Wallis's constant

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ABSTRACT
In this note, we give some formulas related to Wallis's constant

I. Introduction

Wallis's constant is the real solution $\left(x^3 - 2x - 5\right)_1 = 2.09455\ldots$ to the cubic equation $x^3 - 2x - 5 = 0$. It was solved by Wallis (1616-1703) to illustrate Newton's method for numerical equation solving.
Notation: $W = 2.0945514815423\ldots$

In this note, we give some formulas related to Wallis's constant.

II. Formulas

Entry 1.

$$W = \sqrt[3]{\frac{45 - \sqrt{1929}}{18}} + \sqrt[3]{\frac{45 + \sqrt{1929}}{18}} \tag{1}$$

$$W = \sqrt[3]{\frac{5}{2} - \sqrt{\frac{643}{108}}} + \sqrt[3]{\frac{5}{2} + \sqrt{\frac{643}{108}}} \tag{2}$$

$$\frac{1}{W} = \frac{1}{15}\left(-2 + \sqrt[3]{\frac{659 - 15\sqrt{1929}}{2}} + \sqrt[3]{\frac{659 + 15\sqrt{1929}}{2}}\right) \tag{3}$$

Entry 2.

$$W = \sqrt[3]{5 + 2\sqrt[3]{5 + 2\sqrt[3]{5 + \ldots}}} \tag{4}$$

$$W = \sqrt{2 + \frac{5}{\sqrt{2 + \frac{5}{\sqrt{2+\ldots}}}}} \tag{5}$$

$$\frac{1}{W^2} = \left(\frac{1}{5} - \frac{2}{5}\left(\frac{1}{5} - \frac{2}{5}\left(\frac{1}{5} - \ldots\right)^{2/3}\right)^{2/3}\right)^{2/3} \tag{6}$$

Entry 3.

$$c(n) = \sum_{k=\lfloor n/3 \rfloor}^{\lfloor n/2 \rfloor} 2^{3k-n} \cdot 5^{n-2k} \binom{k}{n-2k} \Longrightarrow \frac{c(n+1)}{c(n)} \longrightarrow W \tag{7}$$

$$c(n) = \{1,\ 0,\ 2,\ 5,\ 4,\ 20,\ 33,\ 60,\ 166,\ 285,\ 632,\ \ldots\} \tag{8}$$

Entry 4.

$$\frac{5}{W^3} = \sum_{n=0}^{\infty} \frac{\Gamma\left(1 + \frac{2n}{3}\right)}{\Gamma\left(2 - \frac{n}{3}\right) n!} \left(-\frac{2}{\sqrt[3]{25}}\right)^n \tag{9}$$

Entry 5.

$$\frac{\pi}{8} = \int_0^{(W-2)/2} \frac{5 + 12x + 6x^2}{1 - 20x + 176x^2 + 472x^3 + 448x^4 + 192x^5 + 32x^6}\,dx \tag{10}$$

Entry 6.

$$\pi = 4\tan^{-1}\left(\frac{1}{99}\right) + 8\sum_{n=0}^{\infty} W^{-2n-3}\,2^n \sum_{k=0}^{\lfloor n/3 \rfloor} \frac{(-1)^k\left(2^{-k} + 2^{-3k}\right)}{2k+1}\binom{n-k}{n-3k} \tag{11}$$

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n\,5^{2n+1}}{2n+1}\,W^{-6n-3} + 4\sum_{n=0}^{\infty} W^{-2n-2} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{2k+1}\binom{n}{n-2k} \tag{12}$$

Entry 7.

$$\pi = 6\sum_{n=0}^{\infty} W^{-n-2} \sum_{k=\lfloor n/4 \rfloor}^{\lfloor n/3 \rfloor} \frac{2^{6k-2n}\cdot 5^{4k-n}}{2n-6k+1}\binom{2n-6k}{n-3k}\binom{n-2k}{4k-n} \tag{13}$$

Entry 8.

$$W = 2 + \sum_{n=1}^{\infty} a(n)\,10^{-n} \tag{14}$$

$$a(n) = -6\sum_{k=0}^{n-1} a(k)\,a(n-k-1) - \sum_{k=0}^{n-1}\sum_{m=0}^{n-k-1} a(k)\,a(m)\,a(n-k-m-1)\ ,\ a(0) = 0,\ a(1) = 1 \tag{15}$$

$$a(n) = \{1,\ 0,\ -6,\ -1,\ 72,\ 30,\ -1077,\ -756,\ 17\,976,\ 18\,132,\ -320\,112,\ \ldots\} \tag{16}$$

Entry 9.

$$\pi = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k} (-1)^k\,2^{2n-2m+2}\cdot 5^{2m}\,W^{-2n-2k-2m-5}\binom{n}{k}\binom{k}{m}\left(\frac{15W^2}{2n+2k+2m+3} - \frac{10}{2n+2k+2m+5}\right) \tag{17}$$

$$\pi = \sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{5}\right)^{2n+2}\left(\frac{25}{25+W^6}\right)^{n+1} \sum_{k=0}^{n}\binom{n}{k}(-1)^k \left(\frac{15\,W^{2n+2k+3}}{2n+2k+3}\,{}_2F_1\left(\{n+1,\ 1\},\ \left\{\frac{2n+2k+9}{6}\right\},\ \frac{W^6}{25+W^6}\right) - \frac{10\,W^{2n+2k+1}}{2n+2k+1}\,{}_2F_1\left(\{n+1,\ 1\},\ \left\{\frac{2n+2k+7}{6}\right\},\ \frac{W^6}{25+W^6}\right)\right) \tag{18}$$

Entry 10.

$$\pi = \sum_{n=0}^{\infty} W^{-3n-2} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(25k-3)\,2^{2k+2}\cdot 5^{n-2k}}{\binom{3k}{k}}\binom{n+k}{n-2k} \tag{19}$$

Remarks:

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\right) \tag{20}$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{2k+1}\binom{n}{n-2k} = \left(\frac{1+i}{2}\right)\left(\frac{(1-i)^n - i\,(1+i)^n}{n+1}\right),\ i = \sqrt{-1} \tag{21}$$

$${}_2F_1(\{a,\ b\},\ \{c\},\ z) \text{ is the Gauss hypergeometric function} \tag{22}$$

$$\Gamma(x) \text{ is the Gamma function} \tag{23}$$

$$\lfloor x \rfloor \text{ is the Floor function} \tag{24}$$

III. References

1. Sloane, N.J.A. Sequence A007493

2. Wells, D. The Penguin Dictionary of Curious and Interesting Numbers. Middlesex, England: Penguin Books, p.45, 1986.

3. Weisstein, E.W. “Wallis’s constant”. From MathWorld