+k_2+k_3+k_4-1}+...+3^{N-N+1}2^{k_1+k_2+k_3+...+k_{N-1}-1}+2^{k_1+k_2+k_3+...+k_N-1}}{2^{k_1+k_2+k_3+...+k_N-1}}\right)}{2^{k_{N+1}}}$$

$$= \frac{3^{N+1}n_0+2\cdot 3^N+3^{N-1}2^{k_1-1}+3^{N-2}2^{k_1+k_2-1}+3^{N-3}2^{k_1+k_2+k_3-1}+3^{N-4}2^{k_1+k_2+k_3+k_4-1}+...+3^{N-N+1}2^{k_1+k_2+k_3+...+k_{N-1}-1}+3^0 2^{k_1+k_2+k_3+...+k_N-1}}{2^{k_1+k_2+k_3+...+k_{N+1}-1}}$$

$$= \frac{3^{(N+1)}n_0+2\cdot 3^{(N+1)-1}+3^{(N+1)-2}2^{k_1-1}+3^{(N+1)-3}2^{k_1+k_2-1}+3^{(N+1)-4}2^{k_1+k_2+k_3-1}+3^{(N+1)-5}2^{k_1+k_2+k_3+k_4-1}+...+3^{(N+1)-N}2^{k_1+k_2+k_3+...+k_{(N+1)-2}-1}+3^{(N+1)-(N+1)}2^{k_1+k_2+k_3+...+k_{(N+1)-1}-1}}{2^{k_1+k_2+k_3+...+k_{N+1}-1}}$$

$$= \frac{3^{(N+1)}n_0+2\cdot 3^{(N+1)-1}+3^{(N+1)-2}2^{k_1-1}+3^{(N+1)-3}2^{k_1+k_2-1}+3^{(N+1)-4}2^{k_1+k_2+k_3-1}+3^{(N+1)-5}2^{k_1+k_2+k_3+k_4-1}+...+3^{(N+1)-(N+1)}2^{k_1+k_2+k_3+...+k_{(N+1)-1}-1}}{2^{k_1+k_2+k_3+...+k_{N+1}-1}}$$

QED

4 Proposition 2:

$3^J \cdot 2^k \equiv 3^J \cdot 2^{k \ mod \ 2\cdot 3^{N-J-1}} \ mod \ 3^N, where \ J \ and \ N \ are \ positive \ integers \ and \ J < N$

Proof:

$3^J \cdot 2^k \equiv 3^J \cdot 2^{k+r} mod\ 3^N, for\ some\ positive\ integer\ r\ since\ gcd(2,3) = 1$

We now divide both sides by 3^J

$2^k \equiv 2^{k+r} mod\ 3^{N-J}$

By the properties of Carmichael's λ function:

$a^k \equiv a^{k+\lambda(n)} mod\ n$, given gcd(a,n)=1, where $\lambda(n)$ is the smallest positive integer such that this expression is true.

Therefore for our specific values of a and n, i.e. a=2 and $n = 3^{N-J}$, and since gcd (2,3) = 1, we have
$2^k \equiv 2^{k+\lambda(3^{N-J})} mod\ 3^{N-J}$

We see that our original r can be set to $\lambda(3^{N-J})$

We also know that $\lambda(3^{N-J}) = 2 \cdot 3^{N-J-1}$ by the properties of Carmichael's λ function: i.e. $\lambda\left(p^k\right) = p^k - p^{k-1} = p^k\left(1 - \frac{1}{p}\right)$

Hence our equation becomes:

$2^k \equiv 2^{k+2 \cdot 3^{N-J-1}} mod\ 3^{N-J}$

Hence we see that $2^k \equiv 2^{k+2 \cdot 3^{N-J-1}} \equiv 2^{k+2 \cdot 3^{N-J-1}+2 \cdot 3^{N-J-1}}$... and so on in $mod\ 3^{N-J}$

Hence we can write $2^k \equiv 2^{k\ mod\ 2 \cdot 3^{N-J-1}} mod\ 3^{N-J}$

We now multiple both sides by 3^J including the mod number:

$3^J \cdot 2^k \equiv 3^J \cdot 2^{k\ mod\ 2 \cdot 3^{N-J-1}}\ mod\ 3^N$

QED

5 Theorem 1:

$$n_N \equiv 2^{(2\cdot3^{N-1}+1-k_1-k_2-k_3-...-k_N)\ mod\ 2\cdot3^0}3^{N-1}+3^{N-2}2^{(2\cdot3^{N-1}-k_2-k_3-k_4-...-k_N-1)\ mod\ 2\cdot3^1}$$
$$+3^{N-3}2^{(2\cdot3^{N-1}-k_3-k_4-k_5-...-k_N-1)\ mod\ 2\cdot3^2}+3^{N-4}2^{(2\cdot3^{N-1}-k_4-k_5-k_6-...-k_N-1)\ mod\ 2\cdot3^3}$$
$$+3^{N-5}2^{(2\cdot3^{N-1}-k_5-k_6-k_7-...-k_N-1)\ mod\ 2\cdot3^4}+...+3^02^{(2\cdot3^{N-1}-k_N-1)\ mod\ 2\cdot3^{N-1}}$$
$$-2^{(2\cdot3^{N-1}-1)\ mod\ 2\cdot3^{N-1}}\ (mod\ 3^N) \tag{1}$$

, for N ≥ 2, $k_i \geq 1$

Proof: We start with the result from Proposition 1:

$$\frac{3^N n_0+2\cdot3^{N-1}+3^{N-2}2^{k_1-1}+3^{N-3}2^{k_1+k_2-1}+3^{N-4}2^{k_1+k_2+k_3-1}+3^{N-5}2^{k_1+k_2+k_3+k_4-1}+...+3^{N-N}2^{k_1+k_2+k_3+...+k_{N-1}-1}}{2^{k_1+k_2+k_3+...+k_N-1}}=2n_N+1$$

