# A Symmetrical Reflected Radar Model of Special Relativity using Euclidian Geometry

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## Abstract

A Symmetrical Reflected Radar Model (SRRM) for Special Relativity is described. It analyses radar reflection using Bondi's k-calculus in a symmetrical spacetime diagram (SSD). Complete agreement is shown between the algebra of Special Relativity and its geometrical representation in the SRRM using Euclidian geometry. In the SRRM, the pairs of orthogonal axes; $t'$, $x$, and $t$, $x'$, which all have the same scale, can be considered as a symmetrical pair of Euclidian coordinate systems. The invariance of the spacetime interval in a transformation from the $t'$, $x$ to the $t$, $x'$ system is shown using Pythagoras. Thus, in a transformation from the $t'$, $x$ to the $t$, $x'$ coordinate systems (and vice versa) spacetime in Euclidian.

## Introduction

In 1908 Minkowski [1] introduced the spacetime diagram, now often called the Minkowski diagram.

In 1922, Gruner [2] recognised that the Minkowski diagram is *not* symmetrical and published a symmetrical spacetime diagram (SSD). 1n 1949, Loedel [3] published a similar diagram. SSDs are now often called Loedel Diagrams

In 1965 Bondi [4,5]introduced a reflected radar model of Special Relativity that he called the k-calculus. The algebra of the k-calculus is symmetrical. Bondi illustrated the k-calculus with diagrams of reflected radar in the Minkowski diagram, which is *not* symmetrical.

The symmetry of the algebra of the k-calculus will be illustrated in a symmetrical spacetime model named the Symmetrical Reflected Radar Model (SRRM). The development of this is described in detail by Asquith [6]. It will be shown that there is complete agreement of the algebra of Special Relativity with the SRRM using Euclidian geometry.

The assumptions on which the Symmetrical Reflected Radar Model (SRRM) is based are:

1. The Principle of Relativity. The laws of physics are identical for all inertial observers.
2. The velocity of light in a vacuum is measured the same by all inertial observers.
3. All the axioms and theorems of Euclidian geometry are valid when analysing spacetime. In particular, Cartesian coordinates can be used as a reference system by inertial observers.

4. To construct spacetime diagrams on a two-dimensional surface, with one dimension of time and one of space, it is valid to consider just the direction of relative motion of inertial observers and ignore the other two spatial dimensions.
5. Light and radar are typical examples of electromagnetic radiation. The same principles apply equally to both.
6. Using radar determination of spacetime coordinates, if a radar signal is emitted by an inertial observer at time = (a-b) and, after reflection at an event, received back by the observer at time = (a+b); then the coordinates of the event, as measured by the observer, are time = a ; distance = b.

Method

The model is called the Symmetrical Reflected Radar Model (SRRM).

Light and Radar are collectively referred to as a Signal. In diagrams, Signals are indicated by red lines with arrows. The speed of light is defined as 1.

The SRRM is based on a modification of Gruner's SSD. An orthogonal median inertial system has vertical time, t^ and a horizontal space, x^ axes. Thus, the SRRM shows Signals bisecting the vertical t^ and horizontal x^ axes, at 45 degrees to each..

In the SRRM are inertial observers Peter and Ruby. The time axis of Peter, t, (Peter's worldline) and the time axis of Ruby, t′, (Ruby's worldline) are oblique straight lines diverging symmetrically about the vertical median t^ axis, each at an angle θ. Peter's x axis and Ruby's x′ axis are oblique straight lines diverging symmetrically about the horizontal median x^ axis, each at an angle θ.

The t and t′, x and x′, axes of the SRRM all have the same scale. This is not the same scale as the t^ and x^ axes. The t axis is orthogonal with the x′ axis. The t′ axis is orthogonal with the x axis.

