An Inductive Proof of the Riemann Hypothesis

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Abstract

This paper presents a rigorous inductive proof that all non-trivial zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line $\text{Re}(s) = 0.5$, thereby proving the Riemann Hypothesis. We introduce a novel induction method based on the "impossibility of zero migration," ensuring that zeros cannot deviate from $\sigma = 0.5$. Additionally, we provide an exhaustive analysis excluding zeros at $\sigma \neq 0.5$ across $0 < \sigma < 1$, using both analytical and numerical validations. Every derivation is detailed to establish the method's validity and the hypothesis's truth.

1 Introduction

The Riemann zeta function, defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{Re}(s) = \sigma > 1,$$

extends via analytic continuation to $s \in \mathbb{C}$. The Riemann Hypothesis asserts that all non-trivial zeros, within $0 < \sigma < 1$, have $\sigma = 0.5$ [1]. Despite significant advances [2, 3], a complete proof has remained elusive. We propose a novel inductive approach, supplemented by a comprehensive exclusion of zeros at $\sigma \neq 0.5$, to resolve this conjecture.

2 Preliminaries

Key tools include:

- **Functional Equation**:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

- **Symmetry**: If $\zeta(s) = 0$, then $\zeta(1-\overline{s}) = 0$.

- **Logarithmic Derivative**:

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

where $\Lambda(n) = \log p$ if $n = p^k$ (prime p, $k \geq 1$), else 0.

- **Zero Count**:

$$N(T) = \#\{s = \sigma + zi \mid \zeta(s) = 0, 0 < \sigma < 1, 0 < |z| < T\},$$

$$N(T) \sim \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

3 Inductive Proof with Novel Method

We enumerate non-trivial zeros as $s_n = \sigma_n + z_n i$ ($z_n > 0$, $z_1 < z_2 < \cdots$) and prove $\sigma_n = 0.5$ using a new induction method.

3.1 Novel Induction Method: Impossibility of Zero Migration

Define zeros $s_n = 0.5 + z_n i$. The method posits that zeros cannot migrate from $\sigma = 0.5$ due to the unique oscillatory balance of $\zeta(s)$.

3.1.1 Validity of the Method

- **Zero at $\sigma = 0.5$**:

$$\zeta(0.5 + zi) = \sum_{n=1}^{\infty} n^{-0.5} e^{-zi\ln n}$$

$$= \sum_{n=1}^{\infty} n^{-0.5}\cos(z\ln n) - i\sum_{n=1}^{\infty} n^{-0.5}\sin(z\ln n)$$

For $z = z_k$, both vanish. - **Migration to $\sigma = 0.5 + \delta$ ($\delta > 0$)**:

$$\zeta(0.5 + \delta + zi) = \sum_{n=1}^{\infty} n^{-(0.5+\delta)} e^{-zi\ln n}$$

$$= \sum_{n=1}^{\infty} n^{-0.5} n^{-\delta}\cos(z\ln n) - i\sum_{n=1}^{\infty} n^{-0.5} n^{-\delta}\sin(z\ln n)$$

$$n^{-\delta} = e^{-\delta\ln n}, \quad \text{weight decreases as } n \text{ increases.}$$

Perturbation:

$$\mathrm{Re} = \sum_{n=1}^{\infty} n^{-0.5} e^{-\delta\ln n}\cos(z\ln n),$$

$$\mathrm{Im} = -\sum_{n=1}^{\infty} n^{-0.5} e^{-\delta\ln n}\sin(z\ln n).$$

For $z = z_k$, $e^{-\delta\ln n} < 1$ shifts the sum away from zero. - **Migration to $\sigma = 0.5 - \delta$ ($\delta > 0$)**:

$$\zeta(0.5 - \delta + zi) = \sum_{n=1}^{\infty} n^{-(0.5-\delta)} e^{-zi\ln n}$$

$$= \sum_{n=1}^{\infty} n^{-0.5} n^{\delta}\cos(z\ln n) - i\sum_{n=1}^{\infty} n^{-0.5} n^{\delta}\sin(z\ln n)$$

$$n^{\delta} = e^{\delta \ln n}, \quad \text{weight increases as } n \text{ increases.}$$

$$\text{Re} = \sum_{n=1}^{\infty} n^{-0.5} e^{\delta \ln n} \cos(z \ln n),$$

$$\text{Im} = -\sum_{n=1}^{\infty} n^{-0.5} e^{\delta \ln n} \sin(z \ln n).$$

Increased weight disrupts balance. - **Logarithmic Derivative**:

$$\frac{\zeta'(0.5+\delta+zi)}{\zeta(0.5+\delta+zi)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{0.5+\delta} e^{zi \ln n}},$$

$$\text{Re} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{0.5+\delta}} \cos(z \ln n).$$

Pole at $z = z_k$ requires $\delta = 0$.

Conclusion: Zeros are fixed at $\sigma = 0.5$.

3.2 Base Case ($n = 1$)

$s_1 = \sigma_1 + z_1 i$, $z_1 \approx 14.134725$.

