Investigating prime gaps through zeta behaviour. Investigating the stringent conditions under which the Riemann hypothesis is true

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Contents

Abstract

This paper explores the relationship between prime gaps and the behavior of the Riemann zeta function. We analyze key logarithmic formulations of the zeta function and their decomposition to real and imaginary parts.

A zeta function is formulated that encodes information about Goldbach partitions. An exact prime gap equation is generated. It is shown the prime gaps are determined using the zeroes of the Riemann zeta function. With the exact prime gap formula, an exact formula for counting the number of primes is presented. The mystery of prime numbers is solved.

Keywords

Zeta function for Goldbach partition; Riemann hypothesis proof; prime gaps and singularities in $\zeta(s)$; Alternative zeta formulations; proof of prime gap conjecture; Exact prime gap theorem

Introduction

Prime gaps, the differences between consecutive prime numbers, are a fascinating area of study in number theory, with the distribution of primes being governed by the Prime Number Theorem and related conjectures like the Twin Prime Conjecture [2],[3],[4],[5],[6],[7]. The Riemann zeta function, a complex function, and the Prime Number Theorem are deeply intertwined, with the distribution of primes being intimately connected to the zeros of the zeta function, specifically through the Riemann Hypothesis [8],[9],[10],[11].

Analytical and computational studies of zeta functions, particularly the Riemann zeta function, reveal connections between number theory, prime distribution, and other mathematical and physical fields, with the Riemann Hypothesis being a key unsolved problem [12],[13],[14],[15]. This research aims to investigate the conditions under which the Riemann hypothesis is true.

In this research analysis of key logarithmic formulations of the zeta function and their decomposition to real and imaginary parts will be done

A zeta function will formulated that encodes information about Goldbach partitions. The paper aims at achieving a prime gap formula intricately connected to the zeroes of the Riemann zeta function.

Logarithmic form of the complex variable and its decomposition to real and complex parts. Reformulation of the Riemann zeta function

Consider the logarithmic complex variable $z = \frac{\ln(-\sqrt{x})}{y}$. It can be decomposed into real and imaginary parts at follows: $z = \frac{\ln(-\sqrt{x})}{y} = \frac{\ln(-1)}{y} + \frac{\ln\sqrt{x}}{y} = \frac{ln\sqrt{x}}{y} + i\frac{\pi}{y}$. The Riemann hypothesis requires the real part of it's complex variable to be $1/2$, in which case $y = \ln x$ and $z = \zeta(s) = \frac{1}{2} + \frac{i\pi}{\ln x}$. By this formulation the relationship between $\ln x$ and $\zeta(s)$ is given by $\ln(x) = \frac{i\pi}{\zeta(s) - \frac{1}{2}}$.

If $\zeta(s) - \frac{1}{2} = i\gamma$, then $\ln x = \frac{\pi}{\gamma}$. In the Riemann hypothesis $s = \frac{1}{2} + it$. The two zetas can be reconciled by the transformation: $\frac{\pi}{\ln t} = \frac{\pi}{\ln x}$ or $x = t$.

The number of primes is therefore asymptotically equal to $\frac{t}{\ln t}$.

The first zeta, when reconciled to Riemann zeta is given by $\zeta(s) = \frac{1}{2} + i\frac{\pi}{t}$.

Thus this paper will explore zeroes of alternative zeta formulations.

A zeta function for Goldbach partition

In the paper reference [1] the gap, g between two primes, p_1 and p_2 is given by $g = 2\sqrt{m^2 - p_1 p_2}$ with m representing the mean of the two primes. A logarithmic zeta function encoding information about gaps between primes would therefore be given by $\zeta(X) = \frac{\ln(-\frac{1}{n}\sqrt{m^2 - p_1 p_2})}{m+n}$ where $n = -\frac{g}{2}$.

The decomposition of the Goldbach partition zeta function therefore is $\zeta(X) = \frac{\ln(-\frac{1}{n}\sqrt{m^2 - p_1 p_2})}{m+n} = \frac{\ln\frac{1}{n}\sqrt{m^2 - p_1 p_2}}{m+n} + i\frac{\pi}{m+n}$ and $p_1 \neq p_2$.

Under circumstances in which $p_1 = p_2$ the zeta function $\zeta(X) = \frac{\ln(\sqrt{m^2 - p_1 p_2 + 1})}{m}$ is be used.