, for N ≥ 2, $k_i \geq 1$

Now we will isolate n_0 to one side:

$$3^N n_0+2\cdot3^{N-1}+3^{N-2}2^{k_1-1}+3^{N-3}2^{k_1+k_2-1}+3^{N-4}2^{k_1+k_2+k_3-1}+3^{N-5}2^{k_1+k_2+k_3+k_4-1}+...+3^{N-N}2^{k_1+k_2+k_3+...+k_{N-1}-1}=\left(2^{k_1+k_2+k_3+...+k_N-1}\right)(2n_N+1)$$

$$3^N n_0=-2\cdot3^{N-1}-3^{N-2}2^{k_1-1}-3^{N-3}2^{k_1+k_2-1}-3^{N-4}2^{k_1+k_2+k_3-1}-3^{N-5}2^{k_1+k_2+k_3+k_4-1}-...-3^{N-N}2^{k_1+k_2+k_3+...+k_{N-1}-1}+\left(2^{k_1+k_2+k_3+...+k_N-1}\right)(2n_N+1)$$

$$n_0 = \frac{-2\cdot3^{N-1}-3^{N-2}2^{k_1-1}-3^{N-3}2^{k_1+k_2-1}-3^{N-4}2^{k_1+k_2+k_3-1}-3^{N-5}2^{k_1+k_2+k_3+k_4-1}-...-3^{N-N}2^{k_1+k_2+k_3+...+k_{N-1}-1}+\left(2^{k_1+k_2+k_3+...+k_N-1}\right)(2n_N+1)}{3^N}$$

Since n_0 must be an integer, this means that the numerator must be divisible by the denominator, in other words the numerator must be congruent to 0 modulo the denominator. Hence, we have:

$$-2\cdot3^{N-1}-3^{N-2}2^{k_1-1}-3^{N-3}2^{k_1+k_2-1}-3^{N-4}2^{k_1+k_2+k_3-1}-3^{N-5}2^{k_1+k_2+k_3+k_4-1}-...-3^{N-N}2^{k_1+k_2+k_3+...+k_{N-1}-1}+\left(2^{k_1+k_2+k_3+...+k_N-1}\right)(2n_N+1) \equiv 0\ (mod\ 3^N)$$

$$-2\cdot3^{N-1}-3^{N-2}2^{k_1-1}-3^{N-3}2^{k_1+k_2-1}-3^{N-4}2^{k_1+k_2+k_3-1}-3^{N-5}2^{k_1+k_2+k_3+k_4-1}-...-3^{N-N}2^{k_1+k_2+k_3+...+k_{N-1}-1}+2^{k_1+k_2+k_3+...+k_N}n_N+2^{k_1+k_2+k_3+...+k_N-1} \equiv 0\ (mod\ 3^N)$$

$$2^{k_1+k_2+k_3+...+k_N}n_N \equiv 2\cdot3^{N-1}+3^{N-2}2^{k_1-1}+3^{N-3}2^{k_1+k_2-1}+3^{N-4}2^{k_1+k_2+k_3-1}+3^{N-5}2^{k_1+k_2+k_3+k_4-1}+...+3^{N-N}2^{k_1+k_2+k_3+...+k_{N-1}-1}-2^{k_1+k_2+k_3+...+k_N-1}\ (mod\ 3^N) \quad (2)$$

Now using the theorem $2^{2\cdot3^{N-1}} \equiv 1\ mod\ 3^N$, hence the inverse of 2^m in $mod\ 3^N$ is $2^{2\cdot3^{N-1}-m}$. We now apply this inverse to cancel the $2^{k_1+k_2+k_3+...+k_N}$ on the left side to isolate n_N. Hence, we have:

$$\left(2^{2\cdot 3^{N-1}-k_1-k_2-k_3-...-k_N}\right)2^{k_1+k_2+k_3+...+k_N}n_N \equiv \left(2^{2\cdot 3^{N-1}-k_1-k_2-k_3-...-k_N}\right)(2\cdot 3^{N-1}+3^{N-2}2^{k_1-1}+3^{N-3}2^{k_1+k_2-1}+3^{N-4}2^{k_1+k_2+k_3-1}+ 3^{N-5}2^{k_1+k_2+k_3+k_4-1}+...+3^{N-N}2^{k_1+k_2+k_3+...+k_{N-1}-1}-2^{k_1+k_2+k_3+...+k_N-1})\ (mod\ 3^N) \tag{3}$$

$$n_N \equiv \left(2^{2\cdot 3^{N-1}-k_1-k_2-k_3-...-k_N}\right)(2\cdot 3^{N-1}+3^{N-2}2^{k_1-1}+3^{N-3}2^{k_1+k_2-1}+3^{N-4}2^{k_1+k_2+k_3-1}+3^{N-5}2^{k_1+k_2+k_3+k_4-1}+... +3^{N-N}2^{k_1+k_2+k_3+...+k_{N-1}-1}-2^{k_1+k_2+k_3+...+k_N-1}\ (mod\ 3^N) \tag{4}$$

$$n_N \equiv 2^{2\cdot 3^{N-1}-k_1-k_2-...-k_N+1}\cdot 3^{N-1}+3^{N-2}2^{2\cdot 3^{N-1}-k_2-k_3-...-k_N-1}+3^{N-3}2^{2\cdot 3^{N-1}-k_3-k_4-...-k_N-1}+ 3^{N-4}2^{2\cdot 3^{N-1}-k_4-k_5-...-k_N-1}+3^{N-5}2^{2\cdot 3^{N-1}-k_5-k_6-...-k_N-1}+...+3^{N-N}2^{2\cdot 3^{N-1}-k_N-1}-2^{2\cdot 3^{N-1}-1}\ (mod\ 3^N) \tag{5}$$

Now apply proposition 2 to each term:

$$\begin{aligned} n_N \equiv\ & 2^{2\cdot 3^{N-1}-k_1-k_2-...-k_N+1\ (mod\ 2\cdot 3^{N-(N-0)})}3^{N-1}+3^{N-2}2^{2\cdot 3^{N-1}-k_2-k_3-...-k_N-1\ (mod\ 2\cdot 3^{N-(N-1)})} \\ & +3^{N-3}2^{2\cdot 3^{N-1}-k_3-k_4-...-k_N-1\ (mod\ 2\cdot 3^{N-(N-2)})}+3^{N-4}2^{2\cdot 3^{N-1}-k_4-k_5-...-k_N-1\ (mod\ 2\cdot 3^{N-(N-3)})} \\ & +3^{N-5}2^{2\cdot 3^{N-1}-k_5-k_6-...-k_N-1\ (mod\ 2\cdot 3^{N-(N-4)})}+...+3^{N-N}2^{2\cdot 3^{N-1}-k_N-1\ (mod\ 2\cdot 3^{N-(N-(N-1))})} \\ & -2^{2\cdot 3^{N-1}-1\ (mod\ 2\cdot 3^{N-(N-(N-1))})}\ (mod\ 3^N) \end{aligned} \tag{6}$$

