Submitted exclusively to the London Mathematical Society
doi:10.1112/0000/000000

A Novel Proof of The *abc* Conjecture: It is Easy as abc!

Abdelmajid Ben Hadj Salem

To the memory of my Father who taught me arithmetic
To my wife ***Wahida****, my daughter* ***Sinda*** *and my son* ***Mohamed Mazen***
To Prof. ***A. Nitaj*** *for his work on the abc conjecture*

Abstract

In this paper, we consider the *abc* conjecture. Assuming that the conjecture $c < rad^{1.63}(abc)$ is true, we give the proof that the *abc* conjecture is true.

1. *Introduction and notations*

Let a be a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1.1}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{1.2}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 1. (***abc* Conjecture**): *For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{1.3}$$

where K is a constant depending only of ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [**2**]. It concerned the best example given by E. Reyssat [**2**]:

$$2 + 3^{10}.109 = 23^5 \Longrightarrow c < rad^{1.629912}(abc) \tag{1.4}$$

A conjecture was proposed that $c < rad^2(abc)$ [**3**]. In 2012, A. Nitaj [**4**] proposed the following conjecture:

2000 *Mathematics Subject Classification* 11AXX (primary), 11JXX, 26AXX (secondary)..

Conjecture 2. *Let a, b, c be positive integers relatively prime with $c = a + b$, then:*

$$c < rad^{1.63}(abc) \tag{1.5}$$

$$abc < rad^{4.42}(abc) \tag{1.6}$$

In the following, we assume that the conjecture giving by the equation (1.5) is true that constitutes the key to obtain the proof of the *abc* conjecture and we consider the cases $c > R$ because the *abc* conjecture is verified if $c < R$. For our proof, we proceed by contradiction of the abc conjecture, for $\epsilon \in]0., 0.63[$.

2. *The Proof of the abc conjecture*

Proof. :

2.1. *Trivial Case* $\epsilon \geq (0.63 = \epsilon_0)$.

In this case, we choose $K(\epsilon) = e$ and let a, b, c be positive integers, relatively prime, with $c = a + b$, $1 \leq b < a, R = rad(abc)$, then $c < R^{1+\epsilon_0} \leq K(\epsilon).R^{1+\epsilon} \Longrightarrow c < K(\epsilon).R^{1+\epsilon}$ and the *abc* conjecture is true.

2.2. *Case:* $0 < \epsilon < (0.63 = \epsilon_0)$.

We recall the following proposition [**4**]:

Proposition 2.1. *Let $\epsilon \longrightarrow K(\epsilon)$ the application verifying the abc conjecture, then:*

$$lim_{\epsilon \to 0} K(\epsilon) = +\infty \tag{2.1}$$

We suppose that the abc conjecture is false, then it exists $\epsilon' \in]0, \epsilon_0[$ and for all parameter $K' = K'(\epsilon') > 0$ it exists at least one triplet (a', b', c') so a', b', c' be positive integers relatively prime with $c' = a' + b'$ and c' verifies :

$$c' > K'(\epsilon').{R'}^{1+\epsilon'} \tag{2.2}$$

From the proposition cited above, it follows that $lim_{\epsilon \longrightarrow 0} K'(\epsilon) < +\infty$, we can suppose that $K'(\epsilon)$ is an increasing parameter for $\epsilon \in]0, \epsilon_0[$.

As the parameter K' is arbitrary, we choose $K'(\epsilon) = e^{\epsilon^2}$, it is an increasing parameter. Let :

$$Y_{c'}(\epsilon) = \epsilon^2 + (1+\epsilon) Log R' - Log c', \epsilon \in]0, \epsilon_0[\tag{2.3}$$

About the function $Y_{c'}$, we have:

$$lim_{\epsilon \longrightarrow \epsilon_0} Y_{c'}(\epsilon) = \epsilon_0^2 + Log({R'}^{1+\epsilon_0}/c') = \lambda > 0, \quad as\ c < R^{1+\epsilon_0}$$
$$lim_{\epsilon \longrightarrow 0} Y_{c'}(\epsilon) = -Log(c'/R') < 0, \quad as\ R < c$$

The function $Y_{c'}(\epsilon)$ represents a parabola and it is an increasing function for $\epsilon \in]0, \epsilon_0[$, then the equation $Y_{c'}(\epsilon) = 0$ has one root that we denote ϵ'_1, it follows the equation :

$$e^{\epsilon_1'^2} R'^{\epsilon'_1} = \frac{c'}{R'} \tag{2.4}$$

Discussion about the equation (2.4) above:

We recall the following definition:

Definition 1. The number ξ is called algebraic number if there is at least one polynomial:

$$l(x) = l_0 + l_1 x + \cdots + l_m x^m, \quad l_m \neq 0 \tag{2.5}$$

with integral coefficients such that $l(\xi) = 0$, and it is called transcendental if no such polynomial exists.