Goldbach partition therefore requires solving $\zeta(X) = 0$.

Results

For prime pairs with a gap of 6, using $n = -3$ and $m = p_1 + 3$, the function evaluates as follows:

$$\zeta(s) = \begin{cases} 0.2197 & \text{for } (5,11) \\ 0.0999 & \text{for } (11,17) \\ 0.0646 & \text{for } (17,23) \\ 0.0478 & \text{for } (23,29) \\ 0.0268 & \text{for } (41,47) \end{cases}$$

For prime pairs with a gap of 6, using $n=-2$ and $m=p_1+2$, the function evaluates as follows:

$$\zeta(s) = \begin{cases} 0.2310 & \text{for } (3,7) \\ 0.0990 & \text{for } (7,11) \\ 0.0533 & \text{for } (13,17) \\ 0.0365 & \text{for } (19,23) \end{cases}$$

A further analysis. A Complexity zeta for the Euler product.

Consider the Euler product $\zeta(s)=\prod \frac{p_i^s}{p^s-1}$. The above product generates a zero whenever $s=-\infty$.

We will formulate the complex variable s such that it will always generate a zero at some singularity. If

$\zeta(s)=-\zeta(\frac{1}{X})=\zeta(-\frac{m+n}{\ln(-1/n\sqrt{m^2-p_1p_2}})=\zeta(-\frac{m+n}{i\pi+\ln(1/n\sqrt{m^2-p_1p_2})})$

since n takes a negative value at $\zeta(s)=0$, a further decomposition needs to be done. That is:

$\zeta(s)=-\frac{m+n}{i\pi+\ln(1/n\sqrt{m^2-p_1p_2})}=-\frac{m+n}{2i\pi+\ln(-1/n\sqrt{m^2-p_1p_2})}=-\frac{(m+n)(2i\pi-\ln(-1/n\sqrt{m^2-p_1p_2}))}{-4\pi^2-\ln^2(-1/n\sqrt{m^2-p_1p_2})}=i\frac{m+n}{2\pi}=i\frac{p_1}{2\pi}$

This formulation links prime gaps to sigularities in $\zeta(s)=0$. Zeros are generated when we for any prime gap $n=-\frac{g}{2}$.

It is also observed that $m+n=p_1$.

For twin prime pairs we use $n=-1$ and $m=p_1+1\,|p_2>p_1$.

For gap g between consecutive primes use $n=-g/2$ and $m=p_1+g/2$.

The real part of the zeta of this formulation is

$\frac{(m+n)(\ln(1/n\sqrt{m^2-p_1p_2}))}{-4\pi^2-\ln^2(1/n\sqrt{m^2-p_1p_2})} = 0.$

The imaginary part of the same zeta is

$-\frac{i\pi(m+n)}{-4\pi^2-\ln^2(1/n\sqrt{m^2-p_1p_2})}$.

A nontrivial zero is generated when $\Re(s)=0$
Since $\ln 1/n()$ $\Im(s)=\frac{(m+n)}{2\pi}$.

when these conditions are generated, at the logarithmic for level a singularity is generated since $s=-\infty$ then. The Euler product therefore generates a nontrivial zero.

These results do not contradict Riemann Hypothesis.

Numerical validations

Consider the complex logarithmic $\zeta(s)=-\zeta(\frac{1}{X})=\zeta(-\frac{m+n}{\ln(-1/n\sqrt{m^2-p_1p_2}})$.

When $n=-1$ $m=4$ $p_1=3$ and $p_2=p_1-2n=t$ then $s=-\infty$. The Euler product generates a nontrivial zero.

The imaginary part of the logarithimic complex number is :

$\Im(s)=\frac{m+n}{2\pi}=\frac{3}{2\pi}$. The real part is zero. For all twin primes $q=p+2$

The imaginary part of the logarithmic complex number is :

$\Im(s)=\frac{(m-1)}{2\pi}=\frac{p}{2\pi}$.

$\Re(s)=0$.

For primes
$q=p+2N$

$\Im(s)=\frac{(m-N)}{\pi}=\frac{p}{2\pi}$.

$\Re(s)=0$.

An alternation formulation for zeroes outside the critical strip

Consider the Euler product $\zeta(s)=\prod\frac{p_i^s}{p_i^s-1}$.

