Generalized Dirac delta impulse and determinism obtained from the multivariate Gaussian

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Abstract

In this paper, we will propose to generalize the Dirac delta impulse to several dimensions. This generalization will be done by taking into account the one-dimensional version of the Dirac delta impulse. From a projection of the variance-covariance matrix, located inside the cone of positive semi-definite matrices, onto the boundary of the cone of positive semi-definite matrices having only the last eigenvalue equal to zero, we will make the transition from Gaussian probability theory to determinism.

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1 Introduction

In this paper, we will recall the notion of Dirac delta impulse in the one-dimensional case. This classical definition is made from a limit computed from a one-dimensional Gaussian. We will then generalize this concept to several dimensions from a projection of a variance covariance matrix Σ_{X^2}, initially located inside the cone of semi-definite positive matrices, onto the boundary of the cone of semi-definite positive matrices **having only the last eigenvalue equal to zero**: $\Sigma_{X^2} \longrightarrow \partial S_0^+$.

We will also explain the reason why we have generalized the notion of Dirac delta impulse to several dimensions. From the result obtained in paper [1] page 4, this projection will show the transition between the domain of Gaussian randomness and the determinism.

2 Classical Dirac delta impulse obtained from a Gaussian

Before introducing the generalized Dirac delta impulse, we need to recall the approach to the Dirac delta impulse made by the one-dimensional Gaussian in order to make the analogy later. Recall that the one-dimensional Dirac delta impulse is made by the following limit of the Gaussian:

$$\delta(x) = \lim_{\sigma \longrightarrow 0} [\frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{x^2}{2\sigma^2}\}]$$

In what follows we will generalize this concept to several dimensions.

3 Generalized Dirac delta impulse obtained from the multivariate Gaussian

We define the generalized Dirac delta impulse $\delta(\vec{X}-\vec{\mu}_X)$ from the multivariate Gaussian $\mathcal{N}(\mu_X, \Sigma_{X^2})$ as follows:

$$\boxed{\delta(\vec{X}-\vec{\mu}_X) = \mathcal{P}_{\Sigma_{X^2} \longrightarrow \partial S_0^+}[\mathcal{N}(\vec{\mu}_X, \Sigma_{X^2})] = \mathcal{P}_{\Sigma_{X^2} \longrightarrow \partial S_0^+}[(2.\pi)^{-\frac{n}{2}} |\Sigma_{X^2}|^{-\frac{1}{2}} \exp - \frac{(\vec{X}-\vec{\mu}_X)^t \Sigma_{X^2}^{-1} (\vec{X}-\vec{\mu}_X)}{2}]}$$

To understand the generalized Dirac delta impulse, we must initially consider a random Gaussian vector following the probability law $\mathcal{N}(\mu_X, \Sigma_{X^2})$.

$\mathcal{P}_{\Sigma_{X^2} \longrightarrow \partial S_0^+}$ then corresponds to the projection of the variance covariance matrix, located inside the cone of positive semi-definite matrices, onto the boundary ∂S_0^+ of the cone of positive semi-definite matrices **having only the last eigenvalue equal to zero**. **This projection of the matrix is done by performing the spectral decomposition of the matrix $\Sigma_{X^2} = P.\Lambda.P^t$, by canceling the last eigenvalue of Λ and by returning to the starting basis**.

Onto the boundary of the cone ∂S_0^+, the determinant of the matrix is zero ($|\Sigma_{X^2}| = 0$), the matrix Σ_{X^2} is therefore singular and not invertible. **By making the smallest eigenvalue $\lambda_{min}(\Sigma_{X^2})$ of the matrix Σ_{X^2} tend towards 0, we will show that we have indeed generalized the Dirac delta impulse to several dimensions.(see the analogy)**

As we demonstrated in the paper ([1] page 4) the boundary of the cone of positive semi-definite matrices **having only the last eigenvalue equal to zero** ∂S_0^+ contains **the predictability** and **the determinism**. The projection $\mathcal{P}_{\Sigma_{X^2} \longrightarrow \partial S_0^+}$ therefore expresses the transition from Gaussian probability theory to determinism.

The vector $\vec{X}$ of the multivariate Gaussian $\mathcal{N}(\vec{\mu}_X, \Sigma_{X^2})$ therefore infers randomness while the vector X of the generalized Dirac impulse $\delta(\vec{X}-\vec{\mu}_X)$ infers determinism.

For the multivariate Gaussian, the transition from **randomness** to **determinism** is made with the projection $\mathcal{P}_{\Sigma_{X^2} \longrightarrow \partial S_0^+}$ or by the limit $\lim\limits_{\lambda_{min}(\Sigma_{X^2}) \longrightarrow 0}$.

4 Integration for the generalized Dirac delta impulse

Consider the multivariate Gaussian $Y = \mathcal{N}(\mu_Y, \Sigma_{Y^2})$ projected onto the boundary ∂S_0^+ with the spectral decomposition of its variance covariance matrix: $\Sigma_{Y^2} = P.\Lambda.P^t$.

