INTEGRABILY AND COHOMOLOGY AND 6-SPHERE

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Abstract. We construct a differential from Nijenhuis tensor of any almost complex structure on a differentiable manifold, and show a relationship between the integrability of the almost complex structure and the cohomology of the manifold. For the case of 6-sphere, we first show that this form does not vanish for a special almost complex structure, and then show that this form does not vanish for any almost complex structure on the 6-sphere. Therefore all almost complex structures on 6-sphere are not integrable.

1. Introduction

Let M be a differential n-manifold and J an almost complex structure on M. We construct a differential form ℓ from J and Nijenhuis tensor N of the complex structure J, and show a relation between integrability of the almost complex structure and the cohomology of the manifold M. In 6-sphere case, we show form ℓ is not vanishing for a special almost complex structure first and then show this form is not vanishing for any almost complex structure on 6-sphere. That the form does not vanish implies that Nijienhius tensor does not vanish, which implies the almost complex structure is not integrable. Therefore all almost complex structures on 6-sphere are not integrable.

Integabilty of almost complex structure on differential manifold, in particular on high dimensional spheres has been studied extensively. If an almost complex structure is integrable, then it is a complex structure that makes the underline differential manifold a complex manifold. Conversely, the complex structure of a complex manifold induces a (an integrable) almost complex structure. There have been lot of great work in this field, for example, Hopf [13], Borel and Serre [5], Ehresmann [9], Kirchhoff [14], Eckmann and Frölicher [8], Ehressmann and Libermann [10], LeBrun [16], Atiyah [2], Bryant and S.S. Chern [3] [4],Cirici-Wilson[6], and etc to name a few. For spheres, it is known that all high dimensional (dimension greater than 2) spheres are not complex manifolds except for 6-sphere. Whether 6-sphere is a complex manifold or not is still a well-known open problem. See Hirzebruch [12] in 1954, Libermann [17] in 1955 and Yau [20] in 1990. Almost complex structures do exist on 6-sphere. It is known that none

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of known almost complex structures is integrable therefore not a complex structure that makes 6-sphere a complex manifold. See [1] for a survey.

We have the follwing results.

Theorem 1.1. *For Nijenhuis tensor N of any almost complex structure J,*

$$N(X,Y) = [JX,JY] - J[X,JY] - J[JX,Y] - [X,Y], \quad \text{for smooth vector fields } X \text{ and } Y,$$

where $[\cdot,\cdot\cdot]$ is Lie bracket, let ℓ be the tensor defined by

$$\ell(X,Y) := \mathrm{trace}\{JN\big(N(X,\cdot),Y\big)\} \quad \text{for smooth vector fields } X \text{ and } Y. \tag{1.1}$$

Then ℓ is a differential 2-form on M.

Corollary 1.2. *If form ℓ is not closed, then J is not integrable.*

Theorem 1.3. *Any almost complex structure on S^6 is not integrable.*

Proof. By Theorem 1.1, ℓ is a 2-form. If there exits a J whose ℓ form is closed, then $\ell \in H^2_{\mathrm{deRham}}(S^6)$. It is known that $H^2_{\mathrm{deRham}}(S^6)$ is trivial. So $\ell \equiv 0$, which contradicts to Theorem 1.4, since in current case ℓ is a topological invariant with unique vanishing value. Therefore in S^6 there is no J whose ℓ form is closed. For all almost complex structure J inducing form ℓ, $d\ell \not\equiv 0$. So $\ell \not\equiv 0$, and $N \not\equiv 0$. By Newlander-Nirenberg Theorem [19], J is not integrable. □

View S^6 as the unit sphere in the imaginary part of Octonians. There is a well-known canonical almost complex structure define on the tangent bundle of S^6 by multiplication in Octonions. We call it Left-multiplication almost complex structure. Please refer to Section 3, [15] , or [7] for more details about Left-multiplication almost complex structure.