If we set $s=\frac{-1}{\ln(\sqrt{x^2-p_1p_2-(\frac{p_2-p_1}{2})^2+1})}$,

nontrivial zeroes of a class not belonging to the Riemann hypothesis are generated when $\ln(x^2-p_1p_2-(\frac{p_2-p_1}{2})^2+1)=0$.

The graph (1) below is demonstrates the generation of one such zero.

The real part of the zeta function is however zero.

Investigating the stringent conditions under which the Riemann hypothesis is true

Theorem: Gap between prime

consider the prime p_k. The gap g_k between consecutive prime is given by:

$$\lim_{n\frac{p_k}{g_k}\to\infty}(1+(\frac{g_k}{np_k}))^{(\frac{np_k}{g_k})}=e \tag{1}$$

This means that:

$$\lim_{\frac{np_k}{g_k}\to\infty}(1+(\frac{g_k}{np_k}))=e^{(\frac{g_k}{np_k})} \tag{2}$$

Or

$$\lim_{\frac{np_k}{g_k}\to\infty}\ln(1+(\frac{g_k}{np_k}))=\frac{g_k}{np_k} \tag{3}$$

The above result for example implies $\ln(1.03)\approx 0.03$. It also implies that $\ln(1-0.03)\approx -0.03$,

$\ln(1-0)=0$.

This result follows from Taylor series expansion, $\ln(1\pm x)\approx\pm x$ for small x.

For values around , $|x|<0$ the approximation is very accurate, with an error of less than 0.001.

Riemann hypothesis

In a most general sense, the Riemann function can therefore be reformulated as

$$\zeta(s)=\zeta(\sin^2 p_k^{n_k}+it_k)=\frac{\ln(-(1+\frac{g_k^{m_k}}{p_k^{n_k}})^{\sin^2 p_k^{n_k}})}{\ln(1+\frac{g_k^{m_k})}{p_k^{n_k}})}=\sin^2 p_k^{n_k}+\frac{i\pi}{\ln(1+\frac{g_k^{m_k}}{p_k^{n_k}})} \tag{4}$$

This formulation implies that

$$\ln(1+(\frac{g_k^{m_k}}{p_k^{n_k}}))) = \frac{\pi}{t_k} \quad (5)$$

or

$$1+\frac{g_k^{m_k}}{p_k^{n_k}} = e^{\frac{\pi}{t_k}} \quad (6)$$

or

$$g_k^{m_k} = p_k^{n_k}(e^{\frac{\pi}{t_k}}-1) \quad (7)$$

or

$$g_k = (p_k^{n_k}(e^{\frac{\pi}{t_k}}-1))^{\frac{1}{m_k}} \quad (8)$$

Here t_k represents the k^{th} zero of the Riemann zeta function, while p_k represents the k^{th} prime.

An exact prime gap theorem

The Riemann hypothesis implies that

$$sinp_k^{n_k} = \pm\sqrt{\frac{1}{2}} \quad (9)$$

This means that

$$p_k = \sqrt[n_k]{\frac{\pi(1+8(l_k-1))}{4}} \quad (10)$$

where $l_k=k\geq 1$ is an integer greater or equal to 1.

$$g_k = ((\frac{\pi(1+8(l_k-1))}{4})^{n_k}(e^{\frac{\pi}{t_k}}-1))^{\frac{1}{m_k}} \quad (11)$$

Equation 8 can be written as

$$g_k = p_k^{\frac{n_k}{m_k}}(e^{\frac{\pi}{t_k}}-1)^{\frac{1}{m_k}} \quad (12)$$

By equation 6:

$$m_k = \frac{\ln p_k^{n_k}(e^{\frac{\pi}{t_k}}-1)}{\ln g_k} = \frac{n_k\ln p_k+\ln(e^{\frac{\pi}{t_k}}-1)}{\ln g_k} \quad (13)$$

therefore

$$\frac{n_k}{m_k} = \frac{n_k \ln g_k}{n_k \ln p_k + \ln(e^{\frac{\pi}{t_k}} - 1)} \tag{14}$$

Therefore:

$$g_k = p_k^{\frac{n_k \ln g_k}{n_k \ln p_k + \ln(e^{\frac{\pi}{t_k}} - 1)}} (e^{\frac{\pi}{t_k}} - 1)^{\frac{\ln g_k}{n_k \ln p_k + \ln(e^{\frac{\pi}{t_k}} - 1)}} \tag{15}$$

