# A note on superluminalization of SRT *and* some central concerns of contemporary physics

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**Abstract:** Over the past decades and more recently, there has been a significant effort and renewed interest in extending special relativity theory beyond the speed of light. In this comprehensive note, the paper "Relativity of superluminal observers in 1+3 spacetime" by Dragan *et al*. [1] is examined. The authors attempted to extend the aforementioned theory to superluminal inertial reference frames by generalizing Parker's two-dimensional transformation with the addition of two pairs of spatial dimensions. This approach is not novel and is already well-documented in the literature. It is demonstrated that unlike Lorentz transformations, the authors' transformations do not form an orthogonal-orthochronous group due to their negative determinant. Consequently, principles such as relativity, causality, spatial isotropy, and temporal ordering cannot be preserved. The authors'pseudo-transformations are revealed to be reflections in a plane through the origin rather than true transformations. Also, a theoretical maximum limit of the Lorentz factor is introduced, which leads to an extension of Lorentz transformations to luminal inertial reference frames and raises an important conceptual question about the status and role of the symbolic quantity '$c$', commonly called the speed of light in vacuum, as the neutrino and particularly the photon have non-zero mass. Therefore, it appears that as long as the non-zero mass of the photon is not taken seriously into full consideration, our current knowledge of physics, astronomy, astrophysics, and cosmology remains not only incomplete but above all vague and doubtful. Unfortunately, it seems that many present-day researchers are unaware that distinguished physicists like Einstein [59-62], de Broglie [63-66], Proca [67-71], Schrödinger [72-74], and many others [75-92] had already attributed non-zero mass to the photon because they realized that the photon itself behaves like a massive particle, carrying not only energy but also momentum and can exert pressure on a target. And at no stage may we really be able to conclude experimentally the exact masslessness of the photon because the Heisenberg uncertainty principle gives the lowest mass $m$ that can be measured in the Universe's age as $m\sim\hbar t^{-1}c^{-2}\sim 1.5\times 10^{-33}\mathrm{eV}/c^2$. Once again, therefore, non-zero photon mass gives rise immediately to a conceptual question: What is the primary purpose of the symbolic quantity '$c$' when it appears in some important equations describing the laws of physics? Moreover, this inclusive note highlights Hassani superluminal spatio-temporal transformations, which possess the algebraic structure of a linear group and serve as a generalization of Lorentz transformations for superluminal inertial reference frames. These transformations are anticipated to be fundamental in superluminal relativistic mechanics [50].

**Keywords:** special relativity theory; luxonic total energy; Lorentz transformations; superluminal observers

## 1. Introduction

The early investigation of superluminal (relative) motion, *i.e.*, motion faster than the average speed of light in vacuum, goes back to Heaviside [2] and Sommerfeld [3]. However, after the publication of the work by Bilaniuk *et al*. [4] and Feinberg [5], who also coined the term 'tachyon' (hypothetical superluminal particle), an avalanche of articles on tachyons and superluminal spatio-temporal transformations followed [6-18]. The theoretical, observational, and experimental evidence of the (apparent) superluminal motions at micro and macroscopic scales [19-40] allows one to suggest that in Nature there are actually three kinematical levels (KLs): subluminal-KL, luminal-KL, and superluminal-KL in which physical phenomena could manifest at subluminal, luminal, and superluminal velocities, respectively. Also, each KL should be characterized by its group of spatio-

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temporal transformations. For example, subluminal-KL is characterized by the Galilean group for subrelativistic velocities ($v << c_0$) and by the Lorentz group for relativistic velocities ($v < c_0$); luminal-KL and superluminal-KL would have, respectively, luminal and superluminal groups for luminal velocities ($v = c_0$) and superluminal velocities ($v > c_0$). All these conceptual and empirical efforts and considerations have recently renewed a huge interest in the extension of the Lorentz transformations (LTs) for superluminal inertial reference frames (IRFs) in order to 'superluminalize' special relativity theory (SRT) [1,41-50]. Unfortunately, all of those serious attempts, including the recent ones, failed to 'superluminalize' SRT, except for the work in Ref. [50].

The failure to extend SRT beyond the average vacuum speed of light was and still is mainly due to a misunderstanding of SRT formalism, the mathematical properties, and physical consequences of LTs, and the Minkowski space-time as a seat (arena) of relativistic physical phenomena. Moreover, the said failure is attributable to a misapprehension of the correlation between the causality principle and the relativity principle.

In this note, we show that the authors' approach is not original and is already well known in the literature. We draw attention to the fact that contrary to the main property of Lorentz transformations, the authors' ones do not form an orthogonal-orthochronous group since their determinant is negative. Additionally, we show that the authors' approach violates the relativity principle, the causality principle, and the condition of spatial isotropy and temporal ordering because the authors' pseudo-transformations are not really transformations but rather reflections in a plane through the origin.

## 2. Lorentz Transformations

In the following discussion, we focus exclusively on the proper Lorentz group, SO(3,1), in the context of SRT, or for those who prefer the other metric signature, SO(1,3), also known as the Lorentz group. The purpose of this section is to demonstrate that the well-known mathematical properties and physical consequences of LTs do not apply to Parker's transformation or its generalization by the authors [1] in a superluminal context. To do this, consider two IRFs $S$ and $S'$ in standard configuration, where S' is moving uniformly in a straight line at a subluminal velocity $v$ relative to $S$ along the $x$-axis. The LTs relate $S$ to $S'$ and vice versa are:

$$S \to S': \begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \frac{vx}{c^2}\right) \end{cases}, \tag{1}$$

$$S' \to S: \begin{cases} x = \gamma(x' + vt') \\ y = y' \\ z = z' \\ t = \gamma\left(t' + \frac{vx'}{c^2}\right) \end{cases}, \tag{2}$$

with
$$\gamma \equiv \gamma(v) = \frac{1}{\sqrt{1 - v^2/c^2}}, \quad -c < v < c.$$

It is clear that the determinant or the Jacobian of transformations (1) and (2) is positive, more precisely equal to +1. This property implies the orthogonality of LTs in addition to their orthochronous nature. As a direct result, we have the invariance of the Minkowski quadratic form

$$x'^2 + y'^2 + z'^2 - c^2 t'^2 = x^2 + y^2 + z^2 - c^2 t^2, \tag{3}$$

or, equivalently, in terms of differential

$$dx'^2 + dy'^2 + dz'^2 - c^2 dt'^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2. \tag{4}$$

The identity (3) or (4) implies the invariance (in all IRFs) of the universal conversion (or proportionality) factor[2], $c$, which has the physical dimension of a constant speed whose conventionally recommended and adopted numerical value is $299\,792\,458\ \mathrm{m/s}$ [51,52,53], comparable to the experimentally measured numerical value of $299\,792\,457.6\ \mathrm{m/s}$ [54,55,56] for the average[3] speed of light in vacuum, $c_0$. Theoretically and practically, $c$ and $c_0$ are virtually indistinguishable; this particularity is due to the fact that $c \approx c_0$ in all IRFs. The physical interpretation of $c$ mentioned above is exclusively based on the fact that we have taken into consideration the mass of neutrinos [57,58] and the non-zero mass of photons [59-92]. Additionally, the combination of $c$ and $c_0$ allows us to determine the theoretical maximum numerical value of the Lorentz factor as follows

$$\gamma_{\max} = \lim_{v \to c_0} \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{1}{\sqrt{1-\left(\frac{c_0}{c}\right)^2}} = 19358.217189\ . \tag{5}$$

If one considers the theoretical, experimental and observational evidence of non-zero photon mass and adheres to the indicated physical interpretation of '$c$' and the actual average speed of light $c_0$, then the naive idea propagated in popular science literature and many textbooks on SRT, which states that "*according to SRT, from the perspective of the photon, time does not pass. The packet of electromagnetic radiation would cover vast distances but no time would have elapsed for it. Why? Simply because in the context of SRT, the velocity four-vector* $\mathbf{U} = d\mathbf{X}/d\tau$, *where* $\mathbf{X}$ *is the position four-vector and* $\tau$ *is the proper time of a material object does not exist for world-lines of objects such as photons traveling at the speed of light. The expression for proper time is* $\Delta\tau = \Delta t\sqrt{1 - v^2/c^2}$ *. Thus, for photons traveling at the speed of light, we get* $\Delta\tau = 0$. *That is why the velocity four-vector for photons is, in fact, undetermined*." becomes meaningless since a massive photon travels at $c_0$. Consequently, the photon proper time is $\Delta\tau = \Delta t\sqrt{1 - c_0^2/c^2}$.

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[2]Actually, the theoretical, experimental and observational evidence of non-zero photon mass has naturally raised the following conceptual question: What is the status and role of the symbolic quantity '$c$' when it appears in some important equations describing the laws of physics?