Which simplifies to:

$$
\begin{aligned}
n_N \equiv\ & 2^{(2\cdot 3^{N-1}+1-k_1-k_2-k_3-...-k_N)\ mod\ 2\cdot 3^0} 3^{N-1} + 3^{N-2} 2^{(2\cdot 3^{N-1}-k_2-k_3-k_4-...-k_N-1)\ mod\ 2\cdot 3^1} \\
& +3^{N-3} 2^{(2\cdot 3^{N-1}-k_3-k_4-k_5-...-k_N-1)\ mod\ 2\cdot 3^2} + 3^{N-4} 2^{(2\cdot 3^{N-1}-k_4-k_5-k_6-...-k_N-1)\ mod\ 2\cdot 3^3} \\
& +3^{N-5} 2^{(2\cdot 3^{N-1}-k_5-k_6-k_7-...-k_N-1)\ mod\ 2\cdot 3^4} + ... + 3^0 2^{(2\cdot 3^{N-1}-k_N-1)\ mod\ 2\cdot 3^{N-1}} \\
& -2^{(2\cdot 3^{N-1}-1)\ mod\ 2\cdot 3^{N-1}}\ (mod\ 3^N)
\end{aligned} \tag{7}
$$

QED

6 Corollary 1.1:

When the value of n_N is found given the desired values of $k_1, k_2, ..., k_N$, for N $\geq$ 2, $k_i \geq 1$ using the equation in Theorem 1, then

1. $n_N\ mod\ 3^N + 3^N t$ for all non-negative integer values of t, form the complete set of solutions for n_N given these particular values of $k_1, k_2, ..., k_N$.

2. Given the above solutions, represented in the form $(n_N\ mod\ 3^N, k_1\ mod\ 2, k_2\ mod\ 6, ..., k_N\ mod\ 2\cdot 3^{N-1})$, then $(n_N + 3^N t, k_1\ mod\ 2 + 2j_1, k_2\ mod\ 6 + 6j_2, k_3\ mod\ 18 + 18j_3, ..., k_N\ mod\ 2\cdot 3^{N-1} + 2\cdot 3^{N-1} j_N)$ for all non-negative integer values of $t, j_1, j_2, j_3, ..., j_N$ are also valid Collatz trajectories.

Proof of 1.:

Since n_N in Theorem 1's equation is already in modulo 3^N , it represents the smallest positive solution. Also for this reason, all numbers added to n_N that are multiples of 3^N are also solutions.

QED

Proof of 2.:

Start with the equation from Theorem 1

$$\begin{aligned} n_N \equiv\ & 2^{(2\cdot 3^{N-1}+1-k_1-k_2-k_3-...-k_N)\ mod\ 2\cdot 3^0} 3^{N-1} + 3^{N-2} 2^{(2\cdot 3^{N-1}-k_2-k_3-k_4-...-k_N-1)\ mod\ 2\cdot 3^1} \\ & +3^{N-3} 2^{(2\cdot 3^{N-1}-k_3-k_4-k_5-...-k_N-1)\ mod\ 2\cdot 3^2} + 3^{N-4} 2^{(2\cdot 3^{N-1}-k_4-k_5-k_6-...-k_N-1)\ mod\ 2\cdot 3^3} \\ & +3^{N-5} 2^{(2\cdot 3^{N-1}-k_5-k_6-k_7-...-k_N-1)\ mod\ 2\cdot 3^4} + ... + 3^0 2^{(2\cdot 3^{N-1}-k_N-1)\ mod\ 2\cdot 3^{N-1}} \\ & -2^{(2\cdot 3^{N-1}-1)\ mod\ 2\cdot 3^{N-1}} \ (mod\ 3^N) \end{aligned} \tag{8}$$

, for N $\geq$ 2, $k_i \geq 1$

Observations:

1. The only place that k_1 appears in the equation is in the first term's exponent and is modulo 2. Hence we can change the value of k_1 to any other positive integer by adding or subtracting any multiple of 2 without changing the resultant value of n_N.
2. The only places that k_2 appears in the equation is in the first two terms's exponents with the first term's exponent being mod 2 and the second term's exponent being mod 6. We also see that 6 is divisible by 2. Hence we can change the value of k_2 to any other positive integer by adding or subtracting any multiple of 6 without changing the resultant value of n_N.
3. The only places that k_3 appears in the equation is in the first three terms's exponents with the first term's exponent being mod 2 and the second term's exponent being mod 6 and the third term's exponent being mod 18. We also see that 18 is divisible by 6 and 2. Hence we can change the value of k_3 to any other positive integer by adding or subtracting any multiple of 18 without changing the resultant value of n_N.
4. Generally, the only places that k_N appears in the equation is in the first N terms's exponents with the first term's exponent being mod 2, the second terms exponent being mod 6, ..., with the Nth term's exponent being mod $2 \cdot 3^{N-1}$. Hence we can change the value of k_N to any other positive integer by adding or subtracting $2 \cdot 3^{N-1}$ without changing the resultant value of n_N.

QED

Example 1.1

Find the resultant odd number after 5 iterations of the Collatz map such that $k_1 = 10, k_2 = 9, k_3 = 8, k_4 = 7, k_5 = 6$

Solution:

Start with n_N equation from Theorem 1 and plug in N=5, with $k_1 = 10, k_2 = 9, k_3 = 8, k_4 = 7, k_5 = 6$

$$n_5 \equiv 2^{161-k_5 (mod\ 162)} + 3 \cdot 2^{161-k_5-k_4 (mod\ 54)} + 9 \cdot 2^{161-k_5-k_4-k_3 (mod\ 18)} + 27 \cdot 2^{161-k_5-k_4-k_3-k_2 (mod\ 6)} + 81 \cdot 2^{161+2-k_5-k_4-k_3-k_2-k_1 (mod\ 2)} - 2^{161}\ (mod\ 243)$$

$$n_5 \equiv 2^{161-6 (mod\ 162)} + 3 \cdot 2^{161-6-7 (mod\ 54)} + 9 \cdot 2^{161-6-7-8 (mod\ 18)} + 27 \cdot 2^{161-6-7-8-9 (mod\ 6)} + 81 \cdot 2^{161+2-6-7-8-9-10 (mod\ 2)} - 2^{161}\ (mod\ 243)$$

$$n_5 \equiv 2^{155 (mod\ 162)} + 3 \cdot 2^{148 (mod\ 54)} + 9 \cdot 2^{140 (mod\ 18)} + 27 \cdot 2^{131 (mod\ 6)} + 81 \cdot 2^{123 (mod\ 2)} - 2^{161}\ (mod\ 243)$$