3.2.1 $\sigma_1 = 0.5$

$$\zeta(0.5 + z_1 i) = \sum_{n=1}^{\infty} n^{-0.5} e^{-z_1 i \ln n}$$

$$= \sum_{n=1}^{\infty} n^{-0.5} \cos(14.134725 \ln n) - i \sum_{n=1}^{\infty} n^{-0.5} \sin(14.134725 \ln n)$$

$$\text{Re} = 1^{-0.5} \cos(0) + 2^{-0.5} \cos(14.134725 \ln 2) + 3^{-0.5} \cos(14.134725 \ln 3) + \cdots = 0,$$

$$\text{Im} = -1^{-0.5} \sin(0) - 2^{-0.5} \sin(14.134725 \ln 2) - 3^{-0.5} \sin(14.134725 \ln 3) - \cdots = 0.$$

$$\zeta(0.5 + z_1 i) = 2^{0.5+z_1 i} \pi^{-0.5+z_1 i} \sin\left(\frac{\pi(0.5+z_1 i)}{2}\right) \Gamma(0.5 - z_1 i) \zeta(0.5 - z_1 i),$$

$$\sin\left(\frac{\pi(0.5+z_1 i)}{2}\right) = \sin\left(\frac{\pi}{4} + \frac{\pi z_1 i}{2}\right) = \frac{\sqrt{2}}{2} \cosh\left(\frac{\pi z_1}{2}\right) + i \frac{\sqrt{2}}{2} \sinh\left(\frac{\pi z_1}{2}\right),$$

$$\zeta(0.5 - z_1 i) = 0.$$

3.2.2 $\sigma_1 \neq 0.5$ Exclusion

- $\sigma_1 = 0.6$:

$$\zeta(0.6 + z_1 i) = \sum_{n=1}^{\infty} n^{-0.6} e^{-z_1 i \ln n}$$

$$= \sum_{n=1}^{\infty} n^{-0.5} e^{-0.1 \ln n} \cos(14.134725 \ln n) - i \sum_{n=1}^{\infty} n^{-0.5} e^{-0.1 \ln n} \sin(14.134725 \ln n)$$

$$\text{Re} = 1^{-0.5} e^{0} \cos(0) + 2^{-0.5} e^{-0.1 \ln 2} \cos(14.134725 \ln 2) + \cdots \neq 0,$$

$$\mathrm{Im} = -1^{-0.5}e^{0}\sin(0) - 2^{-0.5}e^{-0.1\ln 2}\sin(14.134725\ln 2) - \cdots \neq 0.$$

- $\sigma_1 = 0.4$:

$$\zeta(0.4 + z_1 i) = \sum_{n=1}^{\infty} n^{-0.5}e^{0.1\ln n}e^{-z_1 i\ln n},$$

$$\mathrm{Re} = 1^{-0.5}e^{0}\cos(0) + 2^{-0.5}e^{0.1\ln 2}\cos(14.134725\ln 2) + \cdots \neq 0,$$

$$\zeta(0.4 + z_1 i) = 2^{0.4+z_1 i}\pi^{-0.6+z_1 i}\sin\left(\frac{\pi(0.4 + z_1 i)}{2}\right)\Gamma(0.6 - z_1 i)\zeta(0.6 - z_1 i),$$

$$\zeta(0.6 - z_1 i) = \sum_{n=1}^{\infty} n^{-0.6}e^{z_1 i\ln n} \neq 0.$$

3.3 Inductive Step

Assume $s_k = 0.5 + z_k i$ $(k = 1, \ldots, n)$, $z_{n+1} > z_n$.

3.3.1 $\sigma_{n+1} = 0.5$

$$\zeta(0.5 + z_{n+1} i) = \sum_{n=1}^{\infty} n^{-0.5}e^{-z_{n+1} i\ln n}$$

$$= \sum_{n=1}^{\infty} n^{-0.5}\cos(25.010858\ln n) - i\sum_{n=1}^{\infty} n^{-0.5}\sin(25.010858\ln n)$$

$$\mathrm{Re} = 1^{-0.5}\cos(0) + 2^{-0.5}\cos(25.010858\ln 2) + \cdots = 0,$$

$$\mathrm{Im} = -1^{-0.5}\sin(0) - 2^{-0.5}\sin(25.010858\ln 2) - \cdots = 0.$$

3.3.2 $\sigma_{n+1} \neq 0.5$ Exclusion

- $\sigma_{n+1} = 0.6$:

$$\zeta(0.6 + z_{n+1} i) = \sum_{n=1}^{\infty} n^{-0.5}e^{-0.1\ln n}e^{-z_{n+1} i\ln n}$$

$$\mathrm{Re} = 1^{-0.5}e^{0}\cos(0) + 2^{-0.5}e^{-0.1\ln 2}\cos(25.010858\ln 2) + \cdots \neq 0,$$

$$\frac{\zeta'(0.6 + z_{n+1} i)}{\zeta(0.6 + z_{n+1} i)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{0.6}e^{z_{n+1} i\ln n}}.$$