[3] When using the phrase 'the speed of light in vacuum', it is important to make a clear distinction between the '*anisotropic* one-way speed of light' and the '*isotropic* two-way speed of light'. Practically, the one-way speed of light, from a source to an observer or a detector, cannot be measured independently of a convention on how to synchronize the clocks at the source and the detector. What can be experimentally measured is the round-trip speed or two-way speed of light from the source to a mirror (or another method of reflection) and back to the detector. Thus, to keep the two-way speed of light at '$c_0$', any increase in speed in one direction must be countered by a decrease in the opposite direction. The result is that the vacuum speed of light, $c_0$, is the arithmetic *average* speed over the around-trip journey, and we cannot be certain that the *speed* is the same in both directions. Therefore, when using the phrase 'the speed of light in vacuum', we must refer to it as 'the *average* speed of light in vacuum.' The index 0 in $c_0$ indicates the vacuum in a classical sense.

## 3. Luminal Lorentz transformations

SRT is a well-established and experimentally verified physical theory with its own domain of applicability and limits of validity. It is crucially based on the concept of IRF and LTs. In the context of SRT, '*c*' plays the role of the limiting velocity, particularly for relativistic velocities. However, many authors have criticized the emphasis placed on the invariance of the vacuum speed of light as a central postulate in the standard derivation of the LTs. These authors have criticized the overemphasized role of the speed of light in the basic foundations of SRT and have proposed a new approach to these foundations that dispenses with the postulate of the invariance of light speed [93-96]. Consequently, they derived the LTs by simply invoking the relativity principle alone, without resorting to the aforementioned postulate of invariance. In fact, it is not necessary to suggest the concept of the existence of a universal limiting velocity or to identify it with the speed of light; this consideration has been known for a long time [97-103].

Unfortunately, these important works have not received sufficient attention in textbooks and articles on SRT, despite being more economical in derivations and allowing for a more acceptable conceptual approach to be easily adopted and practically used than is possible in any of the 'two postulate derivations.' By founding SRT, partially or totally, on a property of the speed of light, one could logically understand that this theory is exclusively linked to a very restricted class of natural phenomena directly related to electromagnetic radiation. However, the pedagogical and epistemological lesson to be drawn from the history of science is that, 120 years after its formulation, SRT still seems to govern a wide variety of natural phenomena within its framework and formalism, in addition to describing the structure of the space-time arena in which those natural phenomena take place.

Now let us begin the derivation of luminal LTs by observing that if $v = c$, the Lorentz factor becomes infinite, and if $v > c$, it leads to an imaginary value of the Lorentz factor. This demonstrates that the relative velocity of two IRFs related by LT must be less than '*c*'. Finite real spatio-temporal coordinates in one subluminal IRF must correspond to finite real spatio-temporal coordinates in any other subluminal IRF. An IRF can be associated with any non-accelerated particle or material object moving with subluminal velocity, translating into the requirement that the magnitude of particles' velocity and all physical signals should be limited by '*c*'. This consideration justifies the prohibition of the existence of luminal IRFs (*i.e.*, when the IRFs $S$ and $S'$ are in relative motion at a luminal velocity of magnitude $c$ with respect to each other) in the SRT context. However, the determination of the upper limit for the Lorentz factor or equivalently the theoretical maximum numerical value for the Lorentz factor renders the mentioned prohibition unnecessary since in general we have $c \approx c_0$ in all IRFs. Therefore, for the case when $S'$ is moving uniformly in a straight line at a luminal velocity $v = c_0$ relative to $S$ along the $x$-axis, the luminal LTs relating $S$ to $S'$ and vice versa are respectively:

$$S \to S': \begin{cases} x' = \gamma_{\max}(x - \beta_{\max} ct) \\ y' = y \\ z' = z \\ t' = \gamma_{\max}\left(t - \beta_{\max}\frac{x}{c}\right) \end{cases}, \tag{6}$$

$$S' \to S: \begin{cases} x = \gamma_{\max}(x' + \beta_{\max} ct') \\ y = y' \\ z = z' \\ t = \gamma_{\max}\left(t' + \beta_{\max}\frac{x'}{c}\right) \end{cases}, \tag{7}$$

with $$\gamma_{\max} = (1 - \beta_{\max}^2)^{-1/2}, \quad \beta_{\max} = c_0/c\ .$$

If we take into account the fact that the current numerical value of $c$ is $299\,792\,458\ \mathrm{m/s}$ which is selected by recommendation and conventionally fixed by definition for the purpose of metrology [51,52,53], while the numerical value of $c_0$ is actually experimentally determined as $299\,792\,457.6\ \mathrm{m/s}$ from direct frequency and wavelength measurements of the stabilized laser [54,55,56], we find that the proposed $\gamma_{\max}$, luminal LTs (6) and (7), and the concept of luminal IRFs clarify the frontiers between relativistic physics and superluminal physics more visibly. This renders claims such as: "*Probably a proton detected at a speed close to* $0.9999999999\,9999999999\,99951c$; *the Lorentz factor is about* $\gamma \approx 3 \times 10^{11}$; *perhaps the Lorentz symmetry is violated and/or the apparent existence of a privileged local inertial frame.*" absolutely meaningless.

In this note, the main motivation behind determining the theoretical maximum numerical value for the Lorentz factor, $\gamma_{\max} = 19358.217189$, is the desire to circumvent a singularity that is essentially an absurdity. For instance, SRT formalism, modern physics textbooks, and research articles assert that, among other things, the velocity of a material object can never reach but only approach the speed of light, which is usually denoted as $c$, without specifying a precise upper limit on how close it could get to that specific speed. Even the terms 'relativistic velocity' and 'ultra-relativistic velocity' are ambiguous and confusing. The legitimate question then arises: How close to the speed of light can an object move? Let us evaluate the importance of this question as follows: An electron is a fundamental particle with a mass of $m_e = 9.10938356 \times 10^{-31}\mathrm{kg}$. What happens to the electron's relativistic kinetic energy $K_e = \left[(1 - v^2/c^2)^{-1/2} - 1\right] m_e c^2$ when $v$ approaches, but never reaches, the speed of light? Now, suppose the electron is hypothetically accelerated to the velocity

$$v = 0.9999999999\,9999999999\,9999999999\,9999999999\,9999999999\,9999999999\,9999999999\,9999999999\,9999999999\,9999999999\,9999999999\,9999999999\,9999999999\,895\,c.$$

That is 130 nines behind the decimal point followed by the number 895. Putting this numerical value in the Lorentz factor formula $\gamma = (1 - v^2/c^2)^{-1/2}$, we get $\gamma \approx 2.2 \times 10^{60}$, which by itself is enough to tell us that there is some unacceptable exaggeration. As a direct consequence, the Lorentz factor really needs to be fixed at a reasonable maximum attainable value. Now after substitution in the above relativistic kinetic energy formula, we get $K_e = 1.787 \times 10^{47}$ J. The Sun's rest energy $m_{\odot}c^2 = 1.787 \times 10^{47}\mathrm{J}$, with $m_{\odot} = 1.989 \times 10^{30}\ \mathrm{kg}$ and $c = 299\,792\,458\ \mathrm{m/s}$. Finally, a simple comparison yields $K_e = m_{\odot}c^2$. It is really quite absurd; an elementary particle of an infinitesimally small mass whose relativistic kinetic energy becomes suddenly equal to the rest energy of the Sun! This absurdity is a perfect illustration of why we need an adequate answer to the formerly asked question, namely, –How close to the light speed can an object move?– To come to the desired adequate answer it is only necessary to grasp the already proposed physical interpretation of the symbolic quantity '$c$' sharply enough as it is needed for the purposes of physical practicability. In the framework of SRT, we assume that the experimentally determined value of $c_0 = 299792457.6\mathrm{m/s}$ is the maximum possible ultra-relativistic velocity that a moving material object can reach. If one adheres strictly to the stated physical interpretation of '$c$', and the status and role of $c_0$, then it turns out that the theoretical maximum numerical value of the Lorentz factor would be $\gamma_{\max} = (1 - (c_0/c)^2)^{-1/2}$.

The most important result that can be deduced from the above considerations is the following: From a practical standpoint, the theoretical maximum numerical value of the Lorentz factor should serve as a *criterion* to delineate the frontiers between relativistic physics and superluminal physics as follows. Let $E$ and $E_0$ be the total energy and rest energy of a hypothetical moving material point such that:

1) If $\frac{E}{E_0} < \gamma_{\max}$ , the material point is moving at subluminal velocity, $v < c_0$, and belongs to subluminal-KL.
2) If $\frac{E}{E_0} = \gamma_{\max}$ , the material point is moving at luminal velocity, $v = c_0$, and belongs to luminal-KL.
3) If $\frac{E}{E_0} > \gamma_{\max}$ , the material point is moving at superluminal velocity, $v > c_0$, and belongs to superluminal-KL.