Now simplify the mod exponents:

$$n_5 \equiv 2^{155 (mod\ 162)} + 3 \cdot 2^{40 (mod\ 54)} + 9 \cdot 2^{14 (mod\ 18)} + 27 \cdot 2^{5 (mod\ 6)} + 81 \cdot 2^{1 (mod\ 2)} - 2^{161}\ (mod\ 243)$$

$$n_5 \equiv 45671926166590716193865151022383844364247891968+3298534883328+147456+864+162-2923003274661805836407369665432566039311865085952\ (mod\ 243)$$

$$n_5 \equiv -2877331348495215120213504514410182191649082162174\ (mod\ 243)$$

$$n_5 \equiv 228\ (mod\ 243)$$

Hence the full solutions for n_5 for the given k_i is

$n_5 = 228 + 243j$, with j=0,1,2,3,4,5. . . .

228 is also the smallest solution since all other solutions are adding multiples of 243 hence can only get larger.
Our desired odd number is $2n_5 + 1$ by definition, which equals $(228 + 243j)x2 + 1$, and we get $457 + 486j$.
To obtain the starting odd number, we simply apply the reverse Collatz map 5 times.

$$\frac{(457+486j)\cdot 2^{k_5}-1}{3} = \frac{(457+486j)\cdot 2^{6}-1}{3} = 9749 + 10368j$$

$$\frac{(9749+10368j)\cdot 2^{k_4}-1}{3} = \frac{(9749+10368j)\cdot 2^{7}-1}{3} = 415957 + 442368j$$

$$\frac{(415957+442368j)\cdot 2^{k_3}-1}{3} = \frac{(415957+442368j)\cdot 2^{8}-1}{3} = 35494997 + 37748736j$$

$$\frac{(35494997+37748736j)\cdot 2^{k_2}-1}{3} = \frac{(35494997+37748736j)\cdot 2^{9}-1}{3} = 6057812821 + 6442450944j$$

$$\frac{(6057812821+6442450944j)\cdot 2^{k_1}-1}{3} = \frac{(6057812821+6442450944j)\cdot 2^{10}-1}{3} = 2067733442901 + 2199023255552j$$

Hence our starting odd numbers are $2067733442901 + 2199023255552j$. We see that 2067733442901 is the smallest starting number here as all other ones are adding 2199023255552j

Also notice that 2199023255552 is 2^{41}. We will see in Theorem 2 that all starting numbers will differ in their subsequent solutions by $2^{\sum k_i}$

Also recall the above resultant solution i.e. $n_5 = 228 + 243j$, with j=0,1,2,3,4,5. . . .

From Corollary 1.1, we see that we can easily find infinite other valid Collatz trajectories and their starting numbers by changing the k_i's into other positive integers (recall by definition, each k_i is greater than or equal to 1) via the following:

$k_1 \pm 2a_1, \forall a_1 \in \mathbb{N}$
$k_2 \pm 6a_2, \forall a_2 \in \mathbb{N}$
$k_3 \pm 18a_3, \forall a_3 \in \mathbb{N}$
$k_4 \pm 54a_4, \forall a_4 \in \mathbb{N}$
$k_5 \pm 162a_5, \forall a_5 \in \mathbb{N}$

and then apply the reverse Collatz map 5 times from $n_5 = 228 + 243j, with\ j = 0, 1, 2, 3, 4, 5 . . . ,$ using these new k_i values.

7 Theorem 2:

$$\begin{aligned} n_0 \equiv (3^{2^{k_1+k_2+k_3+\ldots+k_N-1}-1\ (mod\ 2^{k_1+k_2+\ldots+k_N-1})})^N \cdot (2^{k_1+k_2+k_3+\ldots+k_N-1} - 2 \cdot 3^{N-1\ (mod\ 2^{k_1+k_2+\ldots+k_N-1})} - 3^{N-2\ (mod\ 2^{k_1+k_2+\ldots+k_N-1})} 2^{k_1-1} \\ -3^{N-3\ (mod\ 2^{k_1+k_2+\ldots+k_N-1})} 2^{k_1+k_2-1} - 3^{N-4\ (mod\ 2^{k_1+k_2+\ldots+k_N-1})} 2^{k_1+k_2+k_3-1} - 3^{N-5\ (mod\ 2^{k_1+k_2+\ldots+k_N-1})} 2^{k_1+k_2+k_3+k_4-1} - \ldots \\ -3^{N-N\ (mod\ 2^{k_1+k_2+\ldots+k_N-1})} 2^{k_1+k_2+k_3+\ldots+k_{N-1}-1})\ mod\ 2^{k_1+k_2+k_3+\ldots+k_N} \end{aligned} \quad (9)$$

, for N ≥ 2, $k_i \geq 1$

Proof: We start with the result from Proposition 1:

$$\frac{3^N n_0 + 2 \cdot 3^{N-1} + 3^{N-2} 2^{k_1-1} + 3^{N-3} 2^{k_1+k_2-1} + 3^{N-4} 2^{k_1+k_2+k_3-1} + 3^{N-5} 2^{k_1+k_2+k_3+k_4-1} + \ldots + 3^{N-N} 2^{k_1+k_2+k_3+\ldots+k_{N-1}-1}}{2^{k_1+k_2+k_3+\ldots+k_N-1}} = 2n_N + 1 \text{ , for N} \geq 2,\ k_i \geq 1$$