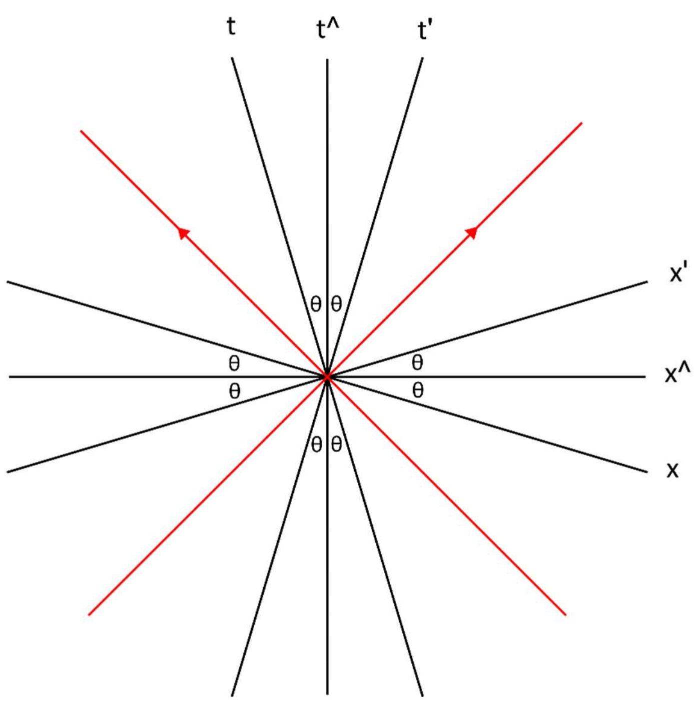


Figure 1. The Symmetrical Spacetime Diagram of the Symmetrical Reflected Radar Model

The scenario for reflecting radar is essentially the same that Bondi used in the k-calculus. In the SRRM, the paths (worldlines) of Peter and Ruby cross at O (Fig 2) where they both set their clocks to zero. Peter sends out a Signal at time (by Peter's clock) t = T after he and Ruby pass each other. Then this Signal will be received by Ruby at time (by Ruby's clock) t′ = kT. From the Principle of Relativity, the linear k factor will apply both ways. Thus, Ruby's response (the reflected Signal) will reach Peter at time (by Peter's clock) $t = k(kT) = k^2T$.

The SRRM then examines the relationship between distance and time, both algebraically and geometrically, for the relative motion between Peter and Ruby.

Results

The linear relationship k, the k factor, is shown geometrically in the SRRM (Fig 2). A Signal is sent by Peter at A, is reflected by Ruby at E and returns to Peter at D. It can be seen that the triangles OAE and OED are similar, each with an angle 2θ, an obtuse angle of (135 – θ) degrees and an acute angle of (45 – θ) degrees. Thus, the ratio of OA to OE is the same as the ratio of OE to OD. This ratio is the k factor.

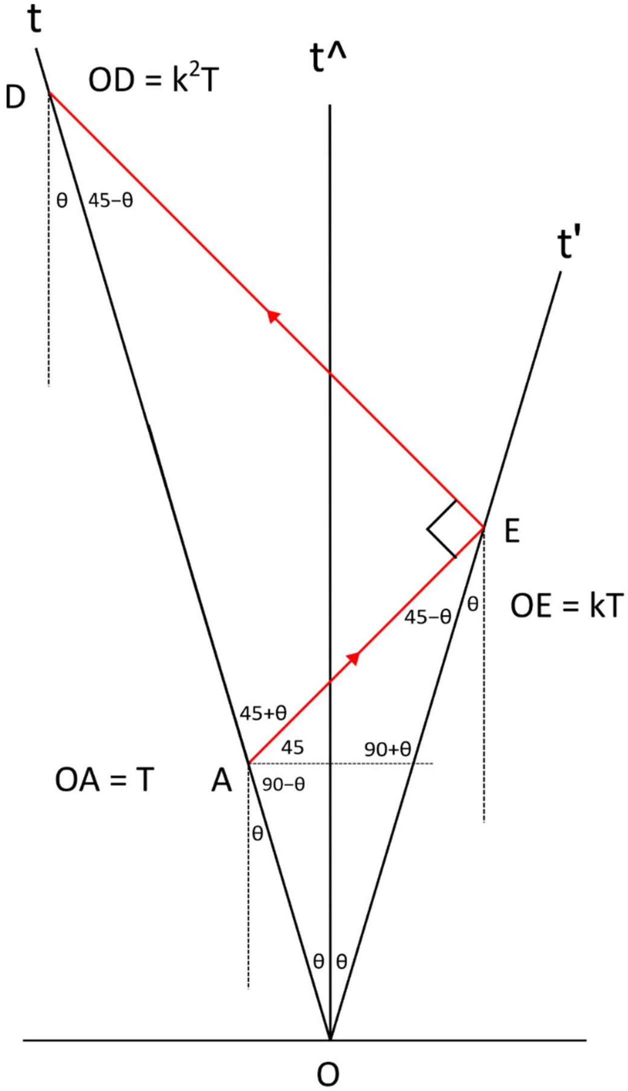


Figure 2 Geometrical representation of the k-calculus showing similar triangles OAE and OED