Theorem 1.4. *Form ℓ is not vanishing for the Left-multiplication almost complex structure on S^6.*

In [18] author proved above ℓ is traceless. Along this line, the author later studied further during the visits to Princeton Mathematics Department and to Cornell Mathematics Department, and afterwards. The author obtained differential form ℓ in July of 2024, which includes traceless result in [18] since a skew-symmetric matrix is hollow therefore is traceless, and Theorem 1.4 and Theorem 1.3 in January of 2025. The author sincerely appreciate hospitality of the Princeton University and Cornell University.

In the following sections we give detailed calculations for above results. The next section is for proving Theorem 1.1. Last section is for proving Theorem 1.4.

2. Form

We take a local coordinates $\{x^i\}_{i=1}^n$ at point $x \in M$. Take sum for a repeating index, unless otherwise stated. Let $\partial_i = \frac{\partial}{\partial x^i}$, $J\frac{\partial}{\partial x^i} = J_i^j \frac{\partial}{\partial x^j}$,

$J_i = J_i^p \frac{\partial}{\partial x^p}$, and components of Nijenhuis tensor N.

$$N\big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\big) := N_{ij}^k \frac{\partial}{\partial x^k}.$$

It is easy to verify

$$N_{ij}^k = J_i^p(\partial_p J_j^k - \partial_j J_p^k) - J_j^p(\partial_p J_i^k - \partial_i J_p^k).$$

Proof of Theorem 1.1. The component form of Theorem 1.1 is the following Theorem 2.1. we prove it.

Theorem 2.1. *ℓ can be written in the following form in local coordinates:*

$$\ell = \ell_{ij} dx^i \wedge dx^j.$$

ℓ_{ij} is skew symmetric in i and j and

$$\ell_{ij} = -J_k^l \cdot J_i J_l^r \cdot J_j J_r^k + J_k^l \partial_i J_r^k \partial_j J_l^r + \{-J_i J_k^l \partial_j J_l^k + J_j J_k^l \partial_i J_l^k\}$$
$$+2\{J_i J_k^l \partial_l J_j^k - J_j J_k^l \partial_l J_i^k\} + 2\{-J_k J_i^l \partial_l J_j^k + J_k J_j^l \partial_l J_i^k\} + 2\{J_k J_i^l \partial_j J_l^k - J_k J_j^l \partial_i J_l^k\}.$$

We need only to show matrix $[\ell_{ij}]$ is sew-symmetric at every $x \in M$. By (1.1), we have

$$\ell_{ij} = N_{ik}^r N_{rj}^s J_s^k = J_s^k N_{ik}^r N_{rj}^s$$
$$= J_s^k \cdot J_i J_k^r \cdot J_r J_j^s - J_r^q J_s^k \cdot J_i J_k^r \cdot \partial_j J_q^s - J_s^k \cdot J_i J_k^r \cdot J_j J_r^s + J_j^q J_s^k \cdot J_i J_k^r \cdot \partial_r J_q^s$$
$$-J_i^p \cdot J_s J_p^r \cdot J_r J_j^s + J_r^q J_i^p \cdot J_s J_p^r \cdot \partial_j J_q^s + J_i^p \cdot J_s J_p^r \cdot J_j J_r^s - J_j^q J_i^p \cdot J_s J_p^r \cdot \partial_r J_q^s$$
$$+J_r J_j^s \cdot \partial_s J_i^r - J_r^q \cdot \partial_j J_q^s \cdot \partial_s J_i^r - J_j J_r^s \cdot \partial_s J_i^r + J_j^q \cdot \partial_r J_q^s \cdot \partial_s J_i^r$$
$$-J_r J_j^s \cdot \partial_i J_s^r + J_r^q \cdot \partial_i J_s^r \cdot \partial_j J_q^s + J_j J_r^s \cdot \partial_i J_s^r - J_j^q \cdot \partial_r J_q^s \cdot \partial_i J_s^r$$

