Some mathematical results of light path models in General Relativity and the decomposition of the null geodesic into gravitational components

Gerald Malczewki[1]

Abstract: This paper is a natural continuation of an earlier paper of light path models and investigates the mathematical implications of geodesic light trajectories within a Schwarzschild metric gravitational field. We focus on a model expressed as infinite Taylor series expansion and its finite cut-off counterpart. A comparison is then made against another existing model that is expressed in closed form not requiring an infinite series and which requires a Jacobian elliptic function. Under some restriction of the mass of the central gravitating body these different models were previously shown to be equivalent. Using these results, some mathematical relationships are then derived. Additionally, we decompose the light path equation into an infinite set of 'gravitational components', somewhat akin to techniques used in Fourier analysis.

A. Introduction.

In **Section B**, we review the Taylor series approach for modeling light paths in a gravitational field. **Section C** introduces an alternative model developed in [6], here referred to as the HK model, which expresses the light path equation in closed form using Jacobian elliptic functions. While not immediately intuitive, prior work [9] has shown its equivalence to the Taylor series model under certain conditions. In **Section D**, we derive mathematical relationships between the two models, leveraging their equivalence to express specific coefficients of the Taylor series expansion in terms of the HK model. These relationships offer insight into the structure of the infinite series. **Section E** introduces a decomposition of the light path equation into an infinite set of "gravitational components." This decomposition, conceptually similar to Fourier series, expresses the light path as a sum of contributing terms, each representing an aspect of the trajectory's curvature. This perspective provides a new way to interpret the gravitational bending of light and its mathematical formulation. We end with a **Summary** section highlighting the main observations and findings.

Familiarity with mathematical functional notation, set notation, and proficiency in basic calculus is assumed. The reader is also assumed to have a basic knowledge of General Relativity (GR) ([3], [4], [7], [10]) and how light reflection is measured (see for example, [5] or section C.2 of [9] as well as the GR references) and some familiarity with impact parameter analysis (see section 6.3 of [10]). A detailed knowledge of the mathematical techniques of GR is not required. There is minimal use of tensors and only in the context of the Schwarzschild metric. Using GR nomenclature, we sometimes refer to the light path in a vacuum as a geodesic or null geodesic.

[1] Email address: MalczewskiGld@gmail.com

As a caution to the reader, the key equations in this paper are drawn from several sources, each with their own notation and conventions. This will necessitate performing multiple sets of notational replacements and transformations which, although adding to the mathematical exposition, will be necessary to demonstrate our assertions. Attempting to show all the derivations of the key equations of each source would result in an excessively long paper and therefore the reader is referred to these sources for the details.

Throughout this paper we assume the gravitating body mass is a symmetric and static spherical mass with uniform density where rotation and charge are not present or negligible, i.e., the metric is Schwarzschild (see endnote [1] for a description).

B. $n < \infty$ and $n = \infty$ models.

We designate the light path equation by the function $\theta = \theta(M/R\,, \rho)$ where θ is the polar angle in spherical coordinates of the light path location. The arguments are M/R and $\rho = R/r$ where r is the radial distance to the center of the body with mass M, and R the distance of closet approach which is also the 'turning point' of the light path as the path proceeds away from the body. The path is perpendicular to the radial line $r = R$. Going forward we will call the ratio M/R the 'mass ratio'. As a reference point, the mass ratio for the sun is $M/R \approx 2.1x10^{-6}$ when R is chosen to be at the surface.

As described in sections B and C of [9], the '$n < \infty$' approximation model (note: we changed the '$k < \infty$' notation used in that paper to '$n < \infty$' in this paper due to indexing notation considerations) of a light path in a vacuum for a Schwarzschild metric is the finite sum

$$\theta_n\left(\frac{M}{R}, \rho\right) = \sum_1^n F_k\left(\frac{M}{R}, \rho\right) + C_n\left(\frac{M}{R}\right) \qquad \text{(B.1)}$$

where $C_n\left(\frac{M}{R}\right)$ is a constant of integration and

$$F_k\left(\frac{M}{R}, \rho\right) = \frac{2k-3}{2^{k-2}}\left[\frac{a_k}{b_k}\sin^{-1}\rho + \frac{(\pm\sqrt{1-\rho}\,)P_k(\rho)}{d_k(1+\rho)^{\frac{2k-3}{2}}}\right]\left(\frac{M}{R}\right)^{k-1} \qquad \text{(B.2)}$$

is the antiderivative of the integral term

$$\int \frac{2k-3}{2^{k-2}} \frac{1}{\sqrt{1-\rho^2}} \left(\left(\frac{1-\rho^3}{1-\rho^2}\right) \frac{M}{R}\right)^{k-1}$$

in a Taylor series resulting from the Schwarzschild metric for a light path. Due to a restriction on the Taylor series, the mass ratio is limited to $0 \leq M/R < 1/3$ which is within the range of most gravitating bodies.

Here, a_k, $b_k \neq 0$, $d_k \neq 0$ are integers and $P_k = P_k(\rho)$ a 'complete' polynomial of degree $2k - 3$ in ρ for $k > 1$. By a 'complete' polynomial of degree $2k - 3$ we mean the polynomial contains terms of every degree up to and including $2k - 3$. Using an integral calculator this form has been confirmed for $k = 2, \ldots\ldots 10$. We emphasize again that since $\theta_n\left(\frac{M}{R}, \rho\right)$ is a finite sum, this model is only an approximation to a true null geodesic path in space-time. The degree of approximation depends on the mass ratio M/R, the index n, and the radial distance r.

In this paper we will slightly weaken the conditions for (B.2) and not require $P_k = P_k(\rho)$ to be complete. Further, we will change (B.2) to the equivalent antiderivative form

$$F_k\left(\frac{M}{R}, \rho\right) = (-1)^{k-1} \frac{2k-3}{2^{k-2}} \left[t_k \cdot \sin^{-1} \rho + \frac{N_k(\rho)}{(1-\rho^2)^{\frac{2k-3}{2}}}\right] \left(\frac{M}{R}\right)^{k-1} \qquad \text{(B.3)}$$

where $N_k(\rho)$ is an incomplete polynomial and $t_k \equiv a_k/b_k$ where $a_2 = 0$. This form will be more convenient when we argue that it holds for all $k \geq 2$ in Appendix B. The coefficient d_k in (B.2) has been absorbed into $N_k(\rho)$. We note that in this form the equation is apparently not valid for $\rho = 1$ $(r = R)$ but we will later argue that $\frac{N_k(\rho)}{(1-\rho^2)^{\frac{2k-3}{2}}} = 0$. For now we simply define $F_k\left(\frac{M}{R}, 1\right) = (-1)^{k-1} \frac{2k-3}{2^{k-2}} \left[t_k \cdot \frac{\pi}{2}\right] \left(\frac{M}{R}\right)^{k-1}$.

The methods employed in Appendix B will imply that, in principle, given any k we can algorithmically compute the corresponding coefficients of the polynomial $N_k(\rho)$. The weakening of the requirement for completeness of $P_k = P_k(\rho)$, replaced by $N_k(\rho)$, will have no material effect on our results.