From (10)

$$n_k = \frac{\ln(\frac{\pi(1+8(l_k-1))}{4})}{\ln p_k} \tag{16}$$

To bring the gap terms togegether equation (15) can be rewritten as:

$$g_k^{\frac{1}{\ln g_k}} = p_k^{\frac{n_k}{n_k \ln p_k + \ln(e^{\frac{\pi}{t_k}} - 1)}} (e^{\frac{\pi}{t_k}} - 1)^{\frac{1}{n_k \ln p_k + \ln(e^{\frac{\pi}{t_k}} - 1)}} \tag{17}$$

This result is significant. The equation (17) constitutes the prime gap theorem.

Table of analysis

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p_k	g_k (Empirical)	g_k (Computed)	t_k (Zeta Zero)	n_k
2	1	1.0000	14.1347	-0.3485
3	2	2.0000	21.0220	1.7801
5	2	2.0000	25.0109	1.6103
7	4	4.0000	30.4249	1.5300
11	2	2.0000	32.9351	1.3574
13	4	4.0000	37.5862	1.3536
17	2	2.0000	40.9187	1.2884
19	4	4.0000	43.3271	1.2911
23	6	6.0000	48.0052	1.2543
29	2	2.0000	49.7738	1.2024

]

These findings suggest that prime gaps are fundamentally governed by the behavior of $\zeta(s)$ zeros, a significant result in analytic number theory.

An exact formulation for counting the number of primes

The mean prime gap, g_m can be defined as:

$$g_m = \frac{\sum (p_k^{\frac{n_k}{m_k}} (e^{\frac{\pi}{t_k}} - 1)^{\frac{1}{m_k}}) + 2}{k - 1} \tag{18}$$

This means

$$k = \pi(p_k) = \frac{\sum (p_k^{\frac{n_k}{m_k}} (e^{\frac{\pi}{t_k}} - 1)^{\frac{1}{m_k}}) + 2}{g_m} \tag{19}$$

Summary and Conclusion

This research establishes a direct relationship between prime gaps and the non-trivial zeroes of the Riemann zeta function. This work Numerically supports the Riemann hypothesis.

This research establishes a prime gap Theorem.

The Theorem strongly implies that prime gaps are not random but instead follow a well-defined deterministic law governed by prime numbers and the nontrivial zeros of the Riemann zeta function.

COPE Disclosure Report

Title of Research: Investigating prime gaps through zeta behaviour. Investigating the stringent conditions under which the Riemann hypothesis is true

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1. Authorship and Contributions

The author, Samuel Bonaya Buya, has solely conducted the research, formulated the mathematical models, performed numerical computations, and written the manuscript. No external contributors or co-authors were involved in the preparation of this work.

2. Ethical Compliance

This research complies with ethical guidelines set by the Committee on Publication Ethics (COPE). The study does not involve human subjects, personal data, or biological samples, and therefore does not require ethical approval from an institutional review board (IRB).

3. Use of AI and Computational Tools

The author used AI-assisted computational tools, including:

Mathematical Software: Python (NumPy, SciPy, SymPy) for symbolic computation and numerical validation.

AI Assistance: ChatGPT was used for generating structured formatting, verifying mathematical expressions, and ensuring clarity in explanations. However, all core research insights, mathematical derivations, and conclusions were independently formulated by the author.

The AI tools were utilized solely as an aid to enhance computational accuracy and presentation clarity, and they did not contribute to the originality of the research ideas.

4. Data Transparency and Reproducibility

The numerical results and graphs presented in this study are derived from publicly available mathematical constants (prime numbers, Riemann zeta function zeros). The computations were carried out using standard mathematical techniques, ensuring reproducibility.

The author has provided a computational framework and methodology that can be independently verified using any standard mathematical software.

5. Conflicts of Interest

The author declares no financial, personal, or professional conflicts of interest that could have influenced the research outcomes or its presentation.

6. Acknowledgments and Funding

No external funding or institutional support was received for this research. The author acknowledges independent efforts in developing the findings presented.

7. Compliance with Journal Policies

This research adheres to the submission guidelines and ethical policies of the target journal. The author affirms that:

The work is original and has not been published or submitted elsewhere.

Proper citation is provided for any referenced material.

The research does not involve plagiarism, falsification, or data fabrication.

