On a Generalization of Perfect Numbers

By

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1. Summary

This paper presents a notion of perfect numbers relative to arithmetical functions: an arithmetical function f produces a set of *f-perfect numbers*. Two among the many examples considered are small "perturbations" of the normal definition; late in these two sequences, odd perfect numbers appear! (Could the situation be similar for the usual perfect numbers?) Also, this paper generalizes amicable pairs and sociable chains. Mysteries and open problems abound for those who like a challenge. There is also *Mathematica* code to start the reader on his own explorations.

2. Preliminaries

In this discussion, *number* means natural number. Functions are called *arithmetical* if their domains are sets of integers. For simplicity, an arithmetical function in this paper will also be integer-valued and include the set of natural numbers in its domain.

3. Perfection is in the eye of the beholder

The following generalization considers perfect numbers relative to an arithmetical function f. Under this scheme, the usual perfect numbers

6, 28, 496, 8128, 33550336, 8589869056 ....

appear as one among many species of "f-perfect numbers".

Let n have proper divisors d_1, ..., d_k. (A *proper divisor* of n is a divisor of n other than n itself.) Recall that n is *perfect* if n is the sum of its proper divisors, that is,

$$n = \sum_{i=1}^{k} d_i$$

which can be written as

$$\mathrm{id}(n) = \sum_{i=1}^{k} \mathrm{id}(d_i),$$

where id is the identity function defined by id(n) = n. This suggests, for a generalization, replacing id by an arbitrary arithmetical function f.

For an arithmetical function f, the *f-perfect numbers* are the numbers n such that

$$f(n) = \sum_{i=1}^{k} f(d_i),$$

where $d_1, ..., d_k$ are the proper divisors of n.

Hence, the ordinary perfect numbers are id-perfect numbers, where id is the identity function.

4. Two canonical perfect number sequences

The sequence of G-perfect numbers with G(n) = n + 1 begins

4, 10, 44, 2336, 8896, 34432, 449295, ...

and the first g-perfect numbers with g(n) = n - 1 are

12, 196, 368, 1696, 30848, 437745....

These appear in the Online Encyclopedia of Integer Sequences (http://www.research.att.com/~njas/sequences/Seis.html) as A066229 and A066230, respectively. Whenever a sequence in this paper appears in the OEIS, its OEIS number appears with it.

For example, consider the G-perfect number 44. The proper divisors of 44 are 1, 2, 4, 11, 22. G(44) = 45 = 23 + 12 + 5 + 3 + 2 = G(22) + G(11) + G(4) + G(2) + G(1), verifying that 44 is G-perfect.

Here is Mathematica code to generate the G-perfect numbers.

```
f[n_] := n + 1;
Do[If[f[n] == Apply[Plus, Map[f, Drop[Divisors[n], -1]]],
   Print[n]], {n, 2, 10^6}]
```

Obviously, it can be modified for other functions to yield other perfect number sequences.

5. In search of a pattern

The (ordinary) even perfect numbers can be generated by Euclid's expression $2^{n-1}(2^n - 1)$, with prime n making $(2^n - 1)$ a prime. Can one find a similar expression for, say, the G-perfect numbers? Euclid's formula was probably discovered by looking for patterns in the prime factorizations of the perfect numbers. For example, the following Mathematica code

```
t = FactorInteger[{6, 28, 496, 8128, 33550336, 8589869056,
   137438691328}]
```

to get the prime factorizations of the first seven perfect numbers yields

```
{{{2, 1}, {3, 1}}, {{2, 2}, {7, 1}}, {{2, 4}, {31, 1}},
  {{2, 6}, {127, 1}}, {{2, 12}, {8191, 1}},
  {{2, 16}, {131071, 1}}, {{2, 18}, {524287, 1}}}.
```

where the factorization of, say, 496 can be read as 2^4 x 31^1 (from{{2, 4},{31, 1}} in the first line). It is readily observed that the perfect numbers are products of powers of 2 and primes 3, 7, 31, 127, 8191,…. In fact, doing a quick search for this prime sequence on the OEIS identifies it as the sequence of Mersenne primes.

A similar experiment on the G-perfect numbers yields

```
{{{2, 2}}, {{2, 1}, {5, 1}}, {{2, 2}, {11, 1}},
 {{2, 5}, {73, 1}}, {{2, 6}, {139, 1}}, {{2, 7}, {269, 1}},
 {{3, 1}, {5, 1}, {7, 1}, {11, 1}, {389, 1}}}
```

Although several G-perfect numbers follow the pattern of a power of 2 times a prime, the first one is a power of 2, and the seventh one does not even have 2 as a prime factor. Also, the first six numbers have at most two prime factors, but the seventh one has five. The only property these numbers seem to have in common is that every prime factor > 2 has multiplicity 1, but even this could be disproved by some larger G-perfect number.

Comparing the means of, say, the first four (n + k)-perfect numbers for k = -3, -2, -1, 0, 1, 2, 3 reveals only more irregular behavior, as in the following list.

k	Mean of first four (n+k)-perfect numbers
3	60.25
2	2533.00
1	598.50
0	2164.50
-1	568.00
-2	37.50
-3	53637.50

6. Odd perfect numbers and other open problems

What is certain from these considerations is that an expression generating G-perfect numbers is far from obvious. One finds a similar situation with the g-perfect numbers. The reader is invited to tackle these apparently difficult open problems.