$C_n = C_n\left(\frac{M}{R}\right)$ in (B.1) is the resulting constant of integration and does not depend on ρ. C_n can be found by using the boundary condition that the polar angle is $\pi/2$ radians when $r = R$ $(\rho = 1)$. It can be readily shown that these constants follow the recursive relationship $C_n = C_{n-1} - F_n(\rho = 1)$ and the summation $C_n\left(\frac{M}{R}\right) = -\frac{\pi}{2}\left[\sum_2^n (-1)^{j-1} \left(\frac{2j-3}{2^{j-2}}\right) t_j (M/R)^{j-1}\right]$.

For the $n < \infty$ model, the *n* value is also referred to as the 'approximation level'. Going forward we will have occasion to set the approximation level to small values of *n* in some of the examples. For this model, the computed trajectory locations will necessarily be approximate. Computational examples and associated graphs of the light paths are detailed in [9].

Notice that when $\rho = 0$, corresponding to $r = \infty$, $F_k\left(\frac{M}{R}, \rho = 0\right)$ reduces to $(-1)^{k-1}\frac{2k-3}{2^{k-2}}[N_k(0)]\left(\frac{M}{R}\right)^{k-1}$ where $N_k(0)$ is the constant term of $N_k(\rho)$. When $\rho = 1$, corresponding to $r = R,\ F_k\left(\frac{M}{R}, \rho = 1\right) = (-1)^{k-1}\frac{2k-3}{2^{k-2}}\left[t_k \cdot \frac{\pi}{2}\right]\left(\frac{M}{R}\right)^{k-1}$ as stated earlier. These results are useful when analyzing asymptotic behavior and finding the constant of integration of the antiderivatives and will also be used in section E to derive a relationship between some of the $F_k\left(\frac{M}{R}, \rho\right)$ coefficients.

Appendix A lists the antiderivatives F_k for $k = 1$ to 5 which exhibit the general form (B.3). For the case $k = 1$ we set $F_1 = \sin^{-1}\rho$, the massless term corresponding to a straight-line path in polar coordinates. The expressions for the constants of integration C_n are also listed. For all $k,$ the justification for the F_k formulas can be determined by solving a set of linear equations using the method described in Appendix B. Alternatively, but not described in this paper, F_k can also be determined using a series of trigonometric substitutions and identities. For either method there does not seem to be any nice recursive relationship among these antiderivatives or some simple dependency on k. Therefore the results in Appendix B show that although each term $F_k\left(\frac{M}{R}, \rho\right)$ can be algorithmically determined, finding a general formula that can generate it for any k is apparently very difficult.

The '$n = \infty$' model of the light path as described in [9] is the infinite sum

$$\theta_\infty\left(\frac{M}{R}, \rho\right) = \sum_1^\infty F_k\left(\frac{M}{R}, \rho\right) + C_\infty\left(\frac{M}{R}\right) \qquad \text{(B.4)}$$

where $C_\infty\left(\frac{M}{R}\right) = \lim_{n\to\infty} C_n\left(\frac{M}{R}\right) = -\frac{\pi}{2}\left[\sum_2^\infty(-1)^{n-1}\left(\frac{2n-3}{2^{n-2}}\right)t_n(M/R)^{n-1}\right]$ is the resulting constant of integration in the limit. $\theta_\infty\left(\frac{M}{R}, \rho\right)$ is the 'true' theoretical path of the light ray (null geodesic) based on General Relativity. However, truncating $\theta_\infty\left(\frac{M}{R}, \rho\right)$ at a sufficiently high cutoff approximation index K, a $K < \infty$ model, gives a polar angle to some specified approximation for a given mass ratio $\frac{M}{R}$ and ρ.

C. Light path equation based on Jacobian elliptic functions – the HK model.

We now turn to the derived light path equation as expressed in equation (80) in [6], where crucially the equation is in closed form, as opposed to the infinite series model $n = \infty$. We will refer to this light path model as the HK (Hioe-Kuebel) model where

$$Q(U_1, \phi) = \frac{(e_1 - e_3)e_2 - (e_2 - e_3)\, e_1\, sn^2(\gamma\phi, k)}{(e_1 - e_3) - (e_2 - e_3)\, sn^2(\gamma\phi, k)}. \qquad \text{(C.1)}$$

We have replaced $1/q = \alpha/r$ in their equation (80) with $Q(U_1, \phi)$ where α is the Schwarzschild radius of the central body with units of length(meters)- see endnote [2]. The mass parameter U_1 is defined to be $U_1 = \alpha/R$. Since $\alpha = 2M$ in the Schwarzschild metric, then $U_1 = 2M/R < 2(1/3) = 2/3$.

The function sn is a Jacobian elliptic function. The e_i are roots of a certain cubic in a differential equation expressing how the radial distance r changes as the angle ϕ varies. This angle is the polar angle in polar coordinates. To avoid confusion with our polar angle θ in spherical coordinates for the $n < \infty$ and $n = \infty$ models we will designate ϕ as the 'vertex' angle, going forward. The e_i roots are in turn functions of α/R. This is all developed in a polar coordinate system with the central body at the origin. In each model the light path propagates in a plane. It is not readily apparent, but this HK model and the $n = \infty$ model are equivalent. Heuristically this can be seen by noting that both models are derived from the same Schwarzschild metric and the GR geodesic equation. A more formal mathematical proof is presented in [9].

The arguments of sn are $\gamma\phi$, the 'amplitude' and k the 'modulus', where γ and k are functions of the roots e_i. Jacobian elliptic functions are a generalization of trigonometric functions which refer to conic sections, the ellipse in particular. For an explanation of these non-elementary functions, including sn, see [1] as well as other online sources such as Wikipedia.

The roots e_i take the values

$$e_1 = (1/2)\left[1 - U_1 + \sqrt{(1 + 2U_1 - 3U_1^2)}\right]$$
$$e_2 = U_1$$
$$e_3 = (1/2)\left[1 - U_1 - \sqrt{(1 + 2U_1 - 3U_1^2)}\right] \qquad \text{(C.2)}$$

Also,

$$\gamma = \sqrt{e_1 - e_3}/2 \text{ and } k = \sqrt{(e_2 - e_3)/(e_1 - e_3)}. \qquad \text{(C.3)}$$

The HK model (C.1) is a function of the vertex angle ϕ and U_1 and outputs the dimensionless value $Q = \alpha/r$ from which the radial distance is determined. Conversely, the $n < \infty$ and $n = \infty$ models (B.1) and (B.4) are functions of radial distance and M/R, giving the polar angles θ_n and θ_∞ respectively. The necessary conversions and transformations to get a consistent framework to prove the equivalency of the two results is given in the proof in Appendix B in [9].

Figure 1 shows the polar graph of a light path computed using equation (C.1) of the HK model. This graph can also be approximated to a high degree of accuracy for the path near the turning point R using equation (B.1) of the $n < \infty$ model for sufficiently large n.

