Investigating prime gaps through zeta behaviour. A reexamination of the Riemann hypothesis

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Abstract

In this research prime gaps will be investigated through their zeta behaviour. A formulation will be presented that links prime gaps to singularities in $\zeta(s)$. This is achieved by identifying a zeta function for Goldbach partition and extending it to the Euler product. A zeta function is formulated that encodes information about Goldbach partitions. We begin the paper by examining the logarithmic form of a complex variable and it's decompostion to real and imaginary parts.

The applications will be extended to Goldbach partition and the Riemann hypothesis.

Keywords

Zeta function for Goldbach partition; Riemann hypothesis; prime gaps and singularities in $\zeta(s)$

Logarithmic form of the complex variable and its decomposition to real and complex parts. Reformulation of the Riemann zeta function

Consider the logarithmic complex variable $z=\frac{\ln(-x)}{y}$. It can be decomposed into real and imaginary parts at follows: $z=\frac{\ln(-x)}{y}$ $=$ $\frac{\ln(-1)}{y}+\frac{\ln x}{y}=\frac{lnx}{y}+i\frac{\pi}{y}$. The Riemann hypothesis requires the real part of it's complex variable to be $1/2$, in which case $y=\ln^2 x$ and $z=\zeta(s)=\frac{1}{2}+\frac{i\pi}{\ln^2 x}$. By this formulation the relationship between $\ln x$ and $\zeta(s)$ is given by $\ln(x)=\sqrt{\frac{i\pi}{\zeta(s)-\frac{1}{2}}}$.

If $\zeta(s)-\frac{1}{2}=i\gamma$, then $\ln x=\sqrt{\frac{\pi}{\gamma}}$. In the Riemann hypothesis $s=\frac{1}{2}+it$. This means that $t=\frac{\pi}{\ln^2 x}$ or $\ln x=\sqrt{\frac{\pi}{t}}$. It also means that $x=e^{\sqrt{\frac{\pi}{t}}}$.

The number of primes is therefore asymptotically equal to $\frac{te^{\sqrt{\frac{\pi}{t}}}}{\pi}$.

Analysis of the zeroes of Riemman zeta function

- The 10^{th} nontrivial zero is $t=49.774$ and is equivalent to $x=53.017$. This means that number of primes is asymptotically equal to $49.774\times e^{\frac{\pi/49.774}{\pi}}\approx 17$. The actual number of primes is 16.
- The first nontrivial zero is $t=14.135$ and is equivalent to $x=17.653$. This means that number of primes is asymptotically equal to $17.653\times e^{\frac{\pi/17.653}{\pi}}\approx 7$. The actual number of primes is 7.
- The second nontrivial zero is $t=21.022$ and is equivalent to $x=24.410$. This means that number of primes is asymptotically equal to $21.022\times e^{\frac{\pi/21.022}{\pi}}\approx 8$. The actual number of primes is 8.
- The third nontrivial zero is $t=25.011$ and is equivalent to $x=28.358$. This means that number of primes is asymptotically equal to $25.011\times e^{\frac{\pi/25.011}{\pi}}\approx 9$. The actual number of primes is 9.
- The fourth notrivial zero is $t=30.425$ and is equivalent to $x=33.735$ and so on.

This means that number of primes is asymptotically equal to $30.425\times e^{\frac{\pi/30.425}{\pi}}\approx 11$. The actual number of primes is 11.

These results show that The Riemann hypothesis predicts the number of primes very accurately

Observations of the analysis

1. The nontrivial zeros are typically associated with oscillations in the error term of the prime number theorem, and the above formula may provide an alternative heuristic connection.
2. The inclusion of an exponential correction factor $e^{\frac{\pi}{t}}$ is intriguing, as it introduces a dependency on in a way that needs further theoretical justification.

3. These results might suggest a deeper structure in how the zeros encode prime distribution beyond standard asymptotics.

A zeta function for Goldbach partition

In the paper reference [1] the gap, "g between two primes, p_1 and p_2 is given by $g=2\sqrt{m^2-p_1p_2}$ where m represents the mean of the two primes. A logarithmic zeta function encoding information about gaps between primes would therefore be given by $\zeta(X)=\frac{\ln(-\frac{1}{n}\sqrt{m^2-p_1p_2})}{m+n}$ where $n=-\frac{g}{2}$. The decomposition of the Goldbach partition zeta function therefore is $\zeta(X)=\frac{\ln(-\frac{1}{n}\sqrt{m^2-p_1p_2})}{m+n}=\frac{\ln\frac{1}{n}\sqrt{m^2-p_1p_2}}{m+n}+i\frac{\pi}{m+n}$ and $p_1\neq p_2$. Where $p_1=p_2$ then $\zeta(X)=\frac{\ln(\sqrt{m^2-p_1p_2+1})}{m}$. Goldbach partition therefore requires solving $\zeta(X)=0$.