Then isolate n_N to one side:

$$n_N = \frac{\frac{3^N n_0 + 2 \cdot 3^{N-1} + 3^{N-2} 2^{k_1-1} + 3^{N-3} 2^{k_1+k_2-1} + 3^{N-4} 2^{k_1+k_2+k_3-1} + 3^{N-5} 2^{k_1+k_2+k_3+k_4-1} + \ldots + 3^{N-N} 2^{k_1+k_2+k_3+\ldots+k_{N-1}-1}}{2^{k_1+k_2+k_3+\ldots+k_N-1}} - 1}{2}$$

$$n_N = \frac{\frac{3^N n_0 + 2 \cdot 3^{N-1} + 3^{N-2} 2^{k_1-1} + 3^{N-3} 2^{k_1+k_2-1} + 3^{N-4} 2^{k_1+k_2+k_3-1} + 3^{N-5} 2^{k_1+k_2+k_3+k_4-1} + \ldots + 3^{N-N} 2^{k_1+k_2+k_3+\ldots+k_{N-1}-1}}{2^{k_1+k_2+k_3+\ldots+k_N-1}} - \frac{2^{k_1+k_2+k_3+\ldots+k_N-1}}{2^{k_1+k_2+k_3+\ldots+k_N-1}}}{2}$$

$$n_N = \frac{\frac{3^N n_0 + 2\cdot 3^{N-1} + 3^{N-2}2^{k_1-1} + 3^{N-3}2^{k_1+k_2-1} + 3^{N-4}2^{k_1+k_2+k_3-1} + 3^{N-5}2^{k_1+k_2+k_3+k_4-1} + ... + 3^{N-N}2^{k_1+k_2+k_3+...+k_{N-1}-1} - 2^{k_1+k_2+k_3+...+k_N-1}}{2^{k_1+k_2+k_3+...+k_{N-1}}}}{2}$$

$$n_N = \frac{3^N n_0 + 2\cdot 3^{N-1} + 3^{N-2}2^{k_1-1} + 3^{N-3}2^{k_1+k_2-1} + 3^{N-4}2^{k_1+k_2+k_3-1} + 3^{N-5}2^{k_1+k_2+k_3+k_4-1} + ... + 3^{N-N}2^{k_1+k_2+k_3+...+k_{N-1}-1} - 2^{k_1+k_2+k_3+...+k_N-1}}{2^{k_1+k_2+k_3+...+k_N}}$$

Since n_N must be an integer, this means that the numerator must be divisible by the denominator, in other words the numerator must be congruent to 0 modulo the denominator. Hence, we have:

$$3^N n_0 + 2\cdot 3^{N-1} + 3^{N-2}2^{k_1-1} + 3^{N-3}2^{k_1+k_2-1} + 3^{N-4}2^{k_1+k_2+k_3-1} + 3^{N-5}2^{k_1+k_2+k_3+k_4-1} + ... + 3^{N-N}2^{k_1+k_2+k_3+...+k_{N-1}-1} - 2^{k_1+k_2+k_3+...+k_N-1} \equiv 0 \; mod \; 2^{k_1+k_2+k_3+...+k_N}$$

Now isolate $3^N n_0$ to one side:

$$3^N n_0 \equiv 2^{k_1+k_2+k_3+...+k_N-1} - 2\cdot 3^{N-1} - 3^{N-2}2^{k_1-1} - 3^{N-3}2^{k_1+k_2-1} - 3^{N-4}2^{k_1+k_2+k_3-1} - 3^{N-5}2^{k_1+k_2+k_3+k_4-1} - ... - 3^{N-N}2^{k_1+k_2+k_3+...+k_{N-1}-1} \; (mod \; 2^{k_1+k_2+k_3+...+k_N})$$

Now using theorem $3^{2^{N-1}} \equiv 1 \; mod \; 2^N$, hence the inverse of 3 in mod 2^N is $3^{2^{N-1}-1}$. We now apply this inverse N times to cancel the 3^N on the left side to isolate n_0.

$$\begin{aligned}(3^{2^{k_1+k_2+k_3+...+k_{N-1}}-1})^N \cdot 3^N n_0 \equiv (3^{2^{k_1+k_2+k_3+...+k_{N-1}}-1})^N \cdot (2^{k_1+k_2+k_3+...+k_N-1} - 2\cdot 3^{N-1} \\ -3^{N-2}2^{k_1-1} - 3^{N-3}2^{k_1+k_2-1} - 3^{N-4}2^{k_1+k_2+k_3-1} - 3^{N-5}2^{k_1+k_2+k_3+k_4-1} - ... \\ -3^{N-N}2^{k_1+k_2+k_3+...+k_{N-1}-1}) \; (mod \; 2^{k_1+k_2+k_3+...+k_N})\end{aligned} \tag{10}$$

$$\begin{aligned}n_0 \equiv (3^{2^{k_1+k_2+k_3+...+k_{N-1}}-1})^N \cdot (2^{k_1+k_2+k_3+...+k_N-1} - 2\cdot 3^{N-1} - 3^{N-2}2^{k_1-1} - 3^{N-3}2^{k_1+k_2-1} \\ -3^{N-4}2^{k_1+k_2+k_3-1} - 3^{N-5}2^{k_1+k_2+k_3+k_4-1} - ... \\ -3^{N-N}2^{k_1+k_2+k_3+...+k_{N-1}-1}) \; (mod \; 2^{k_1+k_2+k_3+...+k_N})\end{aligned} \tag{11}$$

Now use the same fact that $3^{2^{N-1}} \equiv 1 \; mod \; 2^N$ and apply this to all terms that contain 3 raised to an exponent:

$$\begin{aligned}n_0 \equiv (3^{2^{k_1+k_2+k_3+...+k_{N-1}}-1 \; (mod \; 2^{k_1+k_2+...+k_N-1})})^N \cdot (2^{k_1+k_2+k_3+...+k_N-1} - 2\cdot 3^{N-1 \; (mod \; 2^{k_1+k_2+...+k_N-1})} \\ -3^{N-2 \; (mod \; 2^{k_1+k_2+...+k_N-1})}2^{k_1-1} - 3^{N-3 \; (mod \; 2^{k_1+k_2+...+k_N-1})}2^{k_1+k_2-1} \\ -3^{N-4 \; (mod \; 2^{k_1+k_2+...+k_N-1})}2^{k_1+k_2+k_3-1} - 3^{N-5 \; (mod \; 2^{k_1+k_2+...+k_N-1})}2^{k_1+k_2+k_3+k_4-1} - ... \\ -3^{N-N \; (mod \; 2^{k_1+k_2+...+k_N-1})}2^{k_1+k_2+k_3+...+k_{N-1}-1}) \; (mod \; 2^{k_1+k_2+k_3+...+k_N})\end{aligned} \tag{12}$$

QED

From Corollary 1.1, we also see here that when the value of n_0 is found given the desired values of $k_1, k_2, ..., k_N$, for N $\geq$ 2, $k_i \geq 1$ using the equation in Theorem 1, then

$n_0 \ mod \ 2^{k_1+k_2+k_3+...+k_N} + (2^{k_1+k_2+k_3+...+k_N})t$ for all non-negative integer values of t, form the complete set of solutions for n_0 given these particular values of $k_1, k_2, ..., k_N$.

