Proof That π And √2 Are Rational

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ABSTRACT: In this short paper, I've used Wallis' product formula to prove that π is rational, while the product expansion of the cosine function was used to prove that √2 is rational.

I. Proof That π Is Rational

Assume $\frac{\pi}{2} = \frac{r}{n}$, where r and n are very large positive integers.

$$\frac{r}{n} = \frac{\pi}{2}$$

Wallis' Product Formula

$$\prod_{k=1}^{\infty} \frac{2k}{2k+1}\frac{2k}{2k-1} = \lim_{m\to\infty}\prod_{k=1}^{m}\frac{4k^2}{4k^2-1} = \frac{\pi}{2}$$

Hence,

$$r = \lim_{m\to\infty}\prod_{k=1}^{m} 4k^2 = \text{ integer, } \textit{and}$$

$$n = \lim_{m\to\infty}\prod_{k=1}^{m}(4k^2-1) = \text{ integer}$$

Therefore, π is a rational number.

II. Proof That √*2* Is Rational

Assume $\sqrt{2} = \frac{a}{b}$, where a and b are very large positive coprime integers.

Cosine Function Product Formula

$$\cos(\pi x) = \prod_{k=0}^{\infty}\left(1-\frac{4x^2}{(2k+1)^2}\right)$$

$$\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \prod_{k=0}^{\infty}\left(1-\frac{1}{4(2k+1)^2}\right)$$

$$\frac{1}{\sqrt{2}} = \prod_{k=0}^{\infty}\left(\frac{(4k+1)(4k+3)}{(4k+2)^2}\right)$$

By taking the reciprocal of both sides

$$\sqrt{2} = \prod_{k=0}^{\infty}\left(\frac{(4k+2)^2}{(4k+1)(4k+3)}\right)$$

$$\sqrt{2} = \lim_{m\to\infty}\prod_{k=0}^{m}\left(\frac{(4k+2)^2}{(4k+1)(4k+3)}\right)$$

Hence,

$$a = \lim_{m\to\infty}\prod_{k=0}^{m}(4k+2)^2 = \text{ even integer, } \textit{and}$$

$$b = \lim_{m\to\infty}\prod_{k=0}^{m}(4k+1)(4k+3) = \text{ odd integer}$$

By squaring √2

$$2 = \lim_{m\to\infty}\prod_{k=0}^{m}\left(\frac{(4k+2)^4}{(4k+1)^2(4k+3)^2}\right) = \frac{a^2}{b^2}$$

$$a^2 = \lim_{m\to\infty}\prod_{k=0}^{m}(4k+2)^4 = \text{ even integer, } \textit{and}$$

$$b^2 = \lim_{m\to\infty}\prod_{k=0}^{m}(4k+1)^2(4k+3)^2 = \text{ odd integer}$$

Therefore, there is *no* contradiction that a and b are coprime integers, and √*2* is a rational number.

REFERENCES:

[1] https://en.wikipedia.org/wiki/Wallis_product
[2] https://en.wikipedia.org/wiki/Square_root_of_2