A further analysis. A Complexity zeta for the Euler product.

Consider the Euler product $\zeta(s)=\prod\frac{p_i^s}{p^s-1}$. The above product generates a zero whenever $s=-\infty$. We will formulate the complex variable s such that it will always generate a zero at some singularity. If

$$\zeta(s)=-\zeta(\tfrac{1}{X})=\zeta(-\frac{m+n}{\ln(-1/n\sqrt{m^2-p_1p_2}})=\zeta(-\frac{m+n}{i\pi+\ln(1/n\sqrt{m^2-p_1p_2})})=\zeta(-\frac{(m+n)(i\pi-\ln(1/n\sqrt{m^2-p_1p_2}))}{-\pi^2-\ln^2(1/n\sqrt{m^2-p_1p_2})}.$$

This formulation links prime gaps to sigularities in $\zeta(s)=0$. Zero are generated when we for any prime gap $n=-\frac{g}{2}$. It is also observed that $m+n=p_1$. For twin prime pairs we use $n=-1$ and $m=p_1+1\,|p_2>p_1$. For gap g between consecutive primes use $n=-g/2$ and $m=p_1+g/2$.

Reference

[1] Samuel Bonaya Buya and John Bezaleel Nchima (2024). A Necessary and Suffi- cient Condition for Proof of the Binary Gold- bach Conjecture. Proofs of Binary Gold- bach, Andrica and Leg- endre Conjectures. Notes on the Riemann Hypothesis. In- ternational Journal of Pure and Applied Mathematics Research, 4(1), 12-27. doi: 10.51483/IJPAMR.4.1.2024.12-27.

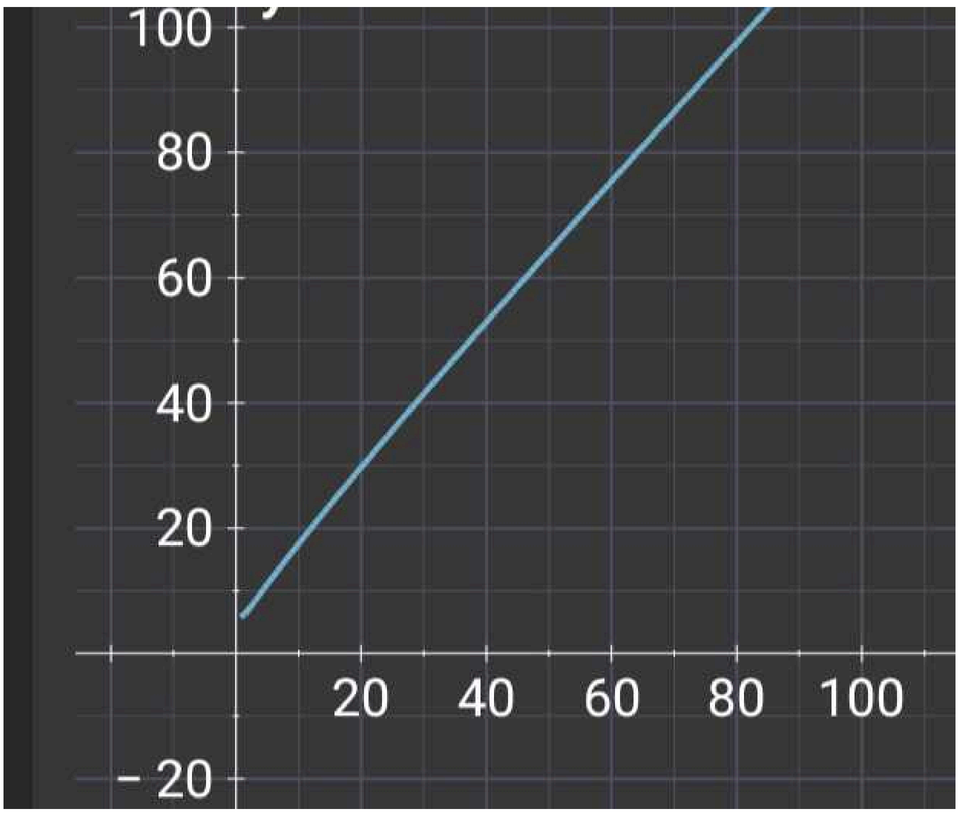

Figure 1: Graph of t against x