However, not all is chaos and darkness: the last numbers appearing in the above G- and g-perfect number lists turn out to be odd. Finally, after initial runs of even perfect numbers, odd perfect numbers appear (even if they are "only" G- or g-perfect)! Could this reflect a similar state of affairs for the usual perfect numbers? At the time of writing, all known perfect numbers are even, but perhaps there is actually an odd one waiting out there in the mysterious realm of the very large….

7. More examples

Obviously, other functions f besides the simple ones mentioned above can be used. If f grows too fast, say $f(n) = n^2$, then no perfect numbers are obtained. At the other extreme, if f grows too slowly, it can generate "too many" perfect numbers. For example, it is not hard to see that the φ-perfect numbers are the powers of 2, and the d-perfect numbers are the squares of primes. (φ(n) denotes Euler's totient function, giving the number of numbers relatively prime to n and less than n; d(n) denotes the number of divisors of n.) The **1**-perfect numbers, where **1** is the constant function mapping each number to 1, are the prime

numbers. (Any nonzero number besides 1 can be used.) The **0**-perfect numbers consist of all numbers > 1.

However, there is a great variety of interesting f. Of course, the choice of which f are interesting is somewhat personal. Nevertheless, most people will agree that there should be relatively few f-perfect numbers. Preferably, the first few f-perfect numbers should share some striking property or pattern, inviting a general conjecture. Or, as in the case of the G- and g-perfect numbers, they should resist simple classification, thereby challenging one to find closed-form generators.

Here are a few more examples. Again, each of these sequences invites a search for a closed-form generator.

Recall that σ(n) denotes the sum of the divisors of n. The sequence of σ-perfect numbers begins

$$198, 608, 11322, 20826, 56608, 3055150.....$$

(A066218). For example, the proper divisors of 198 are 1, 2, 3, 6, 9, 11, 18, 22, 33, 66, 99; and the sum of their σ-values = 1 + 3 + 4 + 12 + 13 + 12 + 39 + 36 + 48 + 144 + 156 = 468 = σ(198). Hence, 198 is σ-perfect. Does an odd σ-perfect number exist?

Let h(n) = σ(φ(n)). The sequence of h-perfect numbers begins

$$2, 88, 328, 5128, 9075, ...$$

(A066226). Again, an odd perfect number appears relatively late in the sequence.

The sequence of j-perfect numbers with j(n) = φ(σ(φ(n))) begins

$$2, 4, 28, 40, 448, ...$$

(A066228). There are no more terms less than 10^5. Are there any more terms? Is this an infinite sequence?

f can even be an arithmetical function defined from a transcendental one. Consider k(n) = Floor(|n sin(n)|) (that is, the greatest integer ≤ |n sin(n)|). The first few k-perfect numbers are

$$3, 6, 10, 34, 50, 91, 222, 364, 1485, 6640, ...$$

(A066245).

Of course, these examples only skim the surface of the ocean of "interesting" perfect number sequences. Perhaps the reader will have fun experimenting with his own functions and discover something intriguing and beautiful in the process.

8. Generalized amicable numbers and sociable chains

f-amicable numbers are defined similarly. Specifically, let D(n) be defined by

$$D(n) = \sum_{i=1}^{k} f(d_i)$$

where d_1, ..., d_k are the proper divisors of n. Then a, b form an *f-amicable pair* if f(b) = D(a) and f(a) = D(b).

It is always delightful to find some striking relationship between apparently unrelated numbers. The discoverer of the amicable pair 220, 284 must have been very pleased. I have had the satisfaction of uncovering three amicable pairs for g(n) = n - 1 and two pairs for G(n) = n + 1. Here are the g-amicable pairs:

<p align="center">100, 110; 1806, 2402; 1872, 3742.</p>

(A066511).
For example, the proper divisors of 100 are 1, 2, 4, 5, 10, 20, 25, 50. g applied to these divisors yields 0, 1, 3, 4, 9, 19, 24, 49, whose sum is 109. So D(100) = g(110). The proper divisors of 110 are 1, 2, 5, 10, 11, 22, 55. g applied to these divisors yields 0, 1, 4, 9, 10, 21, 54, whose sum is 99. So D(110) = g(100). Therefore, 100, 110 form a g-amicable pair.

The G-amicable pairs are

<p align="center">36, 62; 168, 326.</p>

(A066505).

There is only one σ-amicable pair whose members do not exceed 4000: 312, 584. One can modify the above Mathematica code for perfect numbers to deal with amicable pairs. However, exhaustive search for amicable pairs rapidly becomes computationally infeasible.

Can the reader add to these lists?

In general, one can search for *f-sociable number chains*. For example, a *crowd* is a sociable chain of length 3. It is not known whether crowds exist (cf. [W]). Correspondingly, an *f-crowd* is a triple of numbers a, b, c with D(a) = f(b), D(b) = f(c), D(c) = f(a). Can the reader find an f-crowd for an "interesting" function f?

References

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[W] Wells, D. "The Penguin Dictionary of Curious and Interesting Numbers", Penguin Books, 1998.