In figure 3, a signal sent by Peter from A is reflected by Ruby at E and returns to Peter at D.

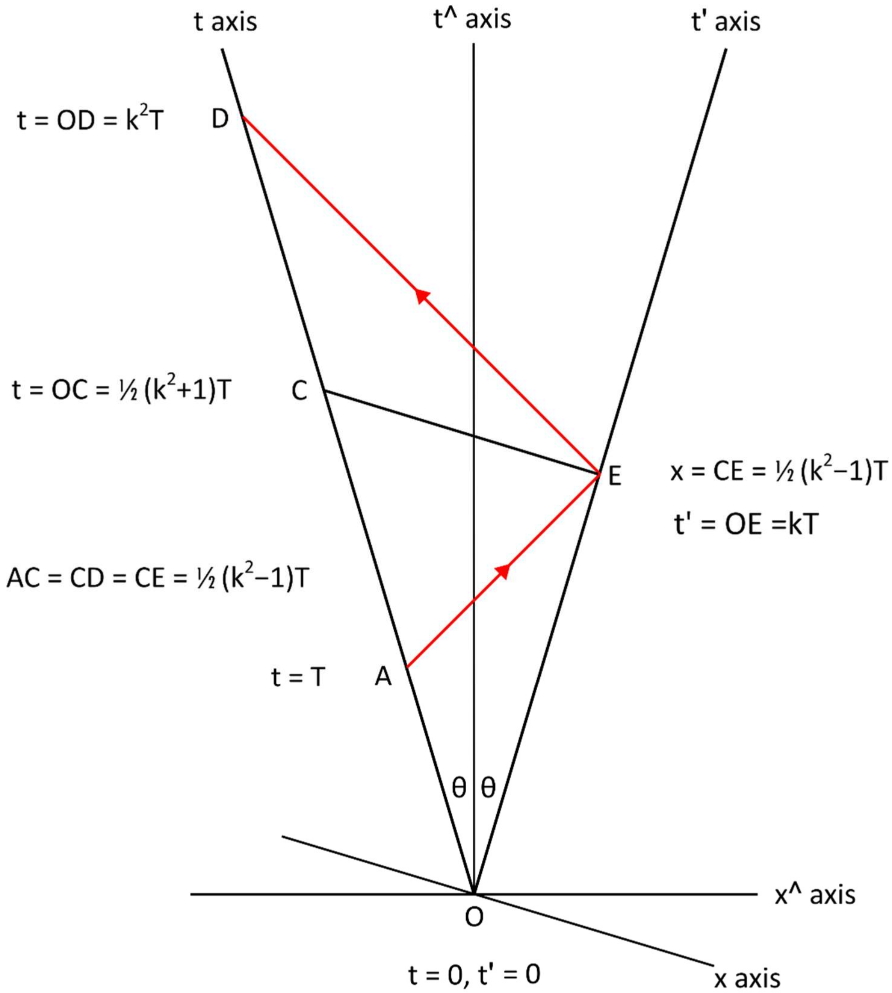


Figure 3 The distances in the SRRM derived from the k-calculus.