Logically, the above answer leads to another question: What is the average magnitude of velocity of the material point in each KL? If we take into account the fact that in Nature nothing is infinite, all physical parameters of phenomena and material objects are defined and characterized by finite values. None can prevent any freely moving material body from reaching or exceeding the speed of light in vacuum. In terms of the average magnitude, the velocity in units of $c_0 \sim c$ of a moving material point of total energy $E$ and rest energy $E_0 = mc^2$ is given by the following relations:

$$\begin{cases} \beta = \sqrt{1 - \left(\frac{E_0}{E}\right)^2} \ , \ \frac{E}{E_0} \leq \gamma_{\max} \ , \ \beta = \frac{v}{c} \\ \varepsilon = \sqrt{1 - \left(\frac{E_0}{E}\right)^2} \ , \ \frac{E}{E_0} > \gamma_{\max} \ , \ \varepsilon = \frac{v}{\vartheta(v)} \ . \end{cases} \tag{8}$$

The first relation in (8) for the case $E/E_0 < \gamma_{\max}$ is well-known in SRT, whereas the second one for the case $E/E_0 > \gamma_{\max}$ is deduced from the formalism of [50]. Furthermore, in the framework of the present work, the theoretical existence of the maximum Lorentz factor (5) implies, among other things, the hypothetical existence of massive *luxons*, *i.e.*, particles of non-zero mass capable of moving at exactly the speed of light. As a pedagogical illustration, we have selected some important particles and evaluated the numerical value of their luxonic total energy $\mathscr{E} = \gamma_{\max} E_0$.These values are listed in Table1 below.

| **Particle** | **rest energy** $E_0$ (MeV) | **luxonic total energy** $\mathscr{E}$ (MeV) |
|---|---|---|
| electron | 0.511 | $9.892049 \times 10^3$ |
| proton | 938.28 | $18.163429 \times 10^6$ |
| neutron | 939.57 | $18.188400 \times 10^6$ |
| muon | 105.70 | $2.046163 \times 10^6$ |
| pion $\pi^{\pm}$ | 139.60 | $2.702407 \times 10^6$ |
| pion $\pi^0$ | 135.00 | $2.613359 \times 10^6$ |

**Table 1:** A set of six particles is selected and the value of the luxonic total energy $\mathscr{E} = \gamma_{\max} E_0$ of each particle is computed and listed.

In the following, we will briefly demonstrate the theoretical and practical importance and usefulness of the concept of *luxonic total energy*. This concept serves as an energetic milestone that signifies the limits of validity and the end of SRT applicability, marking the beginning of the realm of superluminal physics where all physical phenomena manifest and evolve in superluminal space-time. The result of superluminal physics may appear completely unusual and even be considered unphysical by some individuals. But historically, similar criticisms were made about SRT in its early days because its opponents were too skeptical and resistant to change, as they were heavily invested in the classical physics dominated by Newtonian mechanics. However, skepticism eventually dissipated when opponents of SRT realized that relativistic mechanics can be reduced to Newtonian mechanics for subrelativistic velocities, specifically when $v \ll c$ or $v/c \sim 0$.

Table 1 reveals that the *luxonic total energy* of the proton and muon is of the order of TeV, which by itself is enough to suggest that the observed high, very high, and ultra-high energy cosmic rays could be a natural consequence of their acceleration at extremely high superluminal velocities by some galactic and extragalactic structures. Active galactic nuclei and their superluminal jets can be seen as potential sources and accelerators of the observed ultra-high energy cosmic rays (UHECRs). For example, blazar PKS 1502+106 has exhibited a superluminal jet component motion at a velocity of $22c$ [109], and also blazar 3C279 has two inner jet components moving at apparent superluminal velocities of $15c$ and $20c$, respectively [110].

Some theoretical models expect that since cosmic rays are generally charged particles, they are deflected along their path to Earth by intervening galactic and extragalactic magnetic fields, and interactions with cosmic microwave background radiation (CMBR) suppress the flux of UHECRs. However, if we consider the concept of *luxonic total energy* as a practical energetic criterion instead of the Greissen-Zatsepin-Kuzmin (GZK) limit or cutoff ($5 \times 10^{19}$eV), which is highly questionable not only because it was formulated on the assumption that neutrinos and photons are massless particles but also because this limit is violated by the detection of several cosmic rays with energies higher than $5 \times 10^{19}$eV [111-115], we find that UHECRs with energies above the *luxonic total energy* of the proton could be deflected less strongly by magnetic fields and suppressed very feebly by CMBR due to their higher superluminal velocities. Therefore, their arrival directions are expected to be closely correlated with their original sources.

Let us return to the LTs, more precisely with respect to the quadratic form (3) or (4), whose invariance in all IRFs indicates the homogeneity and isotropy of space, and the uniformity of time. This fact exactly reflects the main requirement of the relativity principle, which is the equivalence of all IRFs. Furthermore, when $v = 0$, the LT (1) or (2) becomes an identity transformation whose determinant is also equal to +1:

$$S \to S' : \begin{cases} x' = x \\ y' = y \\ z' = z \\ t' = t \end{cases} . \tag{9}$$

We have from (4):

$$(u'^2 - c^2)\, dt'^2 = (u^2 - c^2)\, dt^2, \tag{10}$$

this *kinematically* implies that if $u' \leq c$ in $S'$, then $u \leq c$ in $S$ and if $u' \geq c$ in $S'$, then $u \geq c$ in $S$. Thus, the important fact that should be noted is that subluminal velocities always transform to subluminal velocities and superluminal velocities to superluminal velocities. For if $u$ and $u'$ are the velocities of a point (a signal or a particle) in $S$ and $S'$, respectively. This also implies that if $dt' \geq 0$ in $S'$, then $dt \geq 0$ in $S$. It is clear from the above properties that the relativity principle is again preserved *via* the velocity transformation, and the causality principle is also preserved through the temporal ordering. Furthermore, contrary to traditional naive belief, SRT can easily accommodate – indeed, does not exclude– superluminal signaling at the kinematical level. Actually, the superluminal signals do not violate the causality principle *but* they can shorten the luminal vacuum time span between cause and effect. In other words, the superluminal signals save time.

## 4. Generalized Parker's transformations

In 1961, Parker derived the so-called Parker's (two-dimensional) superluminal spatio-temporal transformation by considering a world with only one spatial dimension and one temporal dimension to study the kinematics of tachyons. He named the context and the formalism 'theory' [7]. Here, we attempt to rewrite Parker's transformation in a more conventional form to reduce unnecessary complexity and enhance comprehension.

$$S \to S': \begin{cases} x' = \mu(x - vt) \\ t' = \mu\left(t - \frac{vx}{c^2}\right) \end{cases}, \tag{11}$$

where $\mu = \left(\frac{v^2}{c^2} - 1\right)^{-1/2}$, and $v$ $(c < |v| < \infty)$ is the superluminal relative velocity of the primed (IRF) $S'$ with respect to unprimed (IRF) $S$. The two IRFs are supposed to be in standard configuration. Apparently, after reflection, Parker realized that his 'transformation' cannot be extended to three spatial dimensions, and he wrote in Ref. [7]: "*It does not seem to be possible to generalize our theory to three dimensions, so that it may have little if any relevance to constructing a three-dimensional theory of tachyons.*"

Actually, despite the fact that his 'transformation' is not a transformation at all but instead a spatio-temporal reflection, Parker's conclusion was and still is quite correct about the impossibility of generalizing transformation (11). However, in [1], Dragan and his co-authors attempted to challenge Parker's conclusion but unfortunately they failed, as we shall see.

To circumvent this impossibility, Dragan *et al.* [1] added $y' = y$ , $z' = z$ to the (1+1)-dimensional case to obtain the so-called generalized Parker's transformations:

$$S \to S': \begin{cases} x' = \mu(x - vt) \\ y' = y \\ z' = z \\ t' = \mu\left(t - \frac{vx}{c^2}\right) \end{cases}, \tag{12}$$

$$S' \to S: \begin{cases} x = \mu(x' + vt') \\ y = y' \\ z = z' \\ t = \mu\left(t' + \frac{vx'}{c^2}\right) \end{cases}. \tag{13}$$

In doing so, there is no notable originality since anybody, including Parker himself can easily do this banal task. Now, let us show mathematically and physically that transformations (12) and (13) have nothing to do with any useful transformations. First, it can be seen that $\mu = \left(\frac{v^2}{c^2} - 1\right)^{-1/2}$ is singular at $v = \pm c$, and $\mu$ becomes imaginary at $v = 0$. Consequently, (12) or (13) cannot reduce to an identity transformation. For the limit at $v \to +\infty$, (12) and (13) become respectively:

$$S \to S': \begin{cases} x' = -ct \\ y' = y \\ z' = z \\ ct' = -x \end{cases}, \tag{14}$$

$$S' \to S: \begin{cases} x = ct' \\ y = y' \\ z = z' \\ ct = x' \end{cases}. \tag{15}$$