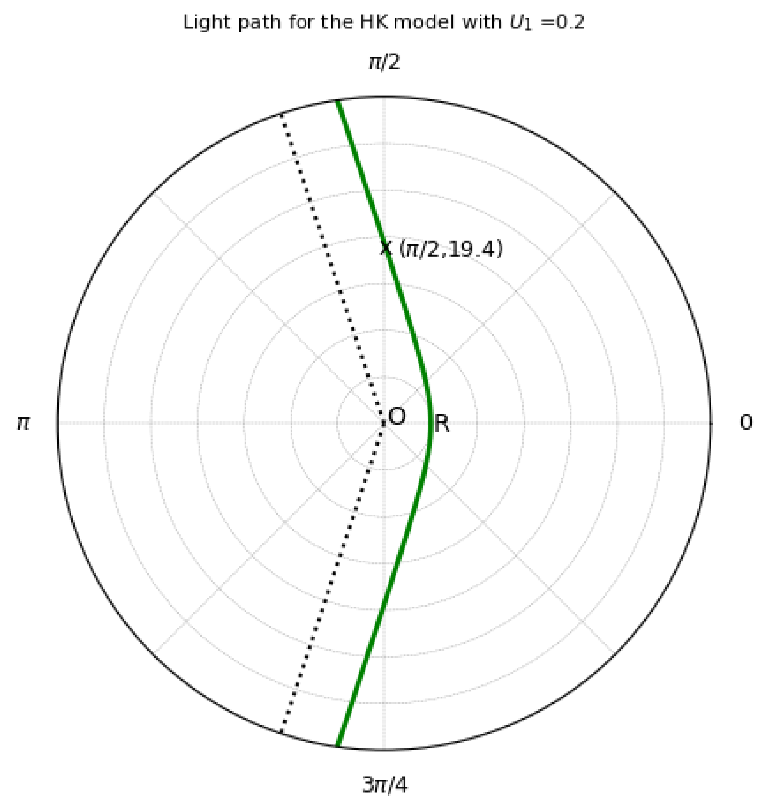

Figure 1. Light path in the HK model for $U_1 = 0.2$ with dotted asymptotic lines out to 35 Schwarzschild radii α. As an example, the location of the light path member $(\phi,\ r_\phi) = (\pi/2\ radians, \approx 19.4\alpha)$ is marked. Each radial unit on this graph equals 5α, the distance of the turning point R from the center of the mass at the origin O. The dotted light path asymptotic lines are located at ≈ 1.9 radians $= 108.4°$ and ≈ 4.4 radians $= 256.6°$. The light path model $n = \infty$ would give identical results.

D. Relationships derived from the two models.

Mathematical relationships for some coefficients in the $F_k\left(\frac{M}{R},\rho\right)$ term of the $n=\infty$ model will now be derived using the fact that the HK model $Q(U_1,\phi)$ of section C is equivalent to the $\theta_\infty\left(\frac{M}{R},\rho\right)$ of section B, where recall that $Q(U_1,\phi)$ is in closed form while $\theta_\infty\left(\frac{M}{R},\rho\right)$ is the sum of an infinite series of terms $F_k\left(\frac{M}{R},\rho\right)$.

First solve $Q(U_1,\phi)$ for $\phi=\phi(U_1,Q)$ to allow the same dimensionality as $\theta_\infty\left(\frac{M}{R},\rho\right)$.

Then $$\phi(U_1,Q) = \left(\frac{1}{\gamma}\right) sn^{-1}\left(\sqrt{\frac{f_{13}\left(1-\frac{e_2}{Q}\right)}{f_{23}\left(1-\frac{e_1}{Q}\right)}},\ k\right) \tag{D.1}$$

where $f_{ij}\equiv e_i-e_j$ and $sn^{-1}=sn^{-1}(x,y)$ is the inverse of the elliptical function $sn=sn(x,y)$ where here x = amplitude and y = modulus. We have taken the positive square root for the amplitude of sn^{-1}.

Note that for our situation $x=\sqrt{\frac{f_{13}\left(1-\frac{e_2}{Q}\right)}{f_{23}\left(1-\frac{e_1}{Q}\right)}}$ is the amplitude, and $\mathrm{y}=k$ is the modulus using a conventional notation which should not be confused with our index k which we have been using for indexing some coefficients and variables. The integral formula for the inverse sn function is

$sn^{-1}(x,y)=\int_0^x\frac{dt}{\sqrt{(1-t^2)(1-yt^2)}}$. This function can also be represented as a power series, see [2]. The integral formula is used to determine the light paths in figures 1-3.

To adjust for the different angle conventions of the two models we note that $\phi+\theta_\infty=\pi/2$. This can be seen by superimposing the three dimensional (x,y,z) coordinate system over the spherical coordinate system (physics convention) and noting that θ_∞ is measured from the z-axis or positive polar axis and the vertex angle ϕ is measured from the y-axis so they are complementary angles. Then

$$\theta_\infty\left(\frac{M}{R},\rho\right)=\frac{\pi}{2}-\phi(U_1,Q) = \frac{\pi}{2}-\left(\frac{1}{\gamma}\right) sn^{-1}\left(\sqrt{\frac{f_{13}\left(1-\frac{e_2}{Q}\right)}{f_{23}\left(1-\frac{e_1}{Q}\right)}},\ k\right). \tag{D.2}$$

Replacing the LHS we have

$$\sum_1^\infty F_k\left(\frac{M}{R},\rho\right)+C_\infty\left(\frac{M}{R}\right) = \frac{\pi}{2} - \left(\frac{1}{\gamma}\right) sn^{-1}\left(\sqrt{\frac{f_{13}\left(1-\frac{e_2}{Q}\right)}{f_{23}\left(1-\frac{e_1}{Q}\right)}},\ k\ \right). \tag{D.3}$$

Expanding the $F_k\left(\frac{M}{R},\rho\right)$ terms we obtain

$$\sin^{-1}\rho+\sum_2^\infty(-1)^{k-1}\frac{2k-3}{2^{k-2}}\left[t_k\sin^{-1}\rho+\frac{N_k(\rho)}{(1-\rho^2)^{\frac{2k-3}{2}}}\right]\left(\frac{M}{R}\right)^{k-1}+\ C_\infty\left(\frac{M}{R}\right) = \frac{\pi}{2} - \left(\frac{1}{\gamma}\right) sn^{-1}\left(\sqrt{\frac{f_{13}\left(1-\frac{e_2}{Q}\right)}{f_{23}\left(1-\frac{e_1}{Q}\right)}},\ k\ \right). \tag{D.4}$$

Examining (D.3) or (D.4), we have an infinite series whose terms are difficult to compute for large k and for which a general formula may not be feasible to construct as argued in Appendix B, so the expression cannot be analytically determined. This makes computation to arbitrary accuracy infeasible using the LHS of (D.4). Yet, on the RHS of the equation we have a closed expression with a numerical value that is readily computed to any degree of accuracy. This result is made possible due to the metric of General Relativity, i.e., gravity, and our two different but equivalent models.