From Figure 3, $OA=T$, $OE=kT$, $OD = k^2T$

$AD = OD - OA = k^2T - T = = (k^2-1)T$. (1)

From assumption 6, the definition of radar coordinates: $CE = AC = CD = ½AD = ½(k^2-1)T$ (2)

Thus $OC = OA + AC = T + ½(k^2-1)T = ½(k^2+1)T$ (3)

The ratio of the lengths is simplified by dividing all lengths by kT

This gives the ratios as $OC = ½(k^2+1)/k$ (4)

$CE = ½(k^2-1)/k$ (5)

$OE = 1$ (6)

From (4) Let $\frac{1}{2}(k^2+1)/k$ be replaced by a (7)
From (5) Let $\frac{1}{2}(k^2-1)/k$ be replaced by b (8)

Thus; OC = a, CE = b, OE =1 (9)

$\text{Cos } 2\theta = OE/OC = 1/a$ (10)
$\text{Sin } 2\theta = CE/OC = b/a$ (11)
$\text{Tan } 2\theta = CE/OE = b$ (12)

From (7) $a = \frac{1}{2}(k^2+1)/k = \frac{1}{2}k + \frac{1}{2}(1/k)$ (13)
From (8) $b = \frac{1}{2}(k^2-1)/k = \frac{1}{2}k - \frac{1}{2}(1/k)$ (14)

Add (13) and (14) $k = a + b$ (15)
(13) minus (14) $1/k = a - b$ (16)

Multiply (15) X (16) $(a + b)(a - b) = k(1/k)$
$a^2 - b^2 = 1$ (17)
$a^2 = b^2 + 1$ (18)

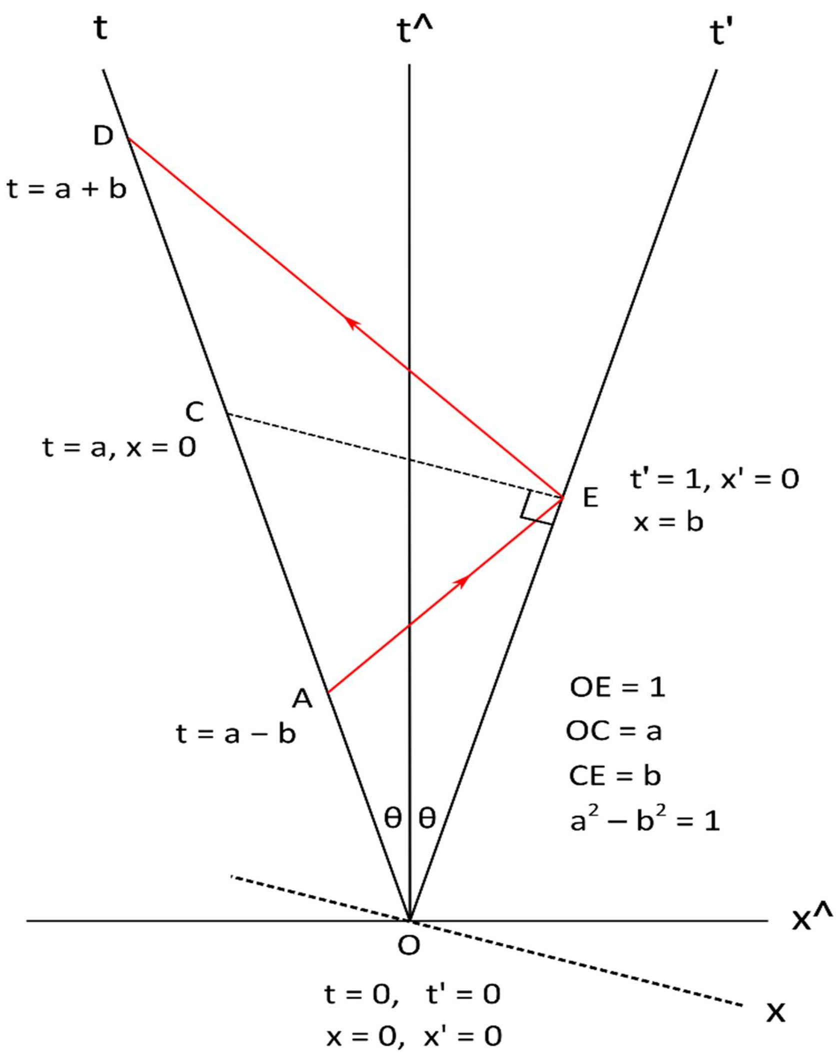


Fig 4 The ratios of the lengths in the SRRM, expressed as a or b based in the length of OE = 1

*The key triangles of Special Relativity and the hyperbola as their locus*

It has been determined that a, b, k and θ are constants related to the relative motion of Peter and Ruby. They are constants, rather than variables because, by definition the relative motion between inertial observers does not change. In a relative motion with a different value of θ; a, b and k change accordingly.