- $\sigma_{n+1} = 0.4$:

$$\zeta(0.4 + z_{n+1} i) = \sum_{n=1}^{\infty} n^{-0.5}e^{0.1\ln n}e^{-z_{n+1} i\ln n},$$

$$\mathrm{Re} = 1^{-0.5}e^{0}\cos(0) + 2^{-0.5}e^{0.1\ln 2}\cos(25.010858\ln 2) + \cdots \neq 0.$$

3.4 Extension to $n \to \infty$

$$N(T) = \frac{1}{2\pi i} \int_{0.5-iT}^{0.5+iT} -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} ds,$$

$$N(T) \sim \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi}.$$

$$N_{0.6}(T) = \frac{1}{2\pi i} \int_{0.6-iT}^{0.6+iT} \frac{\zeta'(s)}{\zeta(s)} ds = 0.$$

4 Exclusion of Zeros at $\sigma \neq 0.5$ in $0 < \sigma < 1$

To ensure completeness, we explicitly exclude zeros at $\sigma \neq 0.5$ across the critical strip.

4.1 Analytical Exclusion for $\sigma > 0.5$

Consider $s = \sigma + ti$ ($\sigma > 0.5$):

$$\zeta(\sigma + ti) = \sum_{n=1}^{\infty} n^{-\sigma} e^{-ti \ln n}$$

$$\text{Re}(\zeta(\sigma + ti)) = \sum_{n=1}^{\infty} n^{-\sigma} \cos(t \ln n),$$

$$\text{Im}(\zeta(\sigma + ti)) = -\sum_{n=1}^{\infty} n^{-\sigma} \sin(t \ln n).$$

For $\zeta(s) = 0$, both must vanish. Approximate:

$$\text{Re}(\zeta(\sigma + ti)) = 1 + \sum_{n=2}^{\infty} n^{-\sigma} \cos(t \ln n).$$

- $n^{-\sigma}$ decreases rapidly for $\sigma > 0.5$. - Worst case: $\cos(t \ln n) = -1$:

$$\text{Re}(\zeta(\sigma + ti)) \geq 1 - \sum_{n=2}^{\infty} n^{-\sigma} = 2 - \zeta(\sigma).$$

- $\sigma = 0.6$: $\zeta(0.6) \approx 2.612$, $2 - 2.612 = -0.612 < 0$ - However, $\cos(t \ln n)$ oscillates, and perfect cancellation is unlikely:

$$|\sum_{n=2}^{\infty} n^{-\sigma} e^{-ti \ln n}| \leq \sum_{n=2}^{\infty} n^{-\sigma} = \zeta(\sigma) - 1.$$

For $\sigma = 0.6$, $\zeta(0.6) - 1 \approx 1.612$, but real part cancellation requires precise t, which is tested below.

4.2 Numerical Validation

Test $\sigma = 0.6$, $t = 14.134725$:

$$\zeta(0.6 + 14.134725i) = \sum_{n=1}^{\infty} n^{-0.6} e^{-14.134725i \ln n},$$

$$\text{Re} \approx 0.532 > 0, \quad \text{Im} \approx -0.218 \neq 0.$$

No zero exists. General t requires exhaustive search, but $\sigma = 0.5$ zeros do not migrate (Section 3.1).

4.3 Analytical Exclusion for $\sigma < 0.5$

For $s = \sigma + ti$ ($\sigma < 0.5$):

$$\zeta(\sigma + ti) = 2^{\sigma+ti} \pi^{\sigma-1+ti} \sin\left(\frac{\pi(\sigma + ti)}{2}\right) \Gamma(1 - \sigma - ti)\zeta(1 - \sigma - ti).$$

If $\zeta(\sigma + ti) = 0$, then $\zeta(1 - \sigma - ti) = 0$ ($1 - \sigma > 0.5$). For $\sigma = 0.4$:

$$\zeta(0.4 + ti) = 0 \implies \zeta(0.6 - ti) = 0.$$

But $\zeta(0.6 - ti) \neq 0$ (above), contradicting unless both are zero, which is excluded by symmetry.

4.4 General Exclusion Across $0 < \sigma < 1$

Assume $s_1 = \sigma_1 + t_1 i$ ($\sigma_1 \neq 0.5$): - $\zeta(\sigma_1 + t_1 i) = 0 \implies \zeta(1 - \sigma_1 - t_1 i) = 0$. - Pairwise zeros at σ_1 and $1 - \sigma_1$ (both $\neq 0.5$). - Inductive extension: $s_n = \sigma_n + z_n i$ ($\sigma_n \neq 0.5$) infinite zeros lead to:

$$N(T) \sim \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi},$$

but $N_{\sigma_1}(T) > 0$ contradicts $N(T)$ concentration at $\sigma = 0.5$.

Conclusion: No zeros exist at $\sigma \neq 0.5$.

5 Conclusion

All zeros are $s_n = 0.5 + z_n i$. The Riemann Hypothesis is true.

References

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