Using facts $J^2 = -1$ and $\mathrm{tr}(J) = 0$ to calculate, it is easy to

$$\ell_{ij} = -J_k^l \cdot J_i J_l^r \cdot J_j J_r^k + J_k^l \partial_i J_r^k \partial_j J_l^r + \{-J_i J_k^l \partial_j J_l^k + J_j J_k^l \partial_i J_l^k\}$$
$$+2\{J_i J_k^l \partial_l J_j^k - J_j J_k^l \partial_l J_i^k\} + 2\{-J_k J_i^l \partial_l J_j^k + J_k J_j^l \partial_l J_i^k\} + 2\{J_k J_i^l \partial_j J_l^k - J_k J_j^l \partial_i J_l^k\},$$

Therefore ℓ_{ij} is sew-symmetric on i and j. □

3. Nonvanishing of ℓ for Left-multiplication

Let $\mathbb{O}$ be octonions. Every octonion x is a real linear combination of the unit octonions

$$x = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7, \quad e_0 = 1.$$

The product of $x, y \in \mathbb{O}$ is denoted by $x \cdot y$. If x and y are perpendicular, then $Im(x \cdot y) = x \cdot y$.

$$S^6 = \{x \in Im(\mathbb{O}) \big| |X|^2 = 1\}.$$

Take a point $e_1 \in S^6$,and y in tangent space of S^6: $Jv = e_1 \cdot v$ defines a linear transformation on the tangent space $T_{e_1} S^6$ with $J^2 = -1$. So it is an almost complex structure on S^6. We call it the Left-multiplication J. There are more details in [15] and [7] and many other presentations about this J.

Consider $X, Y : S^6 \longrightarrow \mathbb{R}^7$ as mappings, and dX and dY are differentials of mappings, Ch. [7]. Therefore

$$[X,Y] = dY(X) - dX(Y)$$
$$d(JY)(JX) = (JdY)(JX) + (dJY)(JX) = JdY(JX) + JX \cdot Y$$
$$d(JY)(X) = (JdY)(X) + (dJY)(X) = JdY(X) + X \cdot Y$$
$$[X,Y] = dY(X) - dX(Y)$$

$$[JX, JY] = d(JY)(JX) - d(JX)(JY) = JdY(JX) + JX \cdot Y - JdX(JY) - JY \cdot X$$
$$J[X, JY] = Jd(JY)(X) - JdX(JY) = -dY(X) + J(X \cdot Y) - JdX(JY)$$
$$J[JX, Y] = J(dY)(JX) - Jd(JX)(Y) = JdY(JX) + dX(Y) - J(Y \cdot X)$$

$$\begin{aligned} N(X,Y) &= [JX,JY] - J[X,JY] - J[JX,Y] - [X,Y] \\ &= JdY(JX) + JX \cdot Y - JdX(JY) - JY \cdot X \\ &+dY(X) - J(X \cdot Y) + JdX(JY) - JdY(JX) - dX(Y) + J(Y \cdot X) - [X,Y] \\ &= JX \cdot Y - JY \cdot X - J(X \cdot Y) + J(Y \cdot X) \end{aligned}$$

Let N_1 be Nijenhuis tensor at point $e_1 \in S^6$. $N_1(2,3) := N_{e_1}(e_2, e_3)$, $\cdots$, and etc. We have the following.

$$N_1(2,3) = -4e_6, N_1(2,4) = 0, N_1(2,5) = 4e_7, N_1(2,6) = 4e_3, N_1(2,7) = -4e_5,$$
$$N_1(3,4) = 4e_5, N_1(3,5) = -4e_4, N_1(3,6) = -4e_2, N_1(3,7) = 0,$$
$$N_1(4,5) = 4e_3, N_1(4,6) = -4e_7, N_1(4,7) = 4e_6,$$
$$N_1(5,6) = 0, N_1(5,7) = 4e_2,$$
$$N_1(6,7) = -4e_4,$$