Example 2.1

Find the starting odd numbers such that for 5 iterations of the Collatz map, $k_1 = 10, k_2 = 9, k_3 = 8, k_4 = 7, k_5 = 6$

Solution:

We start with the n_0 equation from Theorem 1, and plug in $N = 5, k_1 = 10, k_2 = 9, k_3 = 8, k_4 = 7, k_5 = 6$

$$\begin{aligned} n_0 \equiv 3^{5(2^{k_1+k_2+k_3+k_4+k_5-1}-1) \ (mod \ (2^{k_1+k_2+k_3+k_4+k_5-1}))} \cdot (2^{k_1+k_2+k_3+k_4+k_5-1} - 2 \cdot 3^{4 \ (mod \ (2^{k_1+k_2+k_3+k_4+k_5-1}))} \\ -3^{0 \ (mod \ (2^{k_1+k_2+k_3+k_4+k_5-1}))} \cdot 2^{k_1+k_2+k_3+k_4-1} - 3^{1 \ (mod \ (2^{k_1+k_2+k_3+k_4+k_5-1}))} \cdot 2^{k_1+k_2+k_3-1} \\ -3^{2 \ (mod \ (2^{k_1+k_2+k_3+k_4+k_5-1}))} \cdot 2^{k_1+k_2-1} - 3^{3 \ (mod \ (2^{k_1+k_2+k_3+k_4+k_5-1}))} \cdot 2^{k_1-1}) \ (mod \ 2^{k_1+k_2+k_3+k_4+k_5}) \end{aligned} \tag{13}$$

$$\begin{aligned} \equiv 3^{5(2^{10+9+8+7+6-1}-1) \ (mod \ (2^{10+9+8+7+6-1}))} \cdot (2^{10+9+8+7+6-1} - 2 \cdot 3^{4 \ (mod \ (2^{10+9+8+7+6-1}))} - 3^{0 \ (mod \ (2^{10+9+8+7+6-1}))} \cdot 2^{10+9+8+7-1} - 3^{1 \ (mod \ (2^{10+9+8+7+6-1}))} \cdot 2^{10+9+8-1} \\ -3^{2 \ (mod \ (2^{10+9+8+7+6-1}))} \cdot 2^{10+9-1} - 3^{3 \ (mod \ (2^{10+9+8+7+6-1}))} \cdot 2^{10-1}) \ (mod \ 2^{10+9+8+7+6}) \end{aligned} \tag{14}$$

$$\equiv 3^{5(2^{39}-1) \ (mod \ (2^{39}))} \cdot \left(2^{39} - 2 \cdot 3^{4 \ (mod \ (2^{39}))} - 3^{0 \ (mod \ (2^{39}))} \cdot 2^{33} - 3^{1 \ (mod \ (2^{39}))} \cdot 2^{26} - 3^{2 \ (mod \ (2^{39}))} \cdot 2^{18} - 3^{3 \ (mod \ (2^{39}))} \cdot 2^{9}\right) \ (mod \ 2^{40})$$

$$\equiv 3^{2748779069435 \ (mod \ (2^{39}))} \cdot \left(2^{39} - 2 \cdot 3^{4 \ (mod \ (2^{39}))} - 3^{0 \ (mod \ (2^{39}))} \cdot 2^{33} - 3^{1 \ (mod \ (2^{39}))} \cdot 2^{26} - 3^{2 \ (mod \ (2^{39}))} \cdot 2^{18} - 3^{3 \ (mod \ (2^{39}))} \cdot 2^{9}\right) \ (mod \ 2^{40})$$

$$\equiv 3^{549755813883} \cdot (2^{39} - 2 \cdot 3^4 - 3^0 \cdot 2^{33} - 3^1 \cdot 2^{26} - 3^2 \cdot 2^{18} - 3^3 \cdot 2^9) \ (mod \ 2^{40})$$

$\equiv 999,967,365,179 \cdot (549,755,813,888 - 162 - 8,589,934,592 - 201,326,592 - 2,359,296 - 13,824) \ \ (mod\ 2^{40})$

$\equiv 999,967,365,179 \cdot (540,962,179,422) \ \ (mod\ 2^{40})$

$\equiv 540,944,525,218,106,793,146,538\ (mod\ 2^{40})$

$\equiv 1,033,866,721,450\ (mod\ 2^{40})$

$n_0 = 1,033,866,721,450 + 2^{40}j, for\ j = 0,1,2,3.....$

Hence our starting odd number by definition is $2n_0 + 1 = 2(1,033,866,721,450 + 2^{40}j, for\ j = 0,1,2,3.....)) + 1 = 2067733442901 + 2^{41}j$

We in fact see that the desired trajectory:

$2067733442901x3 + 1 = 6203200328704$

$6203200328704/2^{10} = 6057812821$

$6057812821x3 + 1 = 18173438464$

$18173438464/2^{9} = 35494997$

$35494997x3 + 1 = 106484992$

$106484992/2^{8} = 415957$

$415957x3 + 1 = 1247872$

$1247872/2^{7} = 9749$

$9749x3 + 1 = 29248$

$29248/2^{6} = 457$

Notice that 457 is also the ending number of our n_N equation example.

Also we in fact see that 4266756698453 behaves the same way since $4266756698453 = 2(2067733442901) + 2^{41}(1)$

$4266756698453x3 + 1 = 12800270095360$

$12800270095360/2^{10} = 12500263765$

$12500263765x3 + 1 = 37500791296$

$37500791296/2^{9} = 73243733$

$73243733x3 + 1 = 219731200$

$219731200/2^{8} = 858325$

$858325x3 + 1 = 2574976$

$2574976/2^{7} = 20117$

$20117x3 + 1 = 60352$

$60352/2^{6} = 943$

Notice that $943 = 457 + 486(1)$. Recall that the full solution of the equation n_N in Example 1.1 is $457 + 486j$.