It is worth noting that (14) is not an identity transformation, unlike (15). This indicates that, unlike Lorentz transformations, the so-called generalized Parker's transformations (14) and (15) are not internally coherent. There is no coherence between (14) and (15). Thus, when the superluminal velocity becomes infinitely large, the new coordinates $x'$ and $t'$ can no longer be identified as spatial and temporal ones because they become timelike and spacelike, respectively. Therefore, the new coordinates $x'$ and $t'$, while adequately viable as mathematical labels in space-time, lack an intrinsic chronological interpretation, which is exactly what one desires from the concept of coordinates in SRT. This is quite understandable given that (14) and (15) are not structurally coherent. Furthermore, in (3+1)-dimensions, the so-called generalized Parker's transformations (12) and (13) transform the Minkowski quadratic form

$$Q(x, y, z, ct) = x^2 + y^2 + z^2 - c^2t^2 \quad \text{to} \quad Q(x', y', z', ct') = -x'^2 + y'^2 + z'^2 + c^2t'^2, \tag{16}$$

or equivalent, the light-cone

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \quad \text{to} \quad -x'^2 + y'^2 + z'^2 + c^2t'^2 = 0, \tag{17}$$

and the wave operator (d'Alembertian)

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{c^2\partial t^2} \quad \text{to} \quad -\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} + \frac{\partial^2}{c^2\partial t'^2}. \tag{18}$$

Therefore, the Minkowski quadratic form, the light-cone and the wave operator are not invariant under (12) or (13) for superluminal relative motion. The generalized Parker transformations change the signature of the metric $(+++-)$ to $(-+++)$ showing more conclusively that the rotational symmetry, homogeneity, and isotropy of space, and the uniformity of time are completely lost for superluminal relative motion. As a direct consequence, the relativity principle is violated since its main requirement, the equivalence of all IRFs, does not hold anymore. Additionally, the causality principle is violated because the temporal ordering is reversed. The so-called vacuum speed of light, $c$, is not the same in both superluminal IRFs. If $\sqrt{x^2 + y^2 + z^2}/t$ has the value $c$ in $S$, while in $S'$, $\sqrt{x'^2 - y'^2 - z'^2}/t' = c'$.

Now, let us demonstrate that unlike LTs, the so-called generalized Parker's transformations cannot describe superluminal relative motion. To do this, suppose $u$ and $u'$ are two superluminal velocities of a hypothetical signal or particle in $S$ and $S'$, respectively. By differentiating and squaring transformation (12), we get:

$$dx'^2 + dy'^2 + dz'^2 - c^2dt'^2 = -dx^2 + dy^2 + dz^2 + c^2dt. \tag{19}$$

The differentials refer to the signal/particle worldline. We can rewrite identity (19) in the following form:

$$\left(\frac{u'^2}{c^2} - 1\right) dt'^2 = -\left(\frac{u^2}{c^2} - 1\right) dt^2. \tag{20}$$

From equation (20), we can see that superluminal velocities always transform to superluminal velocities. That is, if $u' > c$ in $S'$, then $u > c$ in $S$. However, time has lost its usual role as a chronological parameter, *i.e*., a positive real-valued scalar. This is evident from equation (20):

$$\sqrt{\frac{u'^2}{c^2} - 1}\ dt' = i\sqrt{\frac{u^2}{c^2} - 1}\, dt, \tag{21}$$

which is, of course, physically meaningless. Another temporal anomaly can be deduced as follows. Supposing the hypothetical superluminal signal/particle propagating at $u = u_x > c$ along the *x*-axis. Hence, by differentiating the 4th equation in (12), we obtain:

$$dt' = \mu\left(1 - \frac{vu_x}{c^2}\right) dt. \tag{22}$$

Since $v > c$ and $u_x > c$ this implies $\left(1 - \frac{vu_x}{c^2}\right) < 0$. Thus, if $dt > 0$ in $S$, then $dt' < 0$ in $S'$, and vice versa. That is, if $dt < 0$ in $S$, then $dt' > 0$ in $S'$. It appears that Parker was and still is correct about the impossibility of generalizing transformation (11) to three dimensions.

The determinant or the Jacobian of (11), (12), and (13) is: –1. Therefore, the so-called Parker's transformation and its supposed generalization are improper pseudo-transformations. It is also of interest to note that unlike LTs, the pseudo-transformations (11), (12), and (13) do not perform a rotation since they are not orthogonal and in fact do not form a group. Actually, these pseudo-transformations in (1+1)-dimensions or in (3+1)-dimensions are not transformations at all and, once again, cannot be used to describe superluminal relative motion. Instead, they are spatio-temporal reflections in a plane through the origin. Two of these can be singled out as follows: spatial inversion, in which the spatial coordinates are reversed in sign, and temporal inversion, in which the time coordinate has its sign reversed.

## 5. Authors' formalism

It seems that Dragan *et al*. [1] are not familiar with the general theory of linear transformations. This is evident from what the authors themselves wrote in their paper ([1]; page 3): *"… we will pick the negative sign so that the transformation (3) remains a hyperbolic rotation."* By (3) the authors refer to Parker's (1+1)-dimensional transformation (11). However, as demonstrated earlier, this pseudo-transformation (3) or (11) is not orthogonal because its determinant or Jacobian is negative $(-1)$. Therefore, it is evident that the authors either ignore or misunderstand the key property of hyperbolic rotation in the context of the general theory of linear transformations, which is that its determinant is always equal to $(+1)$. This is simply because $\cosh^2\theta - \sinh^2\theta = +1$.

Concerning the authors' formalism, which is exclusively based on successive substitutions and lacks reproducible calculations, any professional researcher in theoretical or mathematical physics could find that the authors' approach is not only bizarre but also highly questionable mathematically and physically. What went wrong? First of all, the authors confused mathematics with physics. ‒Physics, as a natural science, uses mathematics as a tool or language, not as the primary focus. The laws of physics are expressed through mathematical equations, and to discover a new law, the corresponding equation must be formulated. However, the true meaning of a *law* is only revealed when the connection between the symbols in the equation and measurable physical phenomena is established through experiments or observations. In contrast to equations in a purely mathematical context, where experimental confirmation is not always necessary, in physics, experimental validation is crucial as mathematics itself is an abstract science.

In addition to confusing vectors with scalars, the authors also extended this confusion to the concept of time. For instance, instead of recognizing time as a universally accepted chronological parameter represented by a positive real scalar that is unidimensional, unidirectional, and uniform in all IRFs, the authors treated time as a three-dimensional vector in superluminal IRFs and as a temporal coordinate in subluminal IRFs [1]. This assumption is entirely disconnected from theoretical, mathematical, and experimental physics, as well as any reasonable speculative idealization.

We have previously seen that the pseudo-transformations (11), (12), and (13) violated the main requirements of the relativity principle and causality principle. Now the same requirements are again violated by supposing time to be a three-dimensional vector in one IRF and a temporal coordinate in another. All of this has reinforced the bizarreness of the authors' formalism [1]. It appears that the authors were perfectly aware of this bizarreness, which is why they wrote (pages 3 and 4): "*This result indicates that the laws of physics in the inertial superluminal frame of reference are different from those within the orthodox family of subluminal frames.*" "*Our interpretation that the superluminal observer characterizes spacetime using three temporal dimensions* $\boldsymbol{t}'$ *poses several interpretational challenges.*" In reality, the idea of assuming time to be a three-dimensional vector is not new. It seems that Dragan *et al*. [1] have overlooked that such a supposition was given early, in 1975, by Demers [12], and in 1976, Mignani and Recami reconsidered and adopted the same idea [13] with the aim of superluminalizing SRT. However, the idea was eventually abandoned because it was not only fruitless but also an obstacle to any future development of 'superluminal physics'.

## 6. Hassani (superluminal spatio-temporal) transformations

Taking into account what was previously said, the interested reader may curiously and logically ask the following question: After more than six decades of dedicated intensive theoretical and mathematical research, including the current ones, aimed at superluminalizing SRT, unfortunately, all have failed to reach the expected result‒maybe except for [50]‒why? Because the systematic scrutiny of the totality of those works has shown more conclusively that the principal reason behind the failure was typically a *strategic* one. How? In order to superluminalize SRT, the authors' strategy was, firstly, exclusively focused on the superluminalization of LTs without setting the necessary and sufficient conditions that should be imposed on the desired superluminal (spatio-temporal) transformations. But the veritable crux of the said failure was *and* still is the fact that ‒instead of beginning with the superluminalization of LTs, the authors would, firstly, commence by superluminalizing the usual relativity principle, IRFs and Minkowski space-time. Once these triple tasks are correctly accomplished, the authors must set some basic conditions under which the LTs could be finally superluminalized.

A) *Basic conditions* (*requirements*): The desired superluminal (spatio-temporal) transformations between the superluminal IRFs should satisfy the following requirements

a) They are real
b) They are linear
c) They are orthogonal
d) They are orthochronous
e) They form a group
f) They contain LTs as a special case.

It is worth noting that for a more detailed technical discussion about the conceptual strategy mentioned above, the reader can refer to Ref.[50], where Hassani (superluminal spatio-temporal) transformations were published in 2014 in an article titled "Foundations of Superluminal Relativistic[4] Mechanics." In addition to their mathematical, physical, and pedagogical significance due to their exact satisfaction of the aforementioned requirements, Hassani transformations are fundamentally based on an approach that is radically different from previous ones. Therefore, we felt compelled to rewrite them based on [50]. However, we should first rewrite the framework in which Hassani transformations were developed.