In triangle OEC, it has been shown that, at any θ, $a^2 - b^2 = 1$ (17).

Thus, as θ changes, the relationship between t and x changes such that $t^2 - x^2 = 1$

By the Principle of Relativity, a signal sent by Ruby at time (by Ruby's clock) $t' = a - b$ and reflected by Peter at time (by Peter's clock) $t = 1$ would result in the triangle OGF congruent with the triangle OEC (Figure 5).

Hence, as θ changes, $(t')^2 - (x')^2 = 1$

Thus, the locus of G and E (Figure 5) is described by $t^2 - x^2 = (t')^2 - (x')^2 = 1$

The triangles OEC and OGF in Figure 5 are the key triangles for analysing Special Relativity.

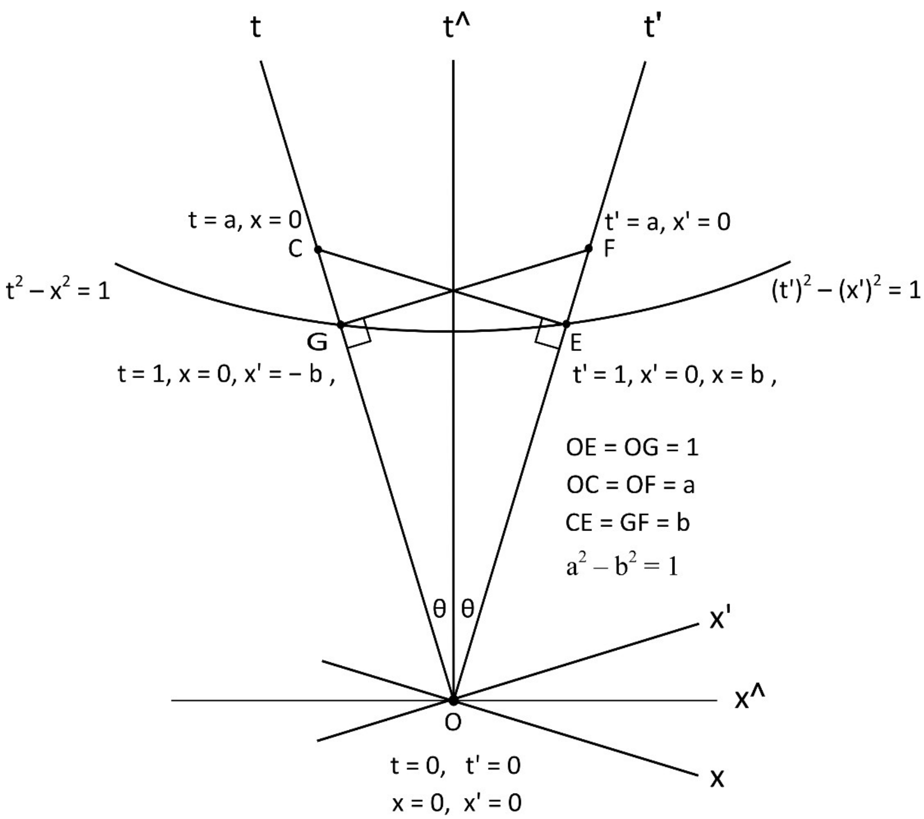


Figure 5. The locus of G and E as θ changes; the hyperbola $t^2 - x^2 = (t')^2 - (x')^2 = 1$

The angle OEC is a right angle formed between OE (the t′ axis) and CE (the x axis).
Similarly, the angle OGF is a right angle formed between OG (the t axis) and FG (the x′ axis).
The orthogonal pairing of axes is shown in Figure 6