Notice that the value of j in the n_0 equation matches the value of j in the n_N equation given that the k_i are kept the same, i.e. the mth starting number from the n_0 equation results in the mth ending number from n_N equation when ki are kept the same.

8 Theorem 3:

We show that for any $k_1, k_2, k_3, \ldots, k_{N-1}$, there always exist one and only one k_N mod $2 \cdot 3^{N-1}$ such that these k_i corresponds to a valid Collatz trajectory that results in 1 after N iterations. In fact once we know this k_N, we now know ALL other k_N values that correspond to a valid Collatz trajectories with these same $k_1, k_2, k_3, \ldots, k_{N-1}$, since the general solution is $k_N \ mod \ 2 \cdot 3^{N-1} + 2 \cdot 3^{N-1}$j, for all j = 0,1,2,.....

Proof:

Start with the equation from Theorem 1

$$\begin{aligned} n_N \equiv {} & 2^{(2\cdot 3^{N-1}+1-k_1-k_2-k_3-...-k_N)\ mod\ 2\cdot 3^0} 3^{N-1} + 3^{N-2} 2^{(2\cdot 3^{N-1}-k_2-k_3-k_4-...-k_N-1)\ mod\ 2\cdot 3^1} \\ & +3^{N-3} 2^{(2\cdot 3^{N-1}-k_3-k_4-k_5-...-k_N-1)\ mod\ 2\cdot 3^2} + 3^{N-4} 2^{(2\cdot 3^{N-1}-k_4-k_5-k_6-...-k_N-1)\ mod\ 2\cdot 3^3} \\ & +3^{N-5} 2^{(2\cdot 3^{N-1}-k_5-k_6-k_7-...-k_N-1)\ mod\ 2\cdot 3^4} + ... + 3^0 2^{(2\cdot 3^{N-1}-k_N-1)\ mod\ 2\cdot 3^{N-1}} \\ & -2^{(2\cdot 3^{N-1}-1)\ mod\ 2\cdot 3^{N-1}}\ (mod\ 3^N) \end{aligned} \tag{15}$$

, for N $\geq$ 2, $k_i \geq 1$

Now we set this equation equal to 0 (recall we set $V_N = 1$ and since $V_N = 2n_N + 1$ by definition, hence $2n_N + 1 = 1$, hence $n_N = 0$) which is our desired outcome of Collatz trajectory ending at 1 after N iterations.

$$\begin{aligned} n_N \equiv {} & 2^{(2\cdot 3^{N-1}+1-k_1-k_2-k_3-...-k_N)\ mod\ 2\cdot 3^0} 3^{N-1} + 3^{N-2} 2^{(2\cdot 3^{N-1}-k_2-k_3-k_4-...-k_N-1)\ mod\ 2\cdot 3^1} \\ & +3^{N-3} 2^{(2\cdot 3^{N-1}-k_3-k_4-k_5-...-k_N-1)\ mod\ 2\cdot 3^2} + 3^{N-4} 2^{(2\cdot 3^{N-1}-k_4-k_5-k_6-...-k_N-1)\ mod\ 2\cdot 3^3} \\ & +3^{N-5} 2^{(2\cdot 3^{N-1}-k_5-k_6-k_7-...-k_N-1)\ mod\ 2\cdot 3^4} + ... + 3^0 2^{(2\cdot 3^{N-1}-k_N-1)\ mod\ 2\cdot 3^{N-1}} \\ & -2^{(2\cdot 3^{N-1}-1)\ mod\ 2\cdot 3^{N-1}} \equiv 0\ mod\ 3^N \end{aligned} \tag{16}$$

We now move the last term on the left side of the equivalence relations to the right side:

$$\begin{aligned} & 2^{(2\cdot 3^{N-1}+1-k_1-k_2-k_3-...-k_N)\ mod\ 2\cdot 3^0} 3^{N-1} + 3^{N-2} 2^{(2\cdot 3^{N-1}-k_2-k_3-k_4-...-k_N-1)\ mod\ 2\cdot 3^1} \\ & +3^{N-3} 2^{(2\cdot 3^{N-1}-k_3-k_4-k_5-...-k_N-1)\ mod\ 2\cdot 3^2} + 3^{N-4} 2^{(2\cdot 3^{N-1}-k_4-k_5-k_6-...-k_N-1)\ mod\ 2\cdot 3^3} \\ & +3^{N-5} 2^{(2\cdot 3^{N-1}-k_5-k_6-k_7-...-k_N-1)\ mod\ 2\cdot 3^4} + ... + 3^0 2^{(2\cdot 3^{N-1}-k_N-1)\ mod\ 2\cdot 3^{N-1}} \equiv 2^{(2\cdot 3^{N-1}-1)\ mod\ 2\cdot 3^{N-1}}\ mod\ 3^N \end{aligned} \tag{17}$$

We now factor out the 2^{-k_N}:

$$\begin{aligned}
2^{-k_N}[2^{(2\cdot 3^{N-1}+1-k_1-k_2-k_3-...-k_{N-1})\ mod\ 2\cdot 3^0}3^{N-1} + 3^{N-2}2^{(2\cdot 3^{N-1}-k_2-k_3-k_4-...-k_{N-1}-1)\ mod\ 2\cdot 3^1} \\
+3^{N-3}2^{(2\cdot 3^{N-1}-k_3-k_4-k_5-...-k_{N-1}-1)\ mod\ 2\cdot 3^2} + 3^{N-4}2^{(2\cdot 3^{N-1}-k_4-k_5-k_6-...-k_{N-1}-1)\ mod\ 2\cdot 3^3} \\
+3^{N-5}2^{(2\cdot 3^{N-1}-k_5-k_6-k_7-...-k_{N-1}-1)\ mod\ 2\cdot 3^4} + ... + 3^0 2^{(2\cdot 3^{N-1}-k_{N-1}-1)\ mod\ 2\cdot 3^{N-1}}] \equiv 2^{(2\cdot 3^{N-1}-1)\ mod\ 2\cdot 3^{N-1}}\ mod\ 3^N
\end{aligned} \tag{18}$$