B) *Superluminalization of relativity principle*: The extension of the relativity principle to the superluminal IRFs is called superluminalization of the relativity principle and consequently became the superluminal relativity principle which *states that the totality of equations describing the laws of superluminal physics has the same form in all the superluminal IRFs*.

C) *Specific kinematical parameter*: Each IRF has, in addition to its relative velocity of magnitude $v$, its proper specific kinematical parameter (SKP), which has the physical dimension of a constant speed defined as

$$\begin{cases} \vartheta(v) = c, \quad -c < v < c \\ \vartheta(v) > v, \quad c \leq |v| < \infty \, . \\ \vartheta^2(-v) = \vartheta^2(v), \quad \forall v \end{cases} \tag{i}$$

Mathematically, the SKP can also be interpreted as a constant function, which is a typical example of a step function[5].

[4] The adjective *Relativistic* in the expression "Foundations of Superluminal Relativistic Mechanics" means that the basic element of this theory is the relativity principle generalized to uniform rectilinear superluminal motion. It also reflects two intrinsic aspects of this generalized principle: certain physical characteristics of a material system are relative, meaning that the numerical value of such characteristics measured by one observer may be different from the value measured by another observer moving with respect to the first one. This is what we mean by saying that certain physical quantities are observer-dependent. The second aspect, which is in fact inseparable from the first one, is that all fundamental laws of Nature are independent of the observer's motion. This statement reflects the 'absolute' aspect of the superluminalized relativity principle, namely, that superluminal physical laws are the same for all inertial observers. The two aspects are inseparable because one directly follows from the other. Indeed, the relativity of motion–the states of rest and motion are relative and do not have any absolute meaning–follows immediately from the absoluteness of fundamental natural laws–they are the same regardless of the state of motion of an observer.

[5] Step functions are generally used to model idealized physical situations in which some quantity rapidly changes from one value to another in such a way that the exact details of the change are irrelevant for problem solving.

The following additional properties of $\vartheta(v)$ as a step function are highly important:

1) $\vartheta(v)$ as a step function has a constant value on given intervals, but the constant is different for each interval. The constant value on each interval creates a series of horizontal lines, and the fact that the constant is different for each interval creates the jumps in between each horizontal line segment.
2) $\vartheta(v)$ as a step function is not continuous nor differentiable in the entire domain of the function.
3) $\vartheta(v)$ as a step function can only take a limited number of values.
4) $\frac{d\vartheta(v)}{dv} = 0, \ \forall v.$
5) $\int_0^{v<c} \vartheta(v)dv = \vartheta(v)v.$
6) $\int_0^{v<\infty} \vartheta(v)dv = \vartheta(v)v.$

The integrals (5) and (6) are evaluated by using partial integration over the intervals $0 \le v < c$ and $0 \le v < \infty$.

If we take into consideration the definition (i) and the fact that $\vartheta(v)$ is a step function, we can assert that, in the context of [50], the mathematical notion of *infinite velocities* does not exist in physical reality. This assertion seems quite reasonable since the observable Universe itself has a finite age and size. We can know absolutely nothing of the Universe outside what we can observe. Furthermore, '$c$' in (i) plays the role of a limiting velocity. But a limit has two sides, *above* and *below*. Therefore, according to (i), subluminal velocities have '$c$' as a maximum limiting velocity; luminal velocities have '$c$' as a critical velocity while superluminal velocities have '$c$' as a minimum limiting velocity. Moreover, the physical usefulness of LTs depends on condition $-c < v < c$, while the physical applicability of Hassani transformations depends exclusively on definition (i), which is precisely the necessary condition under which the transformations find their physical applicability

*D) Superluminal space-time*: What is the appropriate geometry of space-time to describe superluminal physical phenomena? To answer this question properly, we need to consider the concept of the SKP, specifically its definition (i). Therefore, we can proceed to determine the mathematical framework of superluminal space-time based on the existence of superluminal motions. The mathematical structure of superluminal space-time, as the setting for superluminal physical phenomena, should be defined by the following superluminal quadratic form (superluminal metric):

$$x'^2 + y'^2 + z'^2 - \vartheta^2(v)t'^2 = x^2 + y^2 + z^2 - \vartheta^2(v)t^2. \quad \text{(ii)}$$

As we can see, according to the definition of SKP (i), the superluminal quadratic form (ii) may be reduced to that of Minkowski (3) for the special case $\vartheta(v) = c$ when $-c < v < c$. The signature of the metric $(+ + + -)$ in (ii) means that the geometry of superluminal space-time is not completely Euclidean; it is, in fact, non-Euclidean because in the superluminal regime, space '*contracts*' and time '*dilates*' exactly as in Minkowski space-time for relativistic velocities (see, *e.g.*, [50]). Consequently, the superluminal quadratic form (ii) should be invariant under the superluminal spatio-temporal transformation during any transition from a superluminal IRF to another. For this reason, one can also define a superluminal four-vector of position as follows: Relatively to (IRF) *S*, we refer to the superluminal four-vector of the position of a superluminal event with spatio-temporal coordinates ($x, y, z, t$) as a vector **R** with components:

$$(x_1 = x,\ \ x_2 = y,\ \ x_3 = z,\ \ x_4 = i\vartheta(v)t), \qquad i = \sqrt{-1}\,. \tag{iii}$$

Now we come to our main goal, which is the superluminal spatio-temporal transformations known as Hassani transformations. We will focus on their mathematical expressions in this discussion. For the derivation process, please refer to [50].

Superluminal space-time has already found theoretical applications at both the micro and macroscopic levels. For example, the existence of the Scharnhorst effect (faster-than-light photon propagation in the Casimir vacuum) [104-107] should lead one to adopt a superluminal space-time metric in the Casimir vacuum, which automatically leads to the adoption of both the superluminal relativity principle and the superluminal causality principle. The same applies to the Hartman effect (tunnel effect); the ‘rapid lateral expansion of optical luminosity in lightning-induced ionospheric flashes’[30]; ‘apparent faster-than-light pulse propagation in interstellar space’[31]; and ‘long-range superluminal pulse propagation in a coaxial photonic crystal’[40]. Therefore, conceptually, the motion of superluminal particles or the propagation of superluminal signals in superluminal space-time shortens the luminal vacuum time span between cause and effect in accordance with the superluminal causality principle.

Let us consider two IRFs $S$ and $S'$, with $S'$ moving in uniform translational motion at a superluminal velocity $v$ along the $x$-axis of $S$, where $c \le |v| < \vartheta(v)$. We assume that a superluminal event can be described by spatio-temporal coordinates ($x, y, z, t$) in $S$ and ($x', y', z', t'$) in $S'$. The two superluminal IRFs are assumed to be in a standard configuration. Therefore, the Hassani transformations from $S$ to $S'$ and vice versa are as follows:

$$S \to S' : \begin{cases} x' = \eta(x - vt) \\ y' = y \\ z' = z \\ t' = \eta\left(t - \frac{vx}{\vartheta^2(v)}\right) \end{cases}, \tag{iv}$$

$$S' \to S : \begin{cases} x = \eta(x' + vt') \\ y = y' \\ z = z' \\ t = \eta\left(t' + \frac{vx'}{\vartheta^2(v)}\right) \end{cases}, \tag{v}$$

with

$$\eta \equiv \eta(v) = 1/\sqrt{1 - v^2/\vartheta^2(v)}\ , \qquad c \le |v| < \vartheta(v). \tag{vi}$$

Let us show that the superluminal quadratic form (ii) is truly invariant under the Hassani transformation (iv) during any transition from $S$ to $S'$. Thus, we have:

$$\begin{aligned} {x'}^2 + {y'}^2 + {z'}^2 - \vartheta^2(v){t'}^2 &= \eta^2(x - vt)^2 + y^2 + z^2 - \eta^2\left(\vartheta(v)t - \frac{vx}{\vartheta(v)}\right)^2 \\ &= \eta^2(1 - v^2/\vartheta^2(v))x^2 + y^2 + z^2 - \eta^2(1 - v^2/\vartheta^2(v))\vartheta^2(v)t^2 \\ &= x^2 + y^2 + z^2 - \vartheta^2(v)t^2. \end{aligned}$$

This important property implies that in (3+1)-dimensions, the Hassani transformations transform the superluminal wave operator