*Orthogonal pairing of axes in the SRRM*

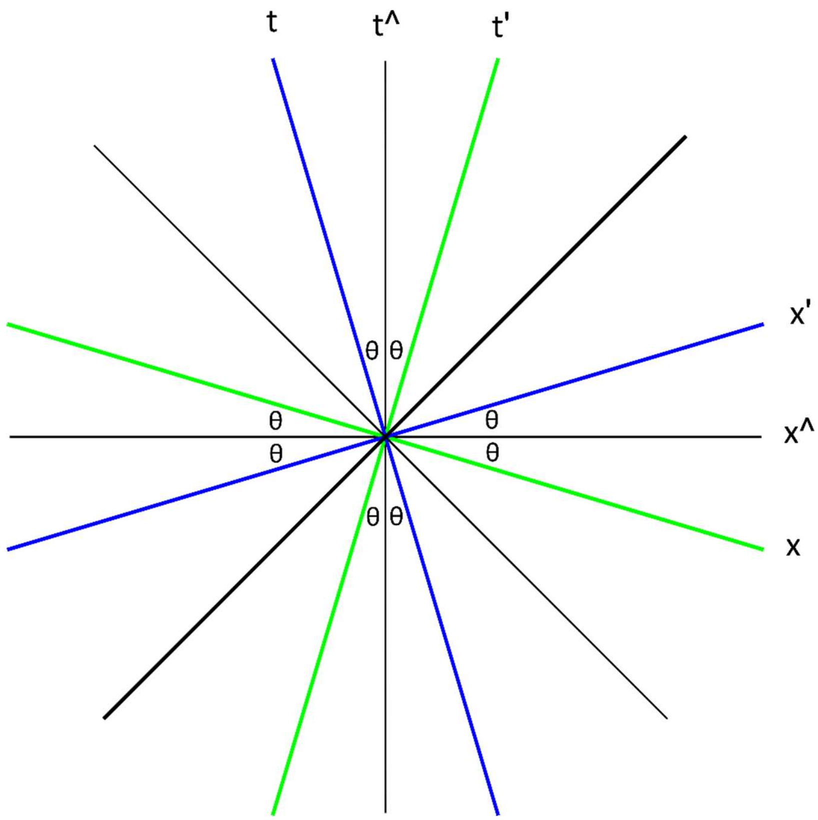


Figure 6. The two orthogonal and symmetric pairs of axes in the SRRM.

As θ varies in Fig 6, the t′ axis remains perpendicular to the x axis. In addition, at any θ, the t′ and x axes have the same scale. Similarly, at any θ, the t axis remains perpendicular to the x′ axis, with the same scale.

Thus, there are two orthogonal and symmetric pairs of axes at any θ. These can be considered as coordinate systems. They are shown in Fig.6 as the green and blue pairs of axes

*Transformation equations derived from the SRRM*

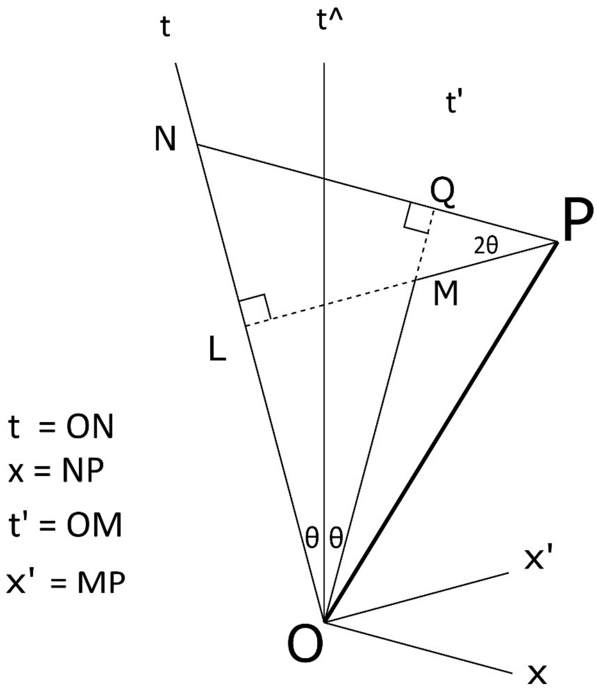


Figure 7. The Spacetime Interval to a point P and the transformation equations in the SRRM

The transformation equations can be derived in the SRRM by considering the coordinates of a point P.