We can get a "leaner" result of (D.4) by evaluating this equation for $\rho = 0, 1$ and the corresponding Q values. Since each resulting equation contains $C_\infty\left(\frac{M}{R}\right)$ we can set them both equal to $C_\infty\left(\frac{M}{R}\right)$ to obtain

$$\sum_2^\infty\ (-1)^{k-1}\frac{2k-3}{2^{k-2}}\left\{[N_k(0)\,]-\left[t_k\cdot\frac{\pi}{2}\right]\right\}\left(\frac{M}{R}\right)^{k-1}=\frac{\pi}{2}-\left(\frac{1}{\gamma}\right) sn^{-1}\left(\sqrt{\frac{f_{13}e_2}{f_{23}e_1}},k\ \right) \tag{D.5}$$

Mass terms appear on both sides of this equation since f_{ij} and e_k depend on U_1. So we can informally say that this equation is the result of gravity, or more precisely, the result of our different equivalent models of a light path in General Relativity.

Since the RHS side of (D.5) is a function of U_1 we write

$$H(U_1)\equiv\frac{\pi}{2}-\left(\frac{1}{\gamma}\right) sn^{-1}\left(\sqrt{\frac{f_{13}e_2}{f_{23}e_1}},k\ \right). \tag{D.6}$$

Note that H is easily computed once we specify the mass ratio or equivalently U_1 where $(1/2)U_1 = M/R$, so H equals some real number.

Using the relationship between U_1 and the mass ratio M/R we can write (D.5) as

$$\sum_2^\infty (-1)^{k-1} \frac{2k-3}{2^{2k-3}} \left\{ [\, N_k(0)] - \left[t_k \cdot \frac{\pi}{2} \right] \right\} U_1^{\;k-1} = \frac{\pi}{2} - \left(\frac{1}{\gamma}\right) sn^{-1}\left(\sqrt{\frac{f_{13} e_2}{f_{23} e_1}}, k \right) \qquad \text{(D.7)}$$

or more compactly as

$$\sum_2^\infty (-1)^{k-1} \frac{2k-3}{2^{2k-3}} \; D_k U_1^{\;k-1} = \; H(U_1) \qquad \text{(D.8)}$$

where $D_k \equiv [\, N_k(0)] - \left[t_k \cdot \frac{\pi}{2} \right]$. This gives us more information about how t_k relates to $N_k(\rho)$.

Equations (D.4) and (D.8) have the basic characteristics of a function expressed as an infinite sum of a Fourier series where each term in the series has coefficients containing a product of the original function and sine and cosine factors. This is used to decompose the original function to a sum of periodic functions. As with our case, it is very difficult to determine a general formula for the kth term of the Fourier infinite sum. By analogy, the light path function can be expressed as an infinite sum of terms where each term can be thought of as a component of the path. In the next section we explore this further.

E. Decomposing the light path equation.

We start with (D.4) and let the RHS be denoted by $H(Q)$:

$$\sin^{-1}\rho + \sum_2^\infty (-1)^{k-1} \frac{2k-3}{2^{k-2}} \left[t_k \sin^{-1}\rho + \frac{N_k(\rho)}{(1-\rho^2)^{\frac{2k-3}{2}}} \right] \left(\frac{M}{R}\right)^{k-1} + \; C_\infty\left(\frac{M}{R}\right) \; = \; H(Q) \qquad \text{(E.1)}$$

where $C_\infty\left(\frac{M}{R}\right) = = -\frac{\pi}{2}\left[\sum_2^\infty (-1)^{n-1}\left(\frac{2n-3}{2^{n-2}}\right) t_n (M/R)^{n-1}\right] = \; -\frac{15\pi}{8}\left(\frac{M}{R}\right)^2 + \frac{15\pi}{8}\left(\frac{M}{R}\right)^3 +$ $\left(-\frac{693\pi}{128}\left(\frac{M}{R}\right)^4\right) + \cdots\cdots\cdots$.

We now "spread out" the individual terms of C_∞ across the infinite sum as follows.

Due to the way we constructed $C_\infty\left(\frac{M}{R}\right)$, it can be rewritten as $\sum_2^\infty C(n)$ with $C(n) \equiv C'(n)\left(\frac{M}{R}\right)^{n-1}$ where $C'(n)$ is the coefficient of the highest power term of $C_n\left(\frac{M}{R}\right)$ with respect to $\frac{M}{R}$. For example, using the results listed in Appendix A we get

$C(1) = 0$
$C(2) = 0$
$C(3) = -\frac{15\pi}{8}\left(\frac{M}{R}\right)^2$
$C(4) = +\frac{15\pi}{8}\left(\frac{M}{R}\right)^3$
$C(5) = -\frac{693\pi}{128}\left(\frac{M}{R}\right)^4$

and in general

$C(n) = C_n - C_{n-1}$ for $n > 1$ where the $\frac{M}{R}$ argument is omitted.

Then (E.1) can be rewritten as

$$\sin^{-1}\rho + \sum_2^\infty \left[F_k(\rho) + C(k)\right] = H(Q) \qquad \text{(E.2)}$$

where $F_k(\rho)$, with the $\frac{M}{R}$ argument omitted, is the form (B.3). Both $F_k(\rho)$ and $C(k)$ have $(M/R)^{k-1}$ as a factor.

Now define $L_k(\rho) \equiv F_k(\rho) + C(k)$ for all k. For example, using the formula for F_3 in Appendix A, we have

$L_3(\rho) \equiv F_3(\rho) + C(3) = (-1)^{k-1}\,\frac{3}{2}\left[(5/2)\sin^{-1}\rho + \frac{N_k(\rho)}{(1-\rho^2)^{\frac{3}{2}}}\right]\left(\frac{M}{R}\right)^2 - \frac{15\pi}{8}\left(\frac{M}{R}\right)^2$ where $N(\rho) = -\left(\frac{1}{2}\right)\rho^5 + \left(\frac{8}{3}\right)\rho^3 - 2\rho^2 - \left(\frac{3}{2}\right)\rho + (\frac{4}{3})$.

Then (E.2) is simply written as

$$\sum_1^\infty L_k(\rho) = H(Q) \qquad \text{(E.3)}$$

where $L_1(\rho) = F_1(\rho) + \mathrm{C}(1) = \sin^{-1}\rho$.

The LHS of (E.3) is just our $n = \infty$ model expressed in a different form. If the infinite sum is instead finite, then it is the $n < \infty$ model. We can view this as a decomposition of the null geodesic path $H(Q)$ into constituent components $L_k(\rho)$. Taking a cue from Fourier series we call the infinite set of $L_k(\rho)$ the decomposition into 'gravitational components', or simply 'components', of the light path. Equation (E.3) can be interpreted as applying the $L_k(\rho)$ components containing mass to a straight-line path resulting in a curved geodesic.

When rewriting (E.3) as $L_1(\rho) = \sin^{-1}\rho = H(Q) + \sum_2^\infty\left(-L_k(\rho)\right)$ it can be interpreted as applying the inverse of the components to the light path to undo the influence of mass and restore the path to a straight-line.