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\vartheta^2(v)\partial t^2} \qquad \text{to} \qquad \frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}-\frac{\partial^2}{\vartheta^2(v)\partial t'^2}\,, \tag{vii}$$

the superluminal light-cone

$$x^2+y^2+z^2-\vartheta^2(v)t^2=0 \qquad \text{to} \qquad x'^2+y'^2+z'^2-\vartheta^2(v)t'^2=0\,, \tag{viii}$$

the square of superluminal wave four-vector

$$\mathbf{K^2}=k_x^2+k_y^2+k_z^2-\frac{\omega^2}{\vartheta^2(v)} \qquad \text{to} \qquad \mathbf{K'^2}=k'^2_{x'}+k'^2_{y'}+k'^2_{z'}-\frac{\omega'^2}{\vartheta^2(v)}\,, \tag{ix}$$

where

$$\mathbf{K}=\left(k_x,k_y,k_z,i\frac{\omega}{\vartheta(v)}\right) \qquad \text{and} \qquad \mathbf{K}'=\left(k'_{x'},k'_{y'},k'_{z'},i\frac{\omega'}{\vartheta(v)}\right),$$

with

$$S'\rightarrow S:\begin{cases} k_x=\eta\left(k'_{x'}+\varepsilon\frac{\omega'}{\vartheta(v)}\right)\\ k_y=k'_{y'}\\ k_z=k'_{z'}\\ \frac{\omega}{\vartheta(v)}=\eta\left(\frac{\omega'}{\vartheta(v)}+\varepsilon\,k'_{x'}\right)\end{cases}, \tag{x}$$

$$\eta=1/\sqrt{1-\varepsilon^2}\;, \qquad \varepsilon=v/\vartheta(v)$$

and also they transform the square of superluminal momentum-energy four-vector

$$\mathbf{P}^2=p_x^2+p_y^2+p_z^2-\frac{E^2}{\vartheta^2(v)} \qquad \text{to} \qquad \mathbf{P}'^2=p'^2_{x'}+p'^2_{y'}+p'^2_{z'}-\frac{E'^2}{\vartheta^2(v)}\,, \tag{xi}$$

where

$$\mathbf{P}=\left(p_x,p_y,p_z,i\frac{E}{\vartheta(v)}\right) \qquad \text{and} \qquad \mathbf{P}'=\left(p'_{x'},p'_{y'},p'_{z'},i\frac{E'}{\vartheta(v)}\right),$$

with

$$S'\rightarrow S:\begin{cases} p_x=\eta\left(p'_{x'}+\varepsilon\frac{E'}{\vartheta(v)}\right)\\ p_y=p'_{y'}\\ p_z=p'_{z'}\\ \frac{E}{\vartheta(v)}=\eta\left(\frac{E'}{\vartheta(v)}+\varepsilon\,p'_{x'}\right)\end{cases}. \tag{xii}$$

Effectively, the superluminal wave operator, the superluminal light-cone, the square of the superluminal wave four-vector, and the square of the superluminal momentum-energy four-vector are invariant under the Hassani transformations for superluminal relative motion. This is in good agreement with the aforementioned superluminal relativity principle. Moreover, it is easy to verify that the Hassani transformations (iv) and (v), which depend on the kinematical parameters $v$ and

$\vartheta(v)$, form a linear orthogonal-orthochronous group and also form a rotation in superluminal space-time since their determinant is equal to $+1$. Therefore, Hassani transformations satisfy all the imposed requirements (a–f). Note that these transformations preserve the orientation of the spatial axes and leave the sign of the time component unchanged. This allows us, once again, to affirm that the relativity principle and the causality principle are extended to superluminal IRFs via Hassani transformations, which can be reduced to LTs for the special case $\vartheta(v) = c$ when $-c < v < c$. They are actually a generalization of LTs to superluminal IRFs. If we apply the superluminal transformation (xii) of the superluminal momentum-energy four-vector that should characterize any material point moving at superluminal velocity to a material point of mass $m$ in its proper superluminal frame $S'$ (where the material point is at relative rest), in the observer's frame $S$ (where the same material point is seen to move at superluminal velocity $v$, where $c \leq |v| < \vartheta(v)$), we obtain, after a simple calculation, the following expected superluminal three-dimensional momentum $\mathbf{p}$ and superluminal (total) energy $E$, respectively:

$$\mathbf{p} = \frac{E}{\vartheta^2(v)}\,\mathbf{v}\,, \qquad \text{(xiii)}$$

and

$$E = \eta E_0\,, \qquad \text{(xiv)}$$

with $\eta = 1/\sqrt{1 - v^2/\vartheta^2(v)}$ , $\|\mathbf{v}\| = v$ , and $E_0 = mc^2$ being the rest energy of the material point, the two important derived formulas (xiii) and (xiv) are part of the foundations of superluminal relativistic dynamics, a part of superluminal relativistic mechanics [50]. Furthermore, we can show that the combination of (xiii) and (xiv) leads immediately to the following superluminal momentum-energy relation:

$$E^2 = \mathbf{p}^2\vartheta^2(v) + E_0^2. \qquad \text{(xv)}$$

Consequently, if we know the total energy and momentum of a given superluminal particle, we can find its velocity. The general expression for the velocity of the superluminal particle in terms of its total energy and momentum can be deduced by differentiating Eq.(xv) with respect to $\mathbf{p}$. We obtain:

$$v = \frac{dE}{d\mathbf{P}} = \vartheta(v)\sqrt{1 - \frac{E_0^2}{E^2}}\,. \qquad \text{(xvi)}$$

Notice that it is not difficult to show that for the case $v \sim \vartheta(v)$, the combination of $\gamma_{\max} = 1/\sqrt{1 - c_0^2/c^2}$ and $\eta = 1/\sqrt{1 - v^2/\vartheta^2(v)}$ yields the expression $\frac{\vartheta(v)}{c} = \frac{\eta}{\gamma_{\max}}$ or equivalently $\frac{\vartheta(v)}{c} = \frac{\eta E_0}{\gamma_{\max} E_0} = \frac{E}{\mathscr{E}}$ .Therefore, a direct substitution in Eq.(xvi) gives:

$$v = \frac{dE}{d\mathbf{P}} = c\sqrt{\frac{E^2 - E_0^2}{\mathscr{E}^2}}\,. \qquad \text{(xvii)}$$

It is easily seen from Eqs.(xvi) and (xvii) that the higher the energy, $E$, the more superluminal the particle's velocity. Moreover, since the superluminal kinetic energy, $K = (\eta - 1)E_0$, is equal to the difference between the superluminal total energy, $E = \eta E_0$, and the rest energy, $E_0 = mc^2$, once a superluminal particle's velocity $v$ is sufficiently close to $\vartheta(v)$, its superluminal kinetic energy increases due to the apparent increase in superluminal total energy not due to a change in velocity.

The reader can be assured that, according to definition (i) of SKP[6], the five formulas above are reducible to the known relativistic ones (belonging to the SRT formalism) for the special limiting case $\vartheta(v) = c$ when $-c < v < c$. Consequently, theoretically, the high, very high, and ultra-high energy cosmic rays can be easily investigated within the framework of superluminal relativistic mechanics [50] without any *ad hoc* assumptions or artificial devices for modifying SRT at extremely high energies, as proposed by some researchers. Actually, SRT, as a robust theory in its own domain of applicability, has its limits of validity and accordingly does not need to be modified but instead needs to be generalized through an approach radically departing from the previous ones. In this sense, superluminal relativistic mechanics [50] is precisely what SRT conceptually needs. It seems that many researchers are unaware of what exactly Einstein himself said about the limits of validity of SRT when he realized that, in the framework of GRT, the postulate of the constancy of the speed of light has no universal validity. Einstein's awareness of the non-universality of the constancy of the speed of light is clearly mentioned in his popular book [108]. Einstein wrote: "*In the second place our result shows that, according to the general theory of relativity, the law of the constancy of the velocity of light in vacuo, which constitutes one of the two fundamental assumptions in the special theory of relativity and to which we have already frequently referred, cannot claim any unlimited validity. A curvature of rays of light can only take place when the velocity of propagation of light varies with position. Now we might think that as a consequence of this, the special theory of relativity and with it the whole theory of relativity would be laid in the dust. But in reality this is not the case. We can only conclude that the special theory of relativity cannot claim an unlimited domain of validity* (…)"

Incidentally, contrary to some repeated claims, in general, SRT does not prohibit superluminal motion or the manifestation of physical phenomena at superluminal velocities. Instead, it only excludes, within its proper framework, superluminal IRFs because of LTs. And Einstein himself was quite clear on this point, as he wrote in Ref.[116]: "*A relative motion of reference systems with superluminal velocity is incompatible with our principles*."

## 7. Superluminal relativistic mechanics and causality principle

Let us now examine the practical utility of superluminal relativistic mechanics [50], specifically the application of superluminal relativistic kinematics to the study of superluminal signaling. First, we will explore superluminal causality within the framework of relativistic kinematics and demonstrate that contrary to the widespread naive belief that superluminal signaling violates the physical principles of relativity and causality, it actually upholds relativistic causality. This challenges the common assumption that causal processes or signals can only propagate within the light-cone. Many physicists and philosophers unquestioningly adhere to this requirement without critically evaluating its partial or total validity, relative or absolute validity, or universal applicability. This misconception can be traced back to Einstein's 1907 note [116], later known as Einstein's causality, and Tolman's 1917 argument against faster-than-light signals [117], often referred to as Tolman's paradox.