If we consider the vectors $\vec{X}$ and $\vec{\mu}_X$ in the normalized eigenvector basis of Σ_{Y^2}:

$\vec{X} = P^{-1}\vec{Y}$ and $\vec{\mu}_X = P^{-1}\mu_Y$

then we obtain the following equality:

$$\boxed{\int_{-\infty}^{\infty} \delta[P.(\vec{X} - \vec{\mu}_X)]d\vec{X} = 1}$$

The Dirac delta impulse in the initial basis therefore passes into the normalized eigenvector basis of the matrix $\Sigma_{X^2} \longrightarrow \partial S_0^+$, then it is transformed with the linear application matrix P into another Dirac delta impulse whose the integral of $[-\infty, +\infty]$ gives 1.

Proof

Consider the Gaussian $\mathcal{N}(\vec{\mu}_Y, \Sigma_{Y^2})$:

The projection $\mathcal{P}_{\Sigma_{Y^2} \longrightarrow \partial S_0^+}$ will be put in the form of a limit in which we make tender the smallest eigenvalue $\lambda_{min}(\Sigma_{Y^2})$ of the matrix Σ_{Y^2} towards 0.

$\delta(\vec{Y} - \vec{\mu_Y})$

$= \mathcal{P}_{\Sigma_{Y^2} \longrightarrow \partial S_0^+} \mathcal{N}(\vec{\mu}_Y, \Sigma_{Y^2})$

$= \lim_{\lambda_{min}(\Sigma_{Y^2}) \longrightarrow 0} \mathcal{N}(\vec{\mu}_Y, \Sigma_{Y^2})$

$$= \lim_{\lambda_{min}(\Sigma_{Y^2}) \longrightarrow 0} [(2.\pi)^{-\frac{n}{2}} |\Sigma_{Y^2}|^{-\frac{1}{2}} \exp - \frac{(\vec{Y} - \vec{\mu}_Y)^t.P^{-t}.\Lambda^{-1}.P^{-1}.(\vec{Y} - \vec{\mu}_Y)}{2}]$$

We pose :

$\vec{X} = P^{-1}\vec{Y}$

$\vec{\mu}_X = P^{-1}\vec{\mu}_Y$

$|\Sigma_{Y^2}| = \lambda_{min}. \prod_{i=1}^{n-1} \lambda_i$

and

$(2\pi)^{-\frac{n}{2}} = (2\pi)^{-\frac{n-1}{2}}.(2\pi)^{-\frac{1}{2}}$

We obtain:

$\delta(\vec{Y} - \vec{\mu_Y})$

$= \delta[P.(\vec{X} - \vec{\mu_X})]$

$$= \lim_{\lambda_{min}(\Sigma_{Y^2}) \longrightarrow 0} [(2\pi)^{-\frac{n-1}{2}}.(2\pi)^{-\frac{1}{2}} (\lambda_{min}. \prod_{i=1}^{n-1} \lambda_i)^{-\frac{1}{2}} \exp - \frac{(\vec{X} - \vec{\mu}_X)^t.\Lambda^{-1}.(\vec{X} - \vec{\mu}_X)}{2}]$$

$$= \lim_{\lambda_{min}(\Sigma_{Y2})\longrightarrow 0}[(2\pi)^{-\frac{n-1}{2}}.(2\pi)^{-\frac{1}{2}}(\lambda_{min}.\prod_{i=1}^{n-1}\lambda_i)^{-\frac{1}{2}}\exp-\frac{(\vec{X}-\vec{\mu}_X)^t_{1...n-1}.\Lambda^{-1}_{(1...n-1X1...n-1)}.(\vec{X}-\vec{\mu}_X)_{1...n-1}}{2}-\frac{1}{2}\frac{(x_n-\mu_n)^2}{\lambda_{min}}]$$

$$= (2\pi)^{-\frac{n-1}{2}}.\prod_{i=1}^{n-1}\lambda_i^{-\frac{1}{2}}.\exp-\frac{(\vec{X}-\vec{\mu}_X)^t_{1...n-1}.\Lambda^{-1}_{(1...n-1X1...n-1)}.(\vec{X}-\vec{\mu}_X)_{1...n-1}}{2}.\lim_{\lambda_{min}(\Sigma_{Y2})\longrightarrow 0}.\frac{1}{\sqrt{2\pi}}.\frac{1}{\sqrt{\lambda_{min}}}.\exp-\frac{1}{2}\frac{(x_n-\mu_n)^2}{\lambda_{min}}$$

As the following multiple integral is equal to 1:

$$\int_{\infty}^{+\infty}(2\pi)^{-\frac{n-1}{2}}.\prod_{i=1}^{n-1}\lambda_i^{-\frac{1}{2}}.\exp-\frac{(\vec{X}-\vec{\mu}_X)^t_{1...n-1}.\Lambda^{-1}_{(1...n-1X1...n-1)}.(\vec{X}-\vec{\mu}_X)_{1...n-1}}{2}dx_1.dx_2...dx_{n-1}$$
$$= 1$$

because it is a probability density integrated over the entire domain.