A visual perspective of the gravitational components is presented in fig. 2 where the graph of $H(Q)$ and the first seven components $L_k(\rho)$ are shown for $U_1 = 0.4$. The straight line $L_1(\rho) = \sin^{-1}\rho$ and the HK light path pass through the turning point R and are the only null geodesics; L_1 is the geodesic when there is no mass present. The paths of our mathematical objects $L_k(\rho)$ for $k > 1$ are not null geodesics and are unphysical. They are shown simply to display their relationship relative to L_1 and the HK path. They do not pass through the turning point R but calculations show they all intersect at $(\pi/2, r = R)$ or $(3\pi/2, r = R)$.

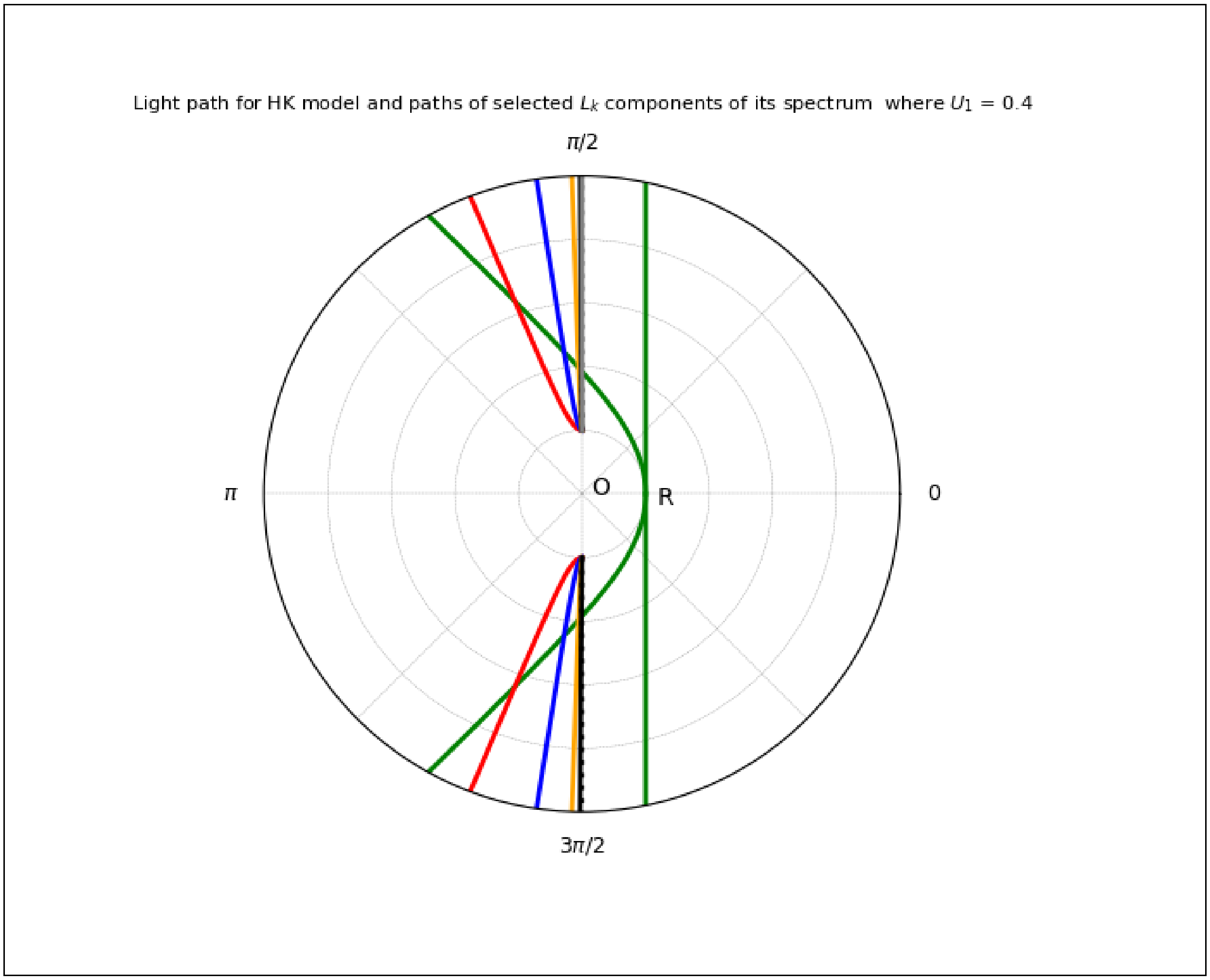

Figure 2. Polar coordinate graph of the light path(green)for $U_1 = 0.4$ and components L_1 (brown) to L_7 out to a radius of $5R$ where R is the turning point of the light path. Both upper and lower branches are plotted. The light path and L_1 (the green straight line) are null geodesics. The remaining L_k components shown are unphysical paths but are shown to display their relation to L_1 and the light path. The L_2 component is shown in red and L_3 in blue. The remaining components are clustered near an angular value of $\pi/2$ radians.

The spread of the L_k components in fig.2 show that L_2(red path), of order $(M/R)^{2-1=1}$ makes the largest individual contribution to the bending of the L_1 light path into the HK curved light path. This is consistent with the well-known calculation of 1.75 arc-seconds for the deflection of the light path passing near the sun. It only requires the first two terms of the Taylor series (for example, see Section C.2 Step E of [8]) where the first term has no mass ratio factor and the second term is to first order in M/R which for the sun is the small value $\approx 2x10^{-6}$.

We can think of the aggregate effect of $\{L_k\}_2^\infty$ as transforming the straight-line light path geodesic into a curved light path. Furthermore, any finite subset of $\{L_k\}_2^\infty$ has the effect of curving the straight-line light to a lesser degree as shown in fig. 3 where the effect of $\{L_k\}_2^7$ and

$\{L_k\}_2^2$ on the straight line path L_1 is shown. This is an alternate view of the standard picture of space-time ‘bending’ in the vicinity of a massive body. In this alternate view, the effect of mass on a path is to “perturb” the straight-line path with the unphysical components $\{L_k\}_2^\infty$. Stretching the analogy even further, we can also compare this effect to that which occurs in quantum particle physics where the state of a particle is transformed via a matrix, which can be viewed as a rotation in an abstract unphysical space.

Recall that our analysis has been restricted to the Schwarzschild metric and a mass ratio $M/R <$ 1/3 or equivalently $U_1 < 2/3$. It would be interesting to extend these results to other gravitational metrics and any mass value.