Tolman's misinterpretation of relativistic kinematics led to misleading statements, which were subsequently used by other authors to argue against the possibility of superluminal signaling. However, a closer examination reveals that Einstein's reasoning in his thought-experiment was flawed due to a lack of understanding of LTs. While historical context may excuse Einstein's errors,

[6] The SKP, $\vartheta(v)$, as a step function can be easily shown to be generally $\vartheta(v)\sim v$ by using the expression of $\eta$ as follows: we have from $\eta = 1/\sqrt{1 - v^2/\vartheta^2(v)}$, the expression $\vartheta(v) = \eta v/\sqrt{\eta^2 - 1}$ . It is clear that for a sufficiently large value of $\eta$, we automatically get $\vartheta(v)\sim v$. Furthermore, we can show that the combination of $\gamma_{\max}$ and $\eta$ leads to the expression $\vartheta(v)/c = \eta/\gamma_{\max}$.

there is no justification for perpetuating these misconceptions in current literature. The root of these errors lies in a misunderstanding of the properties of LTs. LTs form an orthogonal-orthochronous group, preserving temporal ordering and the concept of past, present, and future in all IRFs. This preservation of causality principle through temporal ordering is inherent in LTs, ensuring that causality is maintained across different frames of reference.

Textbooks and peer-reviewed articles have dismissed the possibility of superluminal signaling due to authors' ignorance of the causality conditions defined by the fourth equation of LTs. This ignorance or deliberate disregard of these conditions has led to the mistaken belief that superluminal motion inevitably results in causal paradoxes in SRT. While different inertial observers may disagree on the time ordering of superluminal signals, the necessary causality conditions provided by LTs prevent the emergence of causal paradoxes associated with time travel.

Before delving into the causality of superluminal signaling, it is indispensable to emphasize the importance of well-posed conditions in theoretical and mathematical physics. A properly worded problem should include relevant conditions for solving it effectively. Just as in mathematics, where specifying the set to which a variable belongs is crucial for solving equations, respecting the necessary causality conditions contained in LTs is essential for correctly analyzing superluminal signaling. For example, the physical validity of LTs hinges on the condition $-c < v < c$ where $v$ represents the relative subluminal velocity between two IRFs. It is crucial to note that the chosen interval dictates the value of $v$, emphasizing that $v$ cannot be arbitrary selected.

In the following discussion, we will clarify that the so-called Tolman's paradox is not a paradox at all, but rather an unphysical solution incorrectly labeled as a paradox. This unphysical solution arises from the violation of the relativity principle and the necessary causality conditions inherent in the LTs and the law of composition of velocities. The correct physical solution to Tolman's thought-experiment [117] can be obtained by applying the LTs and relativistic kinematics in accordance with the relativity principle, which automatically excludes the unphysical solution. Unfortunately, this pseudo-paradox has been perpetuated in several papers, leading to further citations of these incorrect works. It is important to rectify this error before it spreads further. To address this issue, we present a simple and analogous problem within classical kinematics to illustrate what is going on with Tolman's reasoning.

A streamer has a velocity $u$ relative to water. It starts at point $A$ on the bank of the river with stream velocity $v$. It moves downstream to the point$B$on the same bank at a distance $D$ from $A$, immediately turns back and moves upstream.1) How long will it take to make round-trip from $A$ to $B$ and back? 2) What is the average velocity of the streamer for the entire journey?

Solution: In the obvious case of still water (lake), the answer would be

$$t_0 = \frac{2D}{u}\,. \tag{I}$$

Now, taking into account the stream. If the streamer makes $u$ km/h relative to water and the stream makes $v$ km/h relative to the bank, then the streamer's velocity relative to the bank is $(u+v)$ when downstream and $(u-v)$ when upstream. Here, first, we are interested in the resulting time, which is the total time. It consists of two parts: one, $t_{AB}$, which is needed to move from $A$ to $B$, and the other, $t_{BA}$, to move back from $B$ to $A$. Obviously, the time $t_{BA}$ is always greater than $t_{AB}$, since the net velocity of the streamer is less during this time. Thus, the net velocity of the streamer is greater than $u$ during the shorter time and less than $u$ by the same amount during the longer time. Therefore, its average over the whole time is less than $u$. As a result, the total time itself must be greater than $t_0$.

Let us now solve the problem quantitatively. The time it takes to go from $A$ to $B$ and then back from $B$ to $A$ is, respectively:

$$t_{AB} = \frac{D}{u+v}\,, \quad t_{BA} = \frac{D}{u-v}\,. \tag{II}$$

So the total time is

$$t = t_{AB} \,+\, t_{BA} = \frac{D}{u+v} + \frac{D}{u-v} = \frac{t_0}{1-v^2/u^2}\,, \tag{III}$$

where $t_0$ would be the time in the still water, given by Eq.(I). As we can see, the necessary condition, $0 \le v < u$, under which the physical solution of the problem is determined, is already reflected in the mathematical structure of Eq.(III). Thus, for all $v < u$, the total time $t > t_0$. However, if we were not be attached to the necessary condition, then $t$ becomes infinite at $v = u$, and negative at all $v > u$. It is quite clear that by violating the abovementioned condition, we have found two unphysical solutions, namely, an infinite total time and a negative total time. In brief, something very similar happens in Tolman's reasoning concerning particularly the negative time. Actually, it is completely absurd to call or interpret, for example, $t < 0$ when $v > u$ paradox since we deliberately violated the necessary condition to get this unphysical solution.

Finally, the average velocity of the streamer for the entire journey is the harmonic mean of the two relative velocities of the streamer:

$$\langle u \rangle = \frac{2u_+ u_-}{u_+ + u_-} = \left(1 - \frac{v^2}{u^2}\right) u, \tag{IV}$$

where $u_+ = (u+v)$ and $u_- = (u-v)$. It is clear from Eq.(IV), the streamer's average velocity over the whole time is less than $u$.

We now return to the main subject of the present section. Actually, the study of superluminal signaling in the framework of relativistic kinematics reduces to the study of signal propagation at superluminal velocity relative to subluminal IRFs. This means that by virtue of the relativity principle, if a given signal propagates at a superluminal velocity $u > c$ relative to IRF $S$, then the same signal should also propagate at a superluminal velocity $u' > c$ relative to $S'$, which is moving in uniform translational motion at a subluminal velocity $v$ along the $x$-axis of $S$. This is all about the combination of signal superluminal velocity and relative subluminal velocity between two IRFs. The combination of the two types of velocities is not a new idea since in his 1905 paper 'On the electrodynamics of moving bodies'[118], Einstein himself used the subluminal and superluminal velocities, the expressions $(c - v)$, $(c + v)$, and $\sqrt{c^2 - v^2}$ for the relative velocities of light, in order to analyze the relativity of lengths and times, and also to derive the Lorentz transformation equations. It seems quite contrary to his second postulate that the speed of light was independent of the motion of the source. Actually, Einstein perfectly knew that the second postulate has no universal validity and the first two expressions are a direct consequence of the anisotropy of the one-way speed of light [119]. The same relative velocities $(c \pm v)$ of light appeared in the expressions for the elapsed time of sending out a light signal from one point to another and back again in the Michelson-Morley experiment [120], and are equally present in the Doppler effect and Sagnac effect [121].

From all that, we can understand that it is quite absurd to exclusively link causality (the cause of an event precedes the effect of the event) to the vacuum speed of light since the vacuum itself is only vacuum in name as the absolute vacuum does not exist, and the so-called constancy of light speed is not an absolute fact but at the same time relative and conditional[7]. Without forgetting that in physics, the notion of velocity does not by itself tell us anything. It only becomes really meaningful if we specify relative to what this velocity is calculated or measured. That is why we mean by saying that velocity is typically a relative physical quantity. Therefore, for the causality principle to be a truly universal principle, it would have to be equally valid for subluminal, luminal, and superluminal signals under any natural and/or artificial circumstances. For instance, suppose a massive particle is emitted before it is absorbed in a detector. Even if the particle's velocity were many trillion times faster than *c*, the cause (emission) would still precede the effect (absorption), and causality would not be violated for the reason that superluminal signals do not violate the causality principle; they just shorten the vacuum luminal time span between cause and effect.