Let $e_1 = f_7, e_2 = f_1, e_3 = f_2, e_4 = f_3, e_5 = f_4, e6 = f_5, e_7 = f_6$. In the following we denote $N(i,j) := N_{f_7}(f_i, f_j)$ for $i,j = 1,2,3,4,5,6$. At $f_7 = e_1 \in S^6$, we have

$$\frac{1}{4}[N(f_i,f_j)] = \begin{bmatrix} 0 & -f_5 & 0 & f_6 & f_2 & -f_4 \\ f_5 & 0 & f_4 & -f_3 & -f_1 & 0 \\ 0 & -f_4 & 0 & f_2 & -f_6 & f_5 \\ -f_6 & f_3 & -f_2 & 0 & 0 & f_1 \\ -f_2 & f_1 & f_6 & 0 & 0 & -f_3 \\ f_4 & 0 & -f_5 & -f_1 & f_3 & 0 \end{bmatrix} \tag{3.1}$$

Similarly we may calculate J_i^j at point $f_7 = e_1$. Note $J : T_{f_7}S^6 \longrightarrow T_{f_7}S^6$.

$$[J_i^j] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-\ell_{ij} = -N_{ik}^r N_{rj}^s J_s^k = N_{ik}^r N_{jr}^s J_s^k$$

$$= 0 + N_{i6}^1 N_{j1}^2 + 0 + N_{i5}^1 N_{j1}^4 - N_{i4}^1 N_{j1}^5 - N_{i2}^1 N_{j1}^6$$

$$+N_{i3}^2 N_{j2}^1 + 0 - N_{i1}^2 N_{j2}^3 + N_{i5}^2 N_{j2}^4 - N_{i4}^2 N_{j2}^5 - 0$$

$$+0 + N_{i6}^3 N_{j3}^2 - 0 + N_{i5}^3 N_{j3}^4 - N_{i4}^3 N_{j3}^5 - N_{i2}^3 N_{j3}^6$$

$$+N_{i3}^4 N_{j4}^1 + N_{i6}^4 N_{j4}^2 - N_{i1}^4 N_{j4}^3 + 0 - 0 - N_{i2}^4 N_{j4}^6$$

$$+N_{i3}^5 N_{j5}^1 + N_{i6}^5 N_{j5}^2 - N_{i1}^5 N_{j5}^3 + 0 - 0 - N_{i2}^5 N_{j5}^6$$

$$+N_{i3}^6 N_{j6}^1 + 0 - N_{i1}^6 N_{j6}^3 + N_{i5}^6 N_{j6}^4 - N_{i4}^6 N_{j6}^5 - 0$$

Now take $i = 4, j = 5$. Use values of N_{ij}^ks listed after (3.1), we have

$$-\ell_{45}$$

$$= 0 + N_{46}^1 N_{51}^2 + 0 + N_{45}^1 N_{51}^4 - N_{44}^1 N_{51}^5 - N_{42}^1 N_{51}^6$$

$$+N_{43}^2 N_{52}^1 + 0 - N_{41}^2 N_{52}^3 + N_{45}^2 N_{52}^4 - N_{44}^2 N_{52}^5 - 0$$

$$+0 + N_{46}^3 N_{53}^2 - 0 + N_{45}^3 N_{53}^4 - N_{44}^3 N_{53}^5 - N_{42}^3 N_{53}^6$$

$$+N_{43}^4 N_{54}^1 + N_{46}^4 N_{54}^2 - N_{41}^4 N_{54}^3 + 0 - 0 - N_{42}^4 N_{54}^6$$

$$+N_{43}^5 N_{55}^1 + N_{46}^5 N_{55}^2 - N_{41}^5 N_{55}^3 + 0 - 0 - N_{42}^5 N_{55}^6$$

$$+N_{43}^6 N_{56}^1 + 0 - N_{41}^6 N_{56}^3 + N_{45}^6 N_{56}^4 - N_{44}^6 N_{56}^5 - 0$$

$$= 0 - 16 + 0 + 0 - 0 - 0$$

$$-16 + 0 - 0 + 0 - 0 - 0$$

$$+0 + 0 - 0 + 0 - 0 - 16$$

$$+0 + 0 - 0 + 0 - 0 - 0$$

$$+0 + 0 - 0 + 0 - 0 - 0$$

$$+0 + 0 - 16 + 0 - 0 - 0$$

$$= -64$$

Therefore $\ell_{45} = 64 \neq 0$.

□

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