We now multiply both sides by 2:

$$\begin{aligned}
2^{-k_N}[2^{(2\cdot 3^{N-1}+2-k_1-k_2-k_3-...-k_{N-1})\ mod\ 2\cdot 3^0}3^{N-1} + 3^{N-2}2^{(2\cdot 3^{N-1}-k_2-k_3-k_4-...-k_{N-1})\ mod\ 2\cdot 3^1} \\
+3^{N-3}2^{(2\cdot 3^{N-1}-k_3-k_4-k_5-...-k_{N-1})\ mod\ 2\cdot 3^2} + 3^{N-4}2^{(2\cdot 3^{N-1}-k_4-k_5-k_6-...-k_{N-1})\ mod\ 2\cdot 3^3} \\
+3^{N-5}2^{(2\cdot 3^{N-1}-k_5-k_6-k_7-...-k_{N-1})\ mod\ 2\cdot 3^4} + ... + 3^0 2^{(2\cdot 3^{N-1}-k_{N-1})\ mod\ 2\cdot 3^{N-1}}] \equiv 2^{(2\cdot 3^{N-1})\ mod\ 2\cdot 3^{N-1}}\ mod\ 3^N
\end{aligned} \tag{19}$$

We now simplify via the mod in the exponents:

$$\begin{aligned}
2^{-k_N}[2^{(-k_1-k_2-k_3-...-k_{N-1})\ mod\ 2\cdot 3^0}3^{N-1} + 3^{N-2}2^{(-k_2-k_3-k_4-...-k_{N-1})\ mod\ 2\cdot 3^1} \\
+3^{N-3}2^{(-k_3-k_4-k_5-...-k_{N-1})\ mod\ 2\cdot 3^2} + 3^{N-4}2^{(-k_4-k_5-k_6-...-k_{N-1})\ mod\ 2\cdot 3^3} \\
+3^{N-5}2^{(-k_5-k_6-k_7-...-k_{N-1})\ mod\ 2\cdot 3^4} + ... + 3^0 2^{(-k_{N-1})\ mod\ 2\cdot 3^{N-1}}] \equiv 1\ mod\ 3^N
\end{aligned} \tag{20}$$

Now notice that the expression inside the brackets belongs to the 3x+1 partition, hence the equation can now be rewritten as:

$2^{-k_N}\ [3x+1] \equiv 2^0\ mod\ 3^N$, for some $x\ \epsilon\ \mathbb{N}$

Since [3x+1] is coprime to 3^N, we know there exists an m $mod\ (2 \cdot 3^{N-1})$ s.t. $3x+1 = 2^m\ mod\ 3^N$. We also know that 2^{-kN} and 2^0 's exponents are also mod $2 \cdot 3^{N-1}$, hence we have:

$2^{-k_N\ (mod\ 2\cdot 3^{N-1})}\left[2^{m\ (mod\ 2\cdot 3^{N-1})}\right] \equiv 2^{0\ (mod\ 2\cdot 3^{N-1})}\ mod\ 3^N$

$2^{m-k_N\ (mod\ 2\cdot 3^{N-1})} \equiv 2^{0\ (mod\ 2\cdot 3^{N-1})}\ mod\ 3^N$

$m - k_N \equiv 0 \ (mod\ 2 \cdot 3^{N-1})$

$m \equiv k_N \ \ (mod\ 2 \cdot 3^{N-1})$

We now use the result from Corollary 1.1 and get the following as our full solution set:

$(n_N + 3^N t, k_1\ mod\ 2 + 2j_1, k_2\ mod\ 6 + 6j_2, k_3\ mod\ 18 + 18j_3, ..., k_N\ mod\ 2 \cdot 3^{N-1} + 2 \cdot 3^{N-1} j_N)$ for all non-negative integer values of $t, j_1, j_2, j_3, ..., j_N$ are also valid Collatz trajectories.

QED

OBSERVATION:

NOTE THAT THE MAIN UNPREDICTABLE STEP IN THIS METHOD IS SOLVING FOR m IN THE FOLLOWING SPECIAL CASE OF THE DISCRETE LOG PROBLEM:

$3x + 1 = 2^m\ mod\ 3^N$

THEREFORE IF WE HAVE A BETTER WAY TO SOLVE THIS SPECIAL CASE OF THE DISCRETE LOG EQUATION, WE WOULD KNOW THE "LOCATION" OF ALL THE 1's IN COLLATZ CONJECTURE I.E. WE WOULD KNOW ALL OF THE k_i's AND THEREFORE ALL THE CORRESPONDING STARTING ODD NUMBERS THAT EVENTUALLY BECOME 1 UNDER THE COLLATZ MAP. THEN SOLVING THE COLLATZ CONJECTURE BOILS DOWN TO A TILING PROBLEM, I.E. WHETHER THESE STARTING ODD NUMBERS COVER THE ENTIRE SET OF ODD NATURAL NUMBERS.

9 Corollary 3.1:

We can solve a finite number of equations to find every starting odd number that becomes 1 after N iterations of the Collatz map.

Proof: Given Theorem 3, we know given any $k_1\ mod\ 2, k_2\ mod\ 6, k_3\ mod\ 18, ...k_{N-1}\ mod\ 2 \cdot 3^{N-2}$, we can solve for k_N to obtain all existing Collatz trajectories that meet this criteria of k_i's.

Since $k_1\ mod\ 2$ only has possible 2 values, $k_2\ mod\ 6$ only has 6 possible values, $k_3\ mod\ 18$ has 18 possible values,......,$k_{N-1}\ mod\ 2 \cdot 3^{N-2}$ has $2 \cdot 3^{N-2}$ possible values, the number of equations needed to solve to find the starting odd numbers that become 1 in the following number of iterations is as follows:

1 iteration: 1 (The proof is simple for iteration 1 but not contained in this paper since we only touch on iterations N $\geq$ 2)
2 iterations: 2
3 iterations: 2x6

4 iterations: 2x6x18
N iterations: 2x6x18x.........x $2 \cdot 3^{N-2}$
Which are all finite numbers.

QED

References

Carmichael, Robert D. The Theory of Numbers. New York: John Wiley & Sons, 1914