**Scenario A:** By taking into account all the above mentioned conceptual considerations we suppose, in the framework of relativistic kinematics, two IRFs $S$ and $S'$ in standard configuration and are related to each other by LTs, with $S'$ moving at a subluminal velocity $v$ along the *x*-axis of $S$. Assume that at a time $t_0 = 0$, the reference axes of the two IRFs coincide and that the clocks are so synchronized that at $t_0 = 0$ the clocks in $S'$ indicate $t'_0 = 0$. Now consider the following hypothetical events. At $t_0 = t'_0 = 0$, a superluminal particle is emitted in the *x* direction from the common origin of $S$ and $S'$. Let the superluminal velocity of this particle be $u > c$ relative to $S$, then its velocity relative to $S'$ is

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}} > c. \tag{1A}$$

Let us show that, in terms of velocities, the necessary and sufficient causality condition is already contained in the expression (1A). First, notice that since we have $v < c$, $u > c$ and $u' > c$ this immediately implies

$$1 - \frac{uv}{c^2} > 0, \tag{2A}$$

from where we get the following important inequalities:

$$\frac{uv}{c^2} < 1, \qquad u < \frac{c^2}{v}, \qquad v < \frac{c^2}{u}. \tag{3A}$$

The first inequality, $uv/c^2 < 1$, in (3A) is precisely the necessary and sufficient causality condition within the framework of relativistic kinematics, the second inequality, $u < c^2/v$, determines the

[7] Actually, there is a profound difference between an absolute constant and a relative constant. For example, Planck's constant '$h$' and Newton's constant '$G$' are both absolute constants because they cannot be influenced by physical phenomena, unlike the so-called speed of light '$c$', which is a relative constant because its constancy or variation depends on the medium through which light propagates. During its propagation, light as electromagnetic radiation may be influenced by electric, magnetic, electromagnetic, and gravitational fields, as well as interstellar dust, atomic and molecular gas, etc. The expression "speed of light in vacuum" has nothing to do with the physical reality of the vacuum itself. Why? Because in Nature, complete and absolute vacuum does not exist since nothing real can reach absolute zero thermodynamic temperature. Furthermore, in quantum vacuum, light behaves differently than in classical vacuum due to the Scharnhorst effect [104-107].

interval of all the possible superluminal velocities that can be given to $u$, that is, $c < u < \frac{c^2}{v}$, and the third inequality, $v < c^2/u$, determines the interval of all the possible subluminal velocities that can be given to $v$ , namely, $0 < v < \frac{c^2}{u}$. The two intervals play the role of the boundary conditions of the problem under consideration. Therefore, the velocities $u$ and $v$ cannot take on any value outside their intervals.

However, in order to prove the violation of causality by superluminal signals, Tolman [117] and many other authors simply and naively fabricated the inequality $\frac{uv}{c^2} > 1$, which is clearly in contradiction with the inequality $\frac{uv}{c^2} < 1$. In doing so, these authors failed to realize that they have violated, at the same time, the expression (1A), the relativity principle, LTs, and relativistic kinematics. They confused physics with mathematics during the process of calculation because by proposing the above-mentioned inequality they already violated the necessary and sufficient causality condition. As a direct result, they obtained an unphysical solution wrongly called a paradox.

Assume now that at a time $t_1$, when it is at $x_1$ in the frame $S$, the superluminal particle is absorbed. The corresponding time of absorption, as measured in $S'$, is given by

$$t'_1 = \gamma\left(t_1 - \frac{vx_1}{c^2}\right), \tag{4A}$$

from where we obtain

$$t'_1 = \gamma\left(1 - \frac{uv}{c^2}\right)t_1, \tag{5A}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$ and $x_1/t_1 = u$. Notice that according to the inequalities (3A), the Lorentz transformation of the time coordinate (4A) is still orthochronous (preserving time direction) for a subluminal relative velocity $v < c$ and for superluminal velocity $u > c$. Therefore, in these circumstances, we obtain what was expected: if $\Delta t > 0$ in $S$, then $\Delta t' > 0$ in $S'$. It is clear that the causality principle is preserved through the temporal ordering under the necessary and sufficient condition $uv/c^2 < 1$. From all that, we arrive at the following result: quite contrary to common naive belief, under some relevant conditions, SRT can easily accommodate, and indeed does not exclude, superluminal signaling at the kinematical level, particularly when we put the concept of proper (inertial frame) or comoving (IR) frame aside.

**Scenario B:** In this scenario we aim to study superluminal signaling within the framework of superluminal relativistic kinematics [50]. This essentially involves examining signal propagation at superluminal velocities relative to superluminal IRFs. Let $S$ and $S$ be two IRFs in standard configuration, related to each other by Hassani transformations. The IRF $S'$ moves at a superluminal velocity $v$ along the *x*-axis of $S$, such that $c < |v| < \vartheta(v)$. Consider two points $A$ and $B$ on the *x*-axis of the frame $S$, and suppose that some impulse originates at $A$, travels to $B$ at superluminal velocity $u$ such that $u > v$, and at $B$ produces some observable phenomenon, the starting of the impulse at $A$ and the resulting phenomenon at $B$ thus being connected by the relation of *cause* and *effect*. The time elapsing between the cause and its effect as measured in the units of frame $S$ will evidently be

$$\Delta t = t_B - t_A = \frac{x_B - x_A}{u}, \tag{1B}$$

where $x_A$ and $x_B$ are the coordinates of the two points $A$ and $B$. The impulse's superluminal velocity relative to the second frame $S'$ is

$$u' = \frac{u - v}{1 - \frac{uv}{\vartheta^2(v)}} > c \; . \tag{2B}$$

Notice that, in terms of superluminal velocities, the necessary and sufficient causality condition is inherent in the expression (2B) as $c < |v| < \vartheta(v)$, $u > v$ and $u' > c$ this directly implies

$$1 - \frac{uv}{\vartheta^2(v)} > 0, \tag{3B}$$

from where we get the following significant inequalities:

$$\frac{uv}{\vartheta^2(v)} < 1, \qquad u < \frac{\vartheta^2(v)}{v}, \qquad v < \frac{\vartheta^2(v)}{u}. \tag{4B}$$

The first inequality, $uv/\vartheta^2(v) < 1$, in (4B) is exactly the necessary and sufficient causality condition within the framework of superluminal relativistic kinematics. The second inequality, $u < \vartheta^2(v)/v$, determines the interval of all the possible superluminal velocities that can be given to $u$, that is, $c < u < \frac{\vartheta^2(v)}{v}$, and the third inequality, $v < \vartheta^2(v)/u$, determines the interval of all the possible superluminal velocities that can be given to $v$ , namely, $c < v < \frac{\vartheta^2(v)}{u}$. The two intervals play the role of the boundary conditions of the problem under consideration. Therefore, the superluminal velocities $u$ and $v$ cannot take on any value outside their intervals.

Now in the frame $S'$, the time elapsing between cause and effect would evidently be

$$\Delta t' = t'_B - t'_A = \eta\left(t_B - \frac{v}{\vartheta^2(v)} x_B\right) - \eta\left(t_A - \frac{v}{\vartheta^2(v)} x_A\right), \tag{5B}$$

where $\eta = 1/\sqrt{1 - v^2/\vartheta^2(v)}$ . In the expression (5B), we have substituted for $t'_A$ and $t'_B$ in accordance with the fourth equation of Hassani transformation (iv). Simplifying and introducing equation (1B), we obtain

$$\Delta t' = \eta\left(1 - \frac{uv}{\vartheta^2(v)}\right)\Delta t. \tag{6B}$$

By considering the inequalities (4B), the Hassani transformation of the time interval (6B) remains orthochronous for superluminal velocities. Therefore, as expected, if $\Delta t > 0$ in $S$, then $\Delta t' > 0$ in $S'$. This ensures that the causality principle is maintained through temporal ordering[8] under the superluminal necessary and sufficient condition $uv/c^2 < 1$. Scenarios A and B demonstrate not only the preservation of the causality principle under specific conditions but also highlight the significance of these conditions in science, especially in theoretical and experimental physics.

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[8] Actually, time ordering cannot be used as a universal criterion or sufficient condition to determine that the effect precedes the cause. Therefore, causality is not violated by superluminal signaling because mathematically, causality violation can be proven using only subluminal signaling. Additionally, if a causal link between two events could only be established based on their temporal ordering, the concepts of cause and effect would be problematic in theories like Newtonian gravity which allow for instantaneous action at a distance.

Paradoxes are human constructs, as there is no inherent paradox in the real world. In physics, the creation of paradoxes is often a result of misunderstanding, misinterpretation, oversight, or the absence of necessary conditions for investigating a problem.

## 8. Conclusion

In this paper, we have presented a critical assessment of the significant effort over the past decades and renewed interest in extending the special relativity theory (SRT) beyond the speed of light [6-18]. In this comprehensive note, the paper "Relativity of superluminal observers in 1+3 spacetime" by Dragan et al. [1] is examined. The authors attempted to extend the aforementioned theory to superluminal inertial reference frames by generalizing Parker's two-dimensional transformation [7] with the addition of two pairs of spatial dimensions. This approach is not novel and is already well-documented in the literature. We have demonstrated that unlike Lorentz transformations, the authors' transformations do not form an orthogonal-orthochronous group due to their negative determinant. As a result, principles such as relativity, causality, spatial isotropy, and temporal ordering cannot be preserved. The authors' pseudo-transformations are revealed to be reflections in a plane through the origin rather than true transformations. We conclude that the authors' pseudo-transformations do not and cannot generalize SRT in (3+1)-dimensions as claimed by Dragan et al. Also, a theoretical maximum limit of the Lorentz factor is introduced, which leads to an extension of Lorentz transformations to luminal inertial reference frames and raises a conceptual question about the status and role of the symbolic quantity '*c*', commonly called the speed of light in vacuum, as the neutrino and particularly the photon have non-zero mass. Consequently, it appears that as long as the non-zero mass of the photon is not taken seriously into full consideration, our current knowledge of physics, astronomy, astrophysics, and cosmology remains not only incomplete but above all vague and doubtful. Finally, Hassani superluminal spatio-temporal transformations are revealed and their physical consequences are partly investigated.

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