$x' = LP - LM = x \operatorname{Cos} 2\theta - t' \operatorname{Sin} 2\theta = x\,(1/a) - t'\,(b/a)$ Thus, $\mathbf{x = ax' + bt'}$ (19)

$t' = OQ - MQ = t \operatorname{Cos} 2\theta - x' \operatorname{Sin} 2\theta = t\,(1/a) - x'\,(b/a)$ $\mathbf{t = at' + bx}$ (20)

$x = NQ + QP = t \operatorname{Sin} 2\theta + x' \operatorname{Cos} 2\theta = t\,(b/a) + x'\,(1/a)$ $\mathbf{x' = ax - bt}$ (21)

$t = OL + LN = t' \operatorname{Cos} 2\theta + x \operatorname{Sin} 2\theta = t'\,(1/a) + x\,(b/a)$ $\mathbf{t' = at - bx}$ (22)

*The spacetime interval OP*

The length of the spacetime interval OP from the origin to point P can be determined by Pythagoras from Figure 5

$$(OQ)^2 + (QP)^2 = (OP)^2 = (OL)^2 + (LP)^2$$

$$(ON\operatorname{Cos}2\theta)^2 + (MP\operatorname{Cos}2\theta)^2 = (OP)^2 = (OM\operatorname{Cos}2\theta)^2 + (NP\operatorname{Cos}2\theta)^2$$

$$\{t^2 + (x')^2\} = \{(t')^2 + x^2\} \quad (23)$$

Thus, if the green and blue pairs of axes are considered as coordinate systems, the interval OP is invariant in the transformation between the green and blue systems, shown by Euclidian geometry.

Discussion

*Development of the SRRM*

The SRRM was developed without any preconceptions about the relationship between distance and time. The constants a, b, k and θ have emerged which are all related to the relative motion. No definition of velocity has been assumed.

*Combination of the coordinate systems when θ is zero*

It can be seen in Figure 6 that, as θ decreases, the time axes of the blue and green coordinate systems rotate symmetrically to approach the t^ axis. When θ is zero the blue t and green t′ axes combine; the blue x and green x′ axes form a straight line. Thus, in the SRRM, when θ is zero, the blue, green and median systems combine and all that is apparent is the orthogonal t^, x^ median system.

If the values x/t′ in the green coordinate system and x′/t in the blue coordinate system are the fundamental measures of velocity, this would not be apparent as θ approaches zero and the coordinate systems combine. The values x/t and x′/t′ become close approximations to x/t′ and x′/t in the orthogonal system where velocities are negligible in relation to the speed of light.

*The limits for 2θ, Tan 2θ and Sin 2θ*

In figure 8, as θ varies, the points E and G form the curve $t^2 - x^2 = (t')^2 - (x')^2 = 1$