We consider the equation (2.4) :

$$c' = K'(\epsilon'_1) R'^{1+\epsilon'_1} \Longrightarrow \frac{c'}{R'} = \frac{\mu'_{c'}}{rad(a'b')} = e^{\epsilon'^2_1} R'^{\epsilon'_1} \tag{2.6}$$

i) - We suppose that $\epsilon'_1 = \beta_1$ is an algebraic number then $\beta_0 = \epsilon'^2_1$ and $\alpha_1 = R'$ are also algebraic numbers. We obtain:

$$\frac{c'}{R'} = \frac{\mu'_{c'}}{rad(a'b')} = e^{\epsilon'^2_1} R'^{\epsilon'_1} = e^{\beta_0}.\alpha_1^{\beta_1} \tag{2.7}$$

From the theorem (see theorem 3, page 196 in [**5**]):

Theorem 2.2. *$e^{\beta_0}\alpha_1^{\beta_1}\dots\alpha_n^{\beta_n}$ is transcendental for any nonzero algebraic numbers $\alpha_1, \dots, \alpha_n, \beta_0, \dots, \beta_n$.*

we deduce that the right member $e^{\beta_0}.\alpha_1^{\beta_1}$ of (2.7) is transcendental, but the term $\dfrac{\mu'_{c'}}{rad(a'b')}$ is an algebraic number, then the contradiction and the hypothesis that the *abc* conjecture is false on $\epsilon \in]0, \epsilon_0[$ is not true. It follows that the *abc* conjecture is true on $\epsilon \in]0, \epsilon_0[$, then for all $\epsilon > 0$.

ii) - We suppose that ϵ'_1 is transcendental, then ϵ'^2_1 is transcendental. If not, ϵ'^2_1 is an algebraic number, it verifies:

$$l(x) = l_{2m}\epsilon'^{2m}_1 + 0 \times \epsilon'^{2m-1}_1 + l_{2(m-1)}\epsilon'^{2(m-1)}_1 + \cdots + l_2\epsilon'^2_1 + 0 \times \epsilon'_1 + l_0 = 0$$

From the definition (2.5) and the equation above, e'_1 is also an algebraic number, then the contradiction with ϵ'_1 a transcendental number.

As $R' > 0$ is an algebraic number, we know that $LogR'$ is transcendental. We rewrite the equation (2.4) as:

$$\frac{c'}{R'} = e^{\epsilon'^2_1} R'^{\epsilon'_1} = e^{\epsilon'^2_1 + \epsilon'_1 LogR'} \tag{2.8}$$

By the theorem of Hermite (page 45, [**5**]) e is transcendental. Let $z = \epsilon'^2_1 + \epsilon'_1 LogR' > 0$:

- If $z \neq 0$, if z is an algebraic number it follows that e^z is transcendental by the theorem of Lindemann (page 51, [**5**]), it follows the contradiction with c'/R' an algebraic number. Then the hypothesis that the *abc* conjecture is false on $\epsilon \in]0, \epsilon_0[$ is not true. It follows that the *abc* conjecture is true on $\epsilon \in]0, \epsilon_0[$, then for all $\epsilon > 0$.

- Now we suppose that z is transcendental. We write e^z as:

$$e^z = \sum_{n=1}^{+\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots + \frac{z^N}{N!} + r(z)$$

$$\text{and } r(z) \ll \frac{z^N}{N!} \quad \text{for } N \text{ very large}$$

Then :

$$R'z^N + R'Nz^{(N-1)} + \cdots + R'N!z + N!(R' - c') = 0$$

It follows that z is an algebraic number $\Longrightarrow$ the contradiction avec z transcendental. Then the hypothesis that the *abc* conjecture is false on $\epsilon \in]0, \epsilon_0[$ is not true. It follows that the *abc* conjecture is true on $\epsilon \in]0, \epsilon_0[$, then for all $\epsilon > 0$.

The proof of the *abc* conjecture is finished. Assuming $c < R^{1+\epsilon_0}$ is true, we obtain that $\forall \epsilon > 0$, $\exists K(\epsilon) > 0$, if $c = a + b$ with a, b, c positive integers relatively coprime, then :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{2.9}$$

and the constant $K(\epsilon)$ depends only of ϵ.

Q.E.D

Ouf, end of the mystery!

□

3. *Conclusion*

Assuming $c < R^{1+\epsilon_0}$ is true, we have given an elementary proof of the *abc* conjecture. We can announce the important theorem:

THEOREM 3.1. *Assuming $c < R^{1+\epsilon_0}$ is true, the abc conjecture is true: For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then:*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{3.1}$$

where K is a constant depending of ϵ.

Acknowledgements. The author is very grateful to Professors Mihăilescu Preda and Gérald Tenenbaum for their comments about errors found in previous manuscripts concerning proofs proposed of the *abc* conjecture.

Author contributions

This is the author contribution text.

Financial disclosure

None reported.

Conflict of interest

The author declares no potential conflict of interests.

ORCID

Abdelmajid Ben Hadj Salem $https://orcid.org/0000-0002-9633-3330$.

References

1. M. Waldschmidt: On the abc Conjecture and some of its consequences, presented at The 6th World Conference on 21st Century Mathematics, Abdus Salam School of Mathematical Sciences (ASSMS), Lahore (Pakistan), March 6-9, 2013.
2. B. De Smit: https://www.math.leidenuniv.nl/ desmit/abc/. Accessed December 2020.
3. P. Mihăilescu: Around ABC, European Mathematical Society Newsletter, **N° 93**, 2014, 29-34.
4. A. Nitaj: Aspects expérimentaux de la conjecture *abc*. Séminaire de Théorie des Nombres de Paris (1993-1994), London Math. Soc. Lecture Note Ser., Vol **n°235**, 1996, 145-156, Cambridge Univ. Press.
5. A. Borisovich Shidlovskii: Transcendental Numbers. Translated from the Russian by Neal Koblitz. (De Gruyter Studies in Mathematics; 12). 488 pages (1989)

Abdelmajid Ben Hadj Salem
Résidence Bousten 8, Mosquée Raoudha,
Bloc B, 1181 Soukra Raoudha
Tunisia

abenhadjsalem@gmail.com