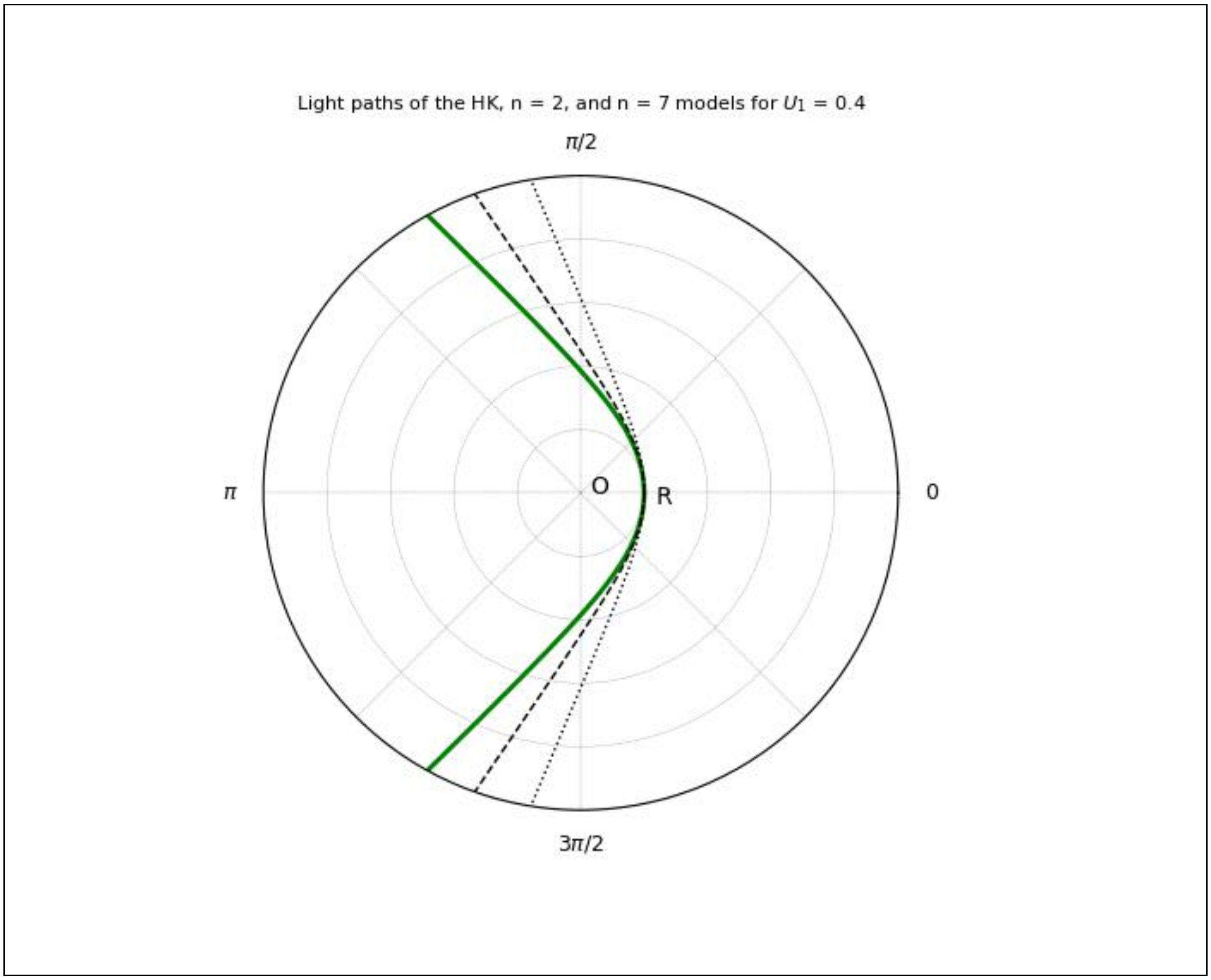

Figure 3. Light paths for the HK (green), $n = 7$ (dashed), and $n = 2$ (dotted) models out to a radial distance of $5R$ where R is the turning point. The $n = 7$ model corresponds to $\{L_k\}_1^7$ and the $n = 2$ model corresponds to $\{L_k\}_1^2$. Since the $n = 7$ path is generated from more L_k gravitational components than the $n = 2$ path it is bent more toward the HK null geodesic. i.e it is a better approximation of the HK light path. If all the components are applied, corresponding to the $n = \infty$ model, then it would produce the HK path.

Summary.

This work expands the understanding of light path equations in GR through a detailed analysis of two equivalent models—one leveraging infinite series and the other employing closed-form Jacobian elliptic functions. By comparing these approaches, the paper derives some mathematical relationships and proposes a decomposition framework that breaks down geodesic trajectories into 'gravitational' components. The findings not only offer deeper insights into the mathematical implications of these models but also open avenues for a new interpretation of the effects of mass on light paths in curved space-time.

Endnotes:

1. The Schwarzschild metric can be represented by the proper time line element $d\tau^2 = (1 - \frac{2m}{r})\, dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 (sin\theta)^2 d\varphi^2$ which contains the gravitational tensor components. This characterizes the curvature of space-time around a spherically symmetric body in a vacuum and is a solution of the Einstein Field Equation for that physical system. Proper time is denoted by τ, m is the mass of the central body in meters, r is radial distance from the center, θ is the polar angle, and φ is the azimuth. In this metric, the Schwarzschild radius of the body is the $2m$ quantity.

2. In GR, mass is frequently converted to units of length in meters (gravitational units'), using the conversion factor G/c^2, as is done in the Schwarzschild metric.

Appendix A: List of antiderivatives F_k and constants of integration C_k.

Antiderivatives:

$$F_1 = \sin^{-1}\rho$$

$$F_2 = -\frac{(-\rho^2 - \rho + 2)}{(1-\rho^2)^{\frac{1}{2}}}\left(\frac{M}{R}\right)$$

$$F_3 = \left(\frac{3}{2}\right)\left[\frac{5}{2}\sin^{-1}\rho + \frac{\left(-\left(\frac{1}{2}\right)\rho^5 + \left(\frac{8}{3}\right)\rho^3 - 2\rho^2 - \left(\frac{3}{2}\right)\rho + \left(\frac{4}{3}\right)\right)}{(1-\rho^2)^{\frac{3}{2}}}\right]\left(\frac{M}{R}\right)^2$$

$$F_4 = \left(\frac{5}{4}\right)\left[-3\sin^{-1}\rho - \frac{\left(\left(-\frac{1}{3}\right)\rho^8 - \left(\frac{8}{3}\right)\rho^6 - \left(\frac{77}{15}\right)\rho^5 + 16\rho^4 + \left(\frac{25}{3}\right)\rho^3 - \left(\frac{61}{3}\right)\rho^2 - 4\rho + \left(\frac{122}{15}\right)\right)}{(1-\rho^2)^{\frac{5}{2}}}\right]\left(\frac{M}{R}\right)^3$$

$$F_5 = \left(\frac{7}{8}\right)\left[\frac{99}{8}\sin^{-1}\rho + \frac{\left(\left(-\frac{1}{4}\right)14\rho^{11} - \left(\frac{11}{8}\right)\rho^9 + 4\rho^8 + \left(\frac{148}{7}\right)\rho^7 - 32\rho^6 - \left(\frac{185}{4}\right)\rho^5 + 64\rho^4 + \left(\frac{157}{4}\right)\rho^3 - 52\rho^2 - \left(\frac{91}{8}\right)\rho + \left(\frac{104}{7}\right)\right)}{(1-\rho^2)^{\frac{7}{2}}}\right]\left(\frac{M}{R}\right)^4$$

$F_k = (-1)^{k-1}\, \frac{2k-3}{2^{k-2}}\left[t_k \sin^{-1}\rho + \frac{N_k(\rho)}{(1-\rho^2)^{\frac{2k-3}{2}}}\right]\left(\frac{M}{R}\right)^{k-1}$ holds for all k where the meaning of the variables and constants shown is defined in section B.