The integral of the generalized Dirac impulse therefore becomes:

$$\int_{-\infty}^{+\infty}\delta[P(\vec{X}-\vec{\mu}_X)]dX_1dX_2...dX_n$$

$$= \int_{-\infty}^{+\infty}\lim_{\lambda_{min}(\Sigma_{Y2})\longrightarrow 0}.\frac{1}{\sqrt{2\pi}}.\frac{1}{\sqrt{\lambda_{min}}}.\exp-\frac{1}{2}\frac{(x_n-\mu_n)^2}{\lambda_{min}}dX_n$$

We will show in what follows that this last limit is always equal to 1 for all values of λ.

We pose $a = \frac{1}{\sqrt{2\lambda_{min}}}$ and $z = (x_n - \mu_n)$ if λ_{min} tends towards 0 then a tends towards infinity.

$$= \int_{-\infty}^{+\infty}\lim_{a\longrightarrow\infty}\frac{a}{\sqrt{\pi}}.\exp\{-a^2z^2\}dz$$

$$= \lim_{a\longrightarrow\infty}\int_{-\infty}^{+\infty}\frac{a}{\sqrt{\pi}}.\exp\{-a^2z^2\}dz$$

We put:

$t = az \Longrightarrow z = \frac{t}{a}$

$dt = a.dz \Longrightarrow dz = \frac{dt}{a}$ if $z \longrightarrow \infty$ then $t \longrightarrow \infty$

We obtain:

$$= \lim_{a\longrightarrow\infty}\int_{-\infty}^{+\infty}\frac{a}{\sqrt{\pi}}.\exp\{-a^2z^2\}dz$$

$$= \lim_{a\longrightarrow\infty}\int_{-\infty}^{+\infty}\frac{a}{\sqrt{\pi}}\exp\{-t^2\}\frac{dt}{a}$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp\{-t^2\}dt$$

However: $\int_{-\infty}^{+\infty}\exp\{-x^2\}dx = \sqrt{\pi}$

we have therefore demonstrated the following equality:

$$\int_{-\infty}^{+\infty}\delta[P(\vec{X}-\vec{\mu}_X)]dX_1dX_2...dX_n = 1$$

5 Analogy between the generalized Dirac delta impulse and the classical Dirac delta impulse

The one-dimensional dirac delta impulse:

$$\delta(x - x_0) = \begin{cases} +\infty & \text{if } x = x_0 \\ 0 & \text{if} x \neq x_0 \end{cases}$$

with

$$\int_{-\infty}^{+\infty} \delta(x - x_0) dx = 1$$

becomes multidimensional the following definition:

$$\delta(\vec{X} - \vec{\mu}_X) = \begin{cases} +\infty & \text{if } \vec{X} = \vec{\mu}_X \\ 0 & \text{if } \vec{X} \neq \vec{\mu}_X \end{cases}$$

with

$$\int_{-\infty}^{\infty} \delta[P.(\vec{X} - \vec{\mu}_X)] d\vec{X} = 1$$

Where P corresponds to the matrix in the spectral decomposition of the variance co-variance matrix Σ_{X^2} projected onto the boundary ∂S_0^+:

$\Sigma_{X^2} = P.\Lambda.P^t$

and

the vectors $\vec{X}$ and $\vec{\mu}_X$ are expressed in the normalized basis of eigenvectors of the matrix projected Σ_{X^2} onto boundary ∂S_0^+.

With several variables, we have another property that is added to the one-dimensional impulse: we have demonstrated that $\delta(\vec{X} - \vec{\mu}_X)$ infers **determinism** (see paper [1] page 4) onto ∂S_0^+ while the multivariate Gaussian $\mathcal{N}(\vec{\mu}_X, \Sigma_{X^2})$ infers **randomness** inside the cone of positive semi-definite matrices.

6 Conclusion

In this paper, the generalization of the Dirac delta impulse was made by taking into account the classical one-dimensional version and the limit of the one-dimensional Gaussian. The limit then became, in a multidimensional case, a projection of a strictly positive-definite variance-covariance matrix onto the boundary of the cone of positive semi-definite matrices having only the last eigenvalue equal to zero. The generalization can also be done by making the last eigenvalue (the smallest) tend towards zeros: $\lim_{\lambda_{min}(\Sigma_{X^2}) \longrightarrow 0}$. This projection, giving the generalized Dirac delta impulse, made the transition between Gaussian randomness and the determinism. The multivariate Gaussian in fact infers randomness, while the generalized Dirac delta impulse obtained by projection infers determinism.

[1] Understanding when the correlations imply the predictability for the multiple Gaussian. Author: Ait-Taleb Nabil. Published on vixra in 2025

[2]Optimal stastical decisions. Author: Morris H.DeGroot. Copyright 1970-2004 John Wiley and sons.

[3]Computing the nearest correlation Matrix-A problem from finance. Author: Nicholas Higham.copyright 2002, The university of Manchester