8. Corrections and Retractions

If any errors are identified post-publication, the author is willing to cooperate with the journal in issuing corrections or retractions as per COPE guidelines.

Author's Declaration: I, Samuel Bonaya Buya, confirm that this research complies with COPE guidelines and that the information provided in this disclosure is accurate to the best of my knowledge.

Signature: ______________________

Date: [6th April 2025]

Reference

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Graphical results

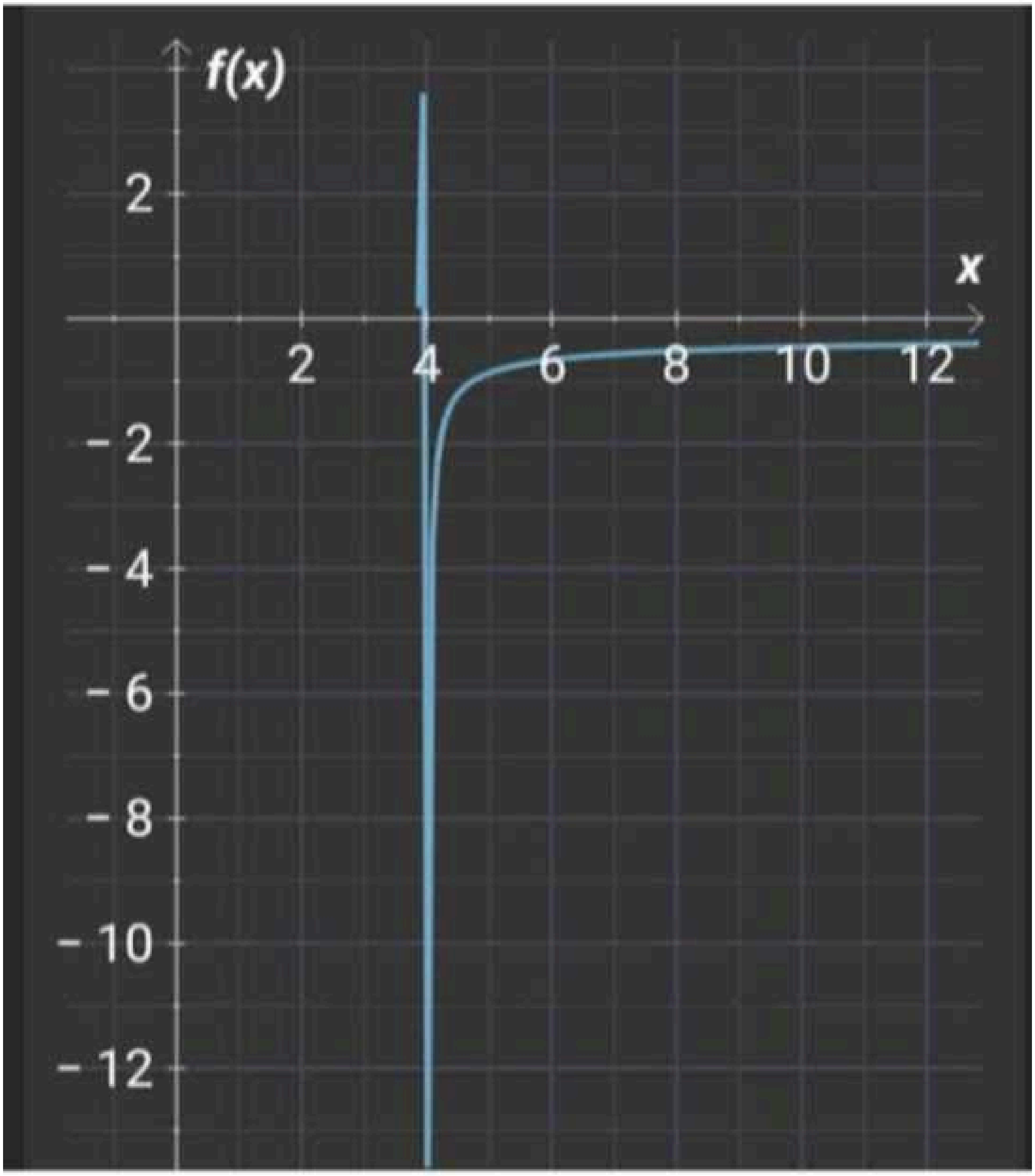

Figure 1: Zeros for

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