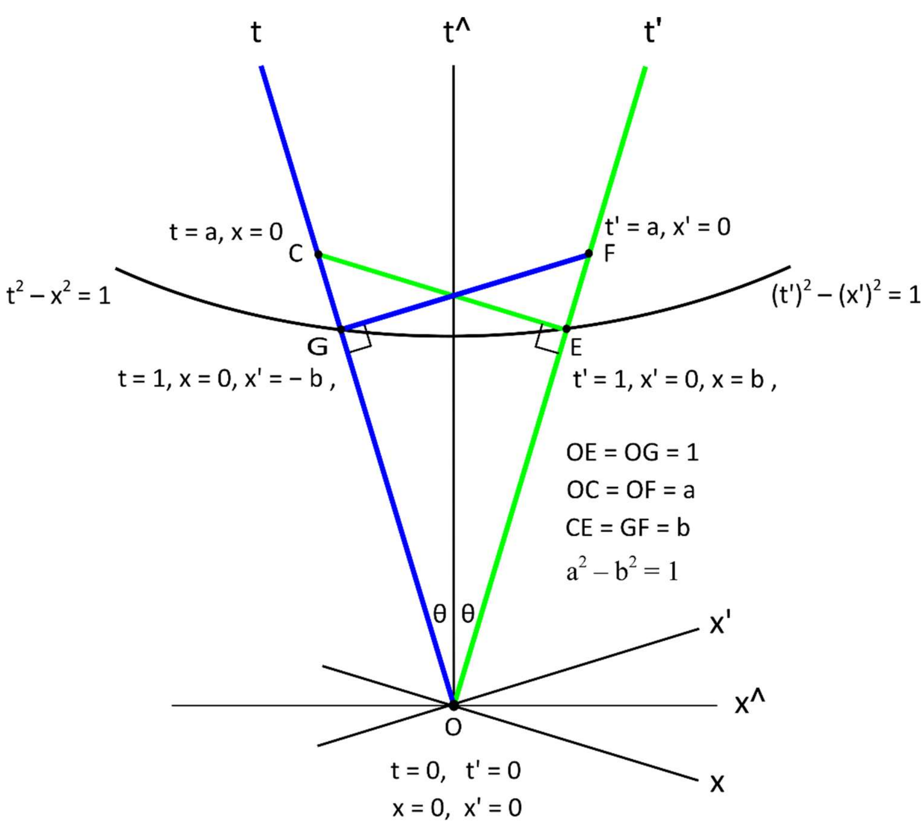


Figure 8. The hyperbola $t^2 - x^2 = (t')^2 - (x')^2 = 1$ formed as θ varies

The curve $t^2 - x^2 = 1$ has abscissae of $t = x$ or $-x$. These are the lines of the light or radar signal at 45 degrees either side of the t^ axis.

These lines are the limit of each angle θ which, thus, has a range 0 - 45 degrees. The angle 2θ, thus, has a range of 0 - 90 degrees. In that range, Tan 2θ increase from zero to infinity.

In the green coordinate system, $x/t' = CE/OE = \tan 2\theta = b$

Similarly, in the blue coordinate system, $x'/t = FG/OG = -\tan 2\theta = -b$

When $\theta = 22.5$ degrees, $2\theta = 45$ degrees. $\tan 2\theta = b = 1$. When $2\theta > 45$ degrees, b increases with a limit of infinity.

In contrast, $x/t = \sin 2\theta = b/a$ cannot exceed one. This is because $a > b$ (18). Put another way, the maximum value for the sine of any angle is 1.

*Invariance of the spacetime interval*

The invariance of the spacetime interval in the transformation between the blue and green coordinate systems in Figure 7 was shown for the interval OP starting at the origin. It is known that, in Euclidian geometry, the result will be true for the interval between any two points regardless of the origin.

Thus, in the transformation between the green and blue coordinate systems,

$$(\Delta t)^2 + (\Delta x')^2 = (\Delta t')^2 + (\Delta x)^2$$

It follows, therefore, that, in the transformation between the green and blue coordinate systems, the spacetime interval is invariant using Euclidian geometry.

*Pairs of coordinate systems*

The basis of the SRRM is that relative motion is always in symmetrical pairs. It follows therefore that any rotation of an axis with relative motion will always be accompanied by a symmetrical rotation. Thus, in relative motion, the symmetrical green and blue coordinate systems will always be formed. Thus a transformation from the green to the blue system is always possible with Euclidian geometry.

<u>Conclusion</u>

The SRRM has emphasised the importance of the pairs of orthogonal axes; t′, x, and t, x′, which all have the same scale. These can be considered as a pair of Euclidian coordinate systems. The invariance of the spacetime interval in a transformation from the x, t′ to the t, x′ coordinate system using Pythagoras suggests that spacetime is Euclidian when analysed in a symmetrical model.

## References


1. Minkowski, H. (1908) “Space and Time”. Address delivered at the 80th Assembly of German Natural Scientists and Physicians, Cologne. In “The Principle of Relativity,” Dover Publications, New York, 1923

https://archive.org/details/principleofrelat00lore_0/page/74/mode/2up

2. Gruner, P. (1922). Elemente der Relativitätstheorie [Elements of the theory of relativity] (in German). Bern: P. Haupt.

3. Loedel, E.(1949) Enseñanza de la Física  Buenos Aires p.495, fig 218
https://archive.org/details/LoedelEnsenanza/page/n527/mode/2up

4. Bondi H (1964) Relativity and Common Sense. *New York: Doubleday & Company.* Also published in 1965 in Great Britain by Heinemann, and reprinted in 1980 by Dover.

https://archive.org/details/relativitycommon0000bond

5. Bondi, H. (1967) Assumption and Myth in Physical Theory. The Tarner lectures delivered at Cambridge in November 1965. Cambridge University Press 1967

https://archive.org/details/assumptionmythin0000bond

6. Asquith, P. (2021) The Velocity Assumption. ISBN 9798772456378 Kindle Direct Publishing