Constants of Integration:

$C_1 = 0$
$C_2 = 0$
$C_3 = C_2 - \frac{15\pi}{8}\left(\frac{M}{R}\right)^2$
$C_4 = C_3 + \frac{15\pi}{8}\left(\frac{M}{R}\right)^3$
$C_5 = C_4 - \frac{693\pi}{128}\left(\frac{M}{R}\right)^4$
$C_n = C_{n-1} - F_n(\rho = 1)$ for n ≥ 2, $F_n(\rho = 1) = (-1)^{n-1}\frac{2n-3}{2^{n-2}}\left[t_n \cdot \frac{\pi}{2}\right]\left(\frac{M}{R}\right)^{n-1}$.

Appendix B: The general form of F_k

With some assumptions to be described, we argue that the general form $F_k = (-1)^{k-1}\frac{2k-3}{2^{k-2}}\left[t_k \sin^{-1}\rho + \frac{N_k(\rho)}{(1-\rho^2)^{\frac{2k-3}{2}}}\right]\left(\frac{M}{R}\right)^{k-1}$ described in section B holds for all k and $0 \leq \rho \leq 1$. An example for some of the derivations is given for F_4.

We first discuss the case $0 \leq \rho < 1$.

As explained in section B, F_k denotes the antiderivative for the kth term of a certain integrand derived from the Schwarzschild metric where the proper time τ is set to zero. See endnote 1. So

$\int\left(\frac{1}{\sqrt{1-\rho^2}}\right)d\rho$ is the first integrand and F_1 the corresponding antiderivative, and

$\int \frac{2k-3}{2^{k-2}}\frac{1}{\sqrt{1-\rho^2}}\left(\left(\frac{1-\rho^3}{1-\rho^2}\right)\frac{M}{R}\right)^{k-1} d\rho$ is the kth integral for $k \geq 2$ and F_k the antiderivative.

Since $\int\left(\frac{1}{\sqrt{1-\rho^2}}\right)d\rho = \sin^{-1}\rho$ we define $F_1 = \sin^{-1}\rho$.

Now consider the case $k \geq 2$ and for now ignore the factors $\frac{2k-3}{2^{k-2}}$ and $\left(\frac{M}{R}\right)^{k-1}$ since the integration element is $d\rho$. Then the remaining integral expressions can be put in the form

$$\int \frac{1}{\sqrt{1-\rho^2}}\left(\frac{1-\rho^3}{1-\rho^2}\right)^{k-1} d\rho = (-1)^{k-1}\int \frac{(\rho^3-1)^{k-1}}{(1-\rho^2)^{\frac{2k-1}{2}}}d\rho = (-1)^{k-1}\left[\frac{N_k(\rho)}{(1-\rho^2)^{\frac{2k-3}{2}}} + t_k\int\left(\frac{1}{\sqrt{1-\rho^2}}\right)d\rho\right] \quad (1)$$

where $N_k(\rho) = \sum_0^{3k-3} p_i\rho^i$ is a polynomial and t_k is some constant. For the justification refer to the techniques cited in [1], section 2.252. This technique is, for example, utilized by the online integral calculator at the URL www.Integral-Calculator.com where the explicit steps in the calculation are displayed for a given k. We are simply generalizing these steps and adding details to show it will result in our desired general form as claimed.

Our immediate goal is to determine t_k and the p_i polynomial coefficients. Ignoring the $(-1)^{k-1}$ factor for now, we can now operate on each side of the equation with $\frac{d}{d\rho}$ and apply differentiation rules to obtain

$$\frac{(\rho^3-1)^{k-1}}{(1-\rho^2)^{\frac{2k-1}{2}}} = \frac{\sum_1^{3k-4} ip_i\rho^{i-1}}{(1-\rho^2)^{\frac{2k-3}{2}}} + \frac{(2k-3)\rho\sum_0^{3k-4} p_i\rho^i}{(1-\rho^2)^{\frac{2k-1}{2}}} + \frac{t_k}{\sqrt{1-\rho^2}}.$$

Now bring everything to a common denominator:

$$\frac{(\rho^3-1)^{k-1}}{(1-\rho^2)^{\frac{2k-1}{2}}} = \frac{\sum_1^{3k-4} ip_i\rho^{i-1}}{(1-\rho^2)^{\frac{2k-1}{2}}}(1-\rho^2) + \frac{(2k-3)\rho\sum_0^{3k-4} p_i\rho^i}{(1-\rho^2)^{\frac{2k-1}{2}}} + \frac{t_k(1-\rho^2)^{k-1}}{(1-\rho^2)^{\frac{2k-1}{2}}}.$$

Then equate the numerators:

$$(\rho^3-1)^{k-1} = \left[\sum_1^{3k-4} ip_i\rho^{i-1}\right](1-\rho^2) + (2k-3)\rho\sum_0^{3k-4} p_i\rho^i + t_k(1-\rho^2)^{k-1}.$$

After restoring the $(-1)^{k-1}$ factor and some additional algebra we have

$$(-1)^{k-1}(\rho^3-1)^{k-1} = (-1)^{k-1}\left[\sum_1^{3k-4} ip_i\rho^{i-1} - \sum_0^{3k-4} ip_i\rho^{i+1} + (2k-3)\sum_0^{3k-4} p_i\rho^{i+1} + t_k(1-\rho^2)^{k-1}\right]. \quad (2)$$

The binomial terms can then be expanded. Then a set of $3k-2$ linear equations in $3k-2$ unknowns (t_k and p_i) are formed by matching up terms with the same power of ρ and canceling these out. Then t_k and p_i are solved for. We will know demonstrate this for a specific value of k but the method can be applied to any value using matrix/determinant calculators.

Let $k = 4$. Then the above procedure will lead to

$$-(\rho^3-1)^3 = -\left[\sum_1^8 ip_i\rho^{i-1} - \sum_0^8 ip_i\rho^{i+1} + 5\sum_0^8 p_i\rho^{i+1} + t_k(1-\rho^2)^3\right]$$

for (2). Matching up terms with the same power of ρ and canceling these out gives the set of linear equations:

$$
\begin{aligned}
1 &= 5p_8 - 8p_8 = -3p_8 \\
0 &= 5p_7 - 7p_7 = -2p_7 \\
0 &= 8p_8 - 6p_6 + 5p_6 = 8p_8 - p_6 \\
-3 &= 7p_7 - 5p_5 + 5p_5 - t_k = 7p_7 - t_k \\
0 &= 6p_6 - 4p_6 + 5p_4 = 6p_6 + p_4 \\
0 &= 5p_5 - 3p_3 + 5p_3 + 3t_k = 5p_5 + 2p_3 + 3t_k \\
3 &= 4p_4 - 2p_2 + 5p_2 = 4p_4 + 3p_2 \\
0 &= 3p_3 - p_1 + 5p_1 - 3c = 3p_3 + 4p_1 - 3t_k \\
0 &= 2p_2 + 5p_0 \\
-1 &= p_1 + t_k.
\end{aligned}
\tag{3}
$$

The first equation corresponds to the ρ^9 terms, the next equation corresponds to the ρ^8 terms, and so on, down to the last equation for the ρ^0 (constant) terms.

Without resorting to software to find a solution we immediately see from examining the equations of (3) with only one variable that $p_8 = -\frac{1}{3}$, $p_7 = 0$. Then a series of simple substitutions leads to the remaining solutions

$$p_0 = \frac{122}{15},\ p_1 = -4,\ p_2 = -\frac{61}{3},\ p_3 = \frac{25}{3},\ p_4 = 16,\ p_5 = -\frac{77}{15},\ p_6 = -\frac{8}{3},\ t_k = 3\,.$$

Our polynomial $N_k(\rho)$ is now determined. Continuing with our example, (1) is now

$$-\int \frac{(\rho^3-1)^3}{(1-\rho^2)^{\frac{7}{2}}} d\rho = \frac{-N_k(\rho)}{(1-\rho^2)^{\frac{5}{2}}} - 3\int\left(\frac{1}{\sqrt{1-\rho^2}}\right) d\rho \text{ or } \int \frac{(1-\rho^3)^3}{(1-\rho^2)^{\frac{7}{2}}} d\rho = \frac{-N_k(\rho)}{(1-\rho^2)^{\frac{5}{2}}} - 3\int\left(\frac{1}{\sqrt{1-\rho^2}}\right) d\rho$$

where

$$N_k(\rho) = \left(-\frac{1}{3}\right)\rho^8 + (-8/3)\rho^6 + \left(-\frac{77}{15}\right)\rho^5 + 16\rho^4 + \left(\frac{25}{3}\right)\rho^3 + \left(-\frac{61}{3}\right)\rho^2 + (-4)\rho + \left(\frac{122}{15}\right).$$

Then

$$F_4 = -3\sin^{-1}\rho \ - \left[\left(-\frac{1}{3}\right)\rho^8 + (-8/3)\rho^6 + \left(-\frac{77}{15}\right)\rho^5 + 16\rho^4 + \left(\frac{25}{3}\right)\rho^3 + \left(-\frac{61}{3}\right)\rho^2 + (-4)\rho + \left(\frac{122}{15}\right)\right]/(1-\rho^2)^{5/2}.$$

which is F_4 in Appendix A after reintroducing the $\frac{2k-3}{2^{k-2}} = \frac{5}{4}$ and $\left(\frac{M}{R}\right)^{k-1=3}$ factors.

Abstracting these steps and performing for the general case leads to our general form (B.3) for $0 \le \rho < 1$ which is

$$F_k\left(\frac{M}{R}, \rho\right) = (-1)^{k-1}\ \frac{2k-3}{2^{k-2}}\left[t_k \sin^{-1}\rho + \frac{N_k(\rho)}{(1-\rho^2)^{\frac{2k-3}{2}}}\right]\left(\frac{M}{R}\right)^{k-1}. \tag{4}$$

This conclusion makes the assumption, which we do not attempt to show, that a solution always exists for the linear equations formed by the set $\{p_i,\ t_k\}$.

We can argue that (4) also holds for $\rho = 1$ since the light trajectory is continuous in the region near $r = R$ $(\rho = 1)$ as it passes through the turning point. Then our model equation should also have this property. Then the term $\frac{N_k(\rho)}{(1-\rho^2)^{\frac{2k-3}{2}}}$ is not a singularity for $\rho = 1$.

Since $\frac{N_k(\rho)}{(1-\rho^2)^{\frac{2k-3}{2}}} = \frac{N_k(\rho)}{\sqrt{1-\rho}\,(1-\rho)^{k-2}\,(1+\rho)^{\frac{2k-3}{2}}} = \frac{\sqrt{1-\rho}\,N_k(\rho)}{(1-\rho)^{k-1}\ (1+\rho)^{\frac{2k-3}{2}}}$, this implies $(1-\rho)^{k-1}$ is a factor of the polynomial $N_k(\rho)$.

Then $\frac{N_k(\rho)}{(1-\rho^2)^{\frac{2k-3}{2}}} = \frac{\sqrt{1-\rho}\ M_k(\rho)}{(1+\rho)^{\frac{2k-3}{2}}}$ for some polynomial $M_k(\rho)$ of degree $2k-3$ and therefore $\frac{N_k(\rho)}{(1-\rho^2)^{\frac{2k-3}{2}}}$ drops out of (3) when $\rho = 1$. Therefore

$$F_k\left(\frac{M}{R},\ \rho = 1\right) = (-1)^{k-1}\ \frac{2k-3}{2^{k-2}}\left[t_k \cdot \frac{\pi}{2}\right]\left(\frac{M}{R}\right)^{k-1}$$

as we had claimed in section B and the general form (4) holds for all ρ in the entire interval $[0,1]$ when $k \ge 2$.

We had previously set $F_1\left(\frac{M}{R},\ \rho\right) = \sin^{-1}\rho$, so $F_1\left(\frac{M}{R},\ \rho = 1\right) = \pi/2$ for this case.

References:

[1] Byrd, Paul F. and Friedman, Morris D., "Handbook of Elliptic Integrals for Engineers and Scientists", 2nd edition Revised. Springer-Verlag. (1971)

[2] Carlson, B. C., "Power series for inverse Jacobian elliptic functions" (PDF). Mathematics of Computation. **77** (263): 1615–1621. (2008)

[3] Carroll, Sean M., "*Spacetime and Geometry: An Introduction to General Relativity*", 1st Edition. Cambridge University Press. (2019)

[4] Einstein, A., "*The Meaning of Relativity"*. Methuen & Co. Ltd. London, Fourth edition, pp 88-89. (1950). This book is based on a series of lectures he gave in 1921 at Princeton University and contains his now famous correct deflection prediction based on General Relativity. The original paper containing the prediction was published in 1916 as "Die Grundlage der allgemeinen Relativitätstheorie", Annalen der Physik, 49.

[5] Goldoni, Emanuelle and Stefanini, Ledo, " A Century of Light-Bending Measurements: Bringing Solar Eclipses into the Classroom", arXiv:2002.01179v1,(2020). In the context of a learning experience for students, describes several key measurements of light deflection from 1919 through 2017.

[6] Hioe, F.T. and Kuebel, David, "Characterizing Planetary Orbits and the Trajectory of Light", arXiv:1001.0031v1. (2009)

[7] Landau, L.D. and Lifschitz, E.M., *"The Classical Theory of Fields"*, section 101, 4rth Revised English Edition. Elsevier. (1975)

[8] Malczewski, G. and Selig, D., "A History of Light Deflection with Newtonian and General Relativity Perspectives", VixRa 2304-0099, version 4. (2024)

[9] Malczewski, G., "A comparison of light path equations in General Relativity using a Taylor series approach vs Jacobian elliptic function", VixRa 2411-006, version 2. (2024)

[10] Wald, Robert M., "*General Relativity*". The University of Chicago Press. (1984)