Solving Recurrence Relations Using Generating Functions

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Abstract. This paper explores the applications of generating functions in solving various recurrence relations. We present the explicit formulas for recurrence relations of different forms, demonstrating step-by-step transformations and manipulations using generating functions. Several cases are examined, including linear and nonlinear recurrences, factorial-based sequences, and Fibonacci-related expressions. The derivations leverage algebraic techniques and characteristic equations to obtain closed-form solutions. The results highlight the power of generating functions in simplifying complex recurrence relations and deriving explicit formulas efficiently.

Solve the recurrence relations using generating functions : $a_n = 3^n - a_{n-1} + 1,\ \ a_0 = 1.$

$$a_n x^n = 3^n x^n - a_{n-1}x^n + x^n = (3x)^n - a_{n-1}x^n + x^n.$$

$$G(x) = \sum_{n=0}^{\infty} a_n x^n.$$

$$\begin{aligned}
G(x) - 1 &= \sum_{n=1}^{\infty} a_n x^n \\
&= \sum_{n=1}^{\infty} ((3x)^n - a_{n-1}x^n + x^n) \\
&= \sum_{n=1}^{\infty} (3x)^n - \sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} x^n \\
&= 3x\sum_{n=1}^{\infty} (3x)^{n-1} - x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} x^{n-1} \\
&= 3x\sum_{n=0}^{\infty} (3x)^{n} - x\sum_{n=0}^{\infty} a_{n}x^{n} + x\sum_{n=0}^{\infty} x^{n} \\
&= \frac{3x}{1-3x} - xG(x) + \frac{x}{1-x}.
\end{aligned}$$

So

$$(1+x)G(x) = \frac{3x}{1-3x} + \frac{x}{1-x} + 1 = \frac{1-3x^2}{(1-3x)(1-x)}.$$

Thus

$$G(x) = \frac{3x}{1-3x} + \frac{x}{1-x} + 1 = \frac{1-3x^2}{(1-3x)(1-x)(1+x)}$$

,i.e.,

$$\begin{aligned}
G(x) &= \frac{3}{4}\frac{1}{1-3x} + \frac{1}{2}\frac{1}{1-x} - \frac{1}{4}\frac{1}{1+x} \\
&= \frac{3}{4}\sum_{n=0}^{\infty}(3x)^n + \frac{1}{2}\sum_{n=0}^{\infty} x^n - \frac{1}{4}\sum_{n=0}^{\infty}(-x)^n \\
&= \sum_{n=0}^{\infty}\left(\frac{3^{n+1}}{4} + \frac{1}{2} - \frac{(-1)^n}{4}\right)x^n
\end{aligned}$$

To conclude

$$a_n = \frac{3^{n+1}}{4} + \frac{1}{2} - \frac{(-1)^n}{4}.$$

Find an explicit formula for a_n if $a_0 = 1$, $a_1 = 1$ and, for all integers $n \geq 2$, $a_n = na_{n-1} + n(n-1)a_{n-2}$.
Let

$$F(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$$

$$\begin{aligned}
F(x) - a_0 - a_1 x &= \sum_{n\geq 2} a_n \frac{x^n}{n!} \\
&= \sum_{n\geq 2} na_{n-1} \frac{x^n}{n!} + \sum_{n\geq 2} n(n-1)a_{n-2} \frac{x^n}{n!} \\
&= \sum_{n\geq 2} a_{n-1} \frac{x^n}{(n-1)!} + \sum_{n\geq 2} a_{n-2} \frac{x^n}{(n-2)!} \\
&= x\sum_{n\geq 2} a_{n-1} \frac{x^{n-1}}{(n-1)!} + x^2 \sum_{n\geq 2} a_{n-2} \frac{x^{n-2}}{(n-2)!} \\
&= x[F(x) - a_0] + x^2 F(x)
\end{aligned}$$

Let $\alpha = \dfrac{-1+\sqrt{5}}{2}$ and $\beta = \dfrac{-1-\sqrt{5}}{2}$. Thus

$$\begin{aligned}
F(x) &= \frac{1}{1-x-x^2} \\
&= -\frac{1}{(\alpha - x)(\beta - x)} \\
&= -\frac{1}{\beta-\alpha}\frac{1}{\alpha - x} + -\frac{1}{\alpha-\beta}\frac{1}{\beta-x} \\
&= -\frac{1}{\beta-\alpha}\frac{1}{\alpha\left(1-\frac{x}{\alpha}\right)} - \frac{1}{\alpha-\beta}\frac{1}{\beta\left(1-\frac{x}{\beta}\right)} \\
&= -\frac{1}{\beta-\alpha}\frac{1}{\alpha}\sum_{n\geq 0}\left(\frac{x}{\alpha}\right)^n - \frac{1}{\alpha-\beta}\frac{1}{\beta}\sum_{n\geq 0}\left(\frac{x}{\beta}\right)^n \\
&= -\frac{1}{\beta-\alpha}\frac{1}{\alpha}\sum_{n\geq 0}\frac{n!}{\alpha^n}\frac{x^n}{n!} - \frac{1}{\alpha-\beta}\frac{1}{\beta}\sum_{n\geq 0}\frac{n!}{\beta^n}\frac{x^n}{n!} \\
&= \sum_{n\geq 0}\left(-\frac{n!}{(\beta-\alpha)\,\alpha^{n+1}} - \frac{n!}{(\alpha-\beta)\,\beta^{n+1}}\right)\frac{x^n}{n!} \\
&= \sum_{n\geq 0}\frac{n!\left(\beta^{n+1}-\alpha^{n+1}\right)}{\sqrt{5}(-1)^{n+1}}\frac{x^n}{n!} \\
&= \sum_{n\geq 0}\frac{n!\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)(-1)^{n+1}}{\sqrt{5}(-1)^{n+1}}\frac{x^n}{n!} \\
&= \sum_{n\geq 0} n! f_{n+1}\frac{x^n}{n!}
\end{aligned}$$

Thus $a_n = n! f_{n+1}$.

Find an explicit formula for a_n if $a_0 = 1$, $a_1 = 2$ and, for all integers $n \geq 2$, $a_n = na_{n-1} + na_{n-2}$.
Let

$$F(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$$

$$\begin{aligned} F(x) - a_0 - a_1 x &= \sum_{n\geq 2} a_n \frac{x^n}{n!} \\ &= \sum_{n\geq 2} na_{n-1} \frac{x^n}{n!} + \sum_{n\geq 2} na_{n-2} \frac{x^n}{n!} \\ &= \sum_{n\geq 2} a_{n-1} \frac{x^n}{(n-1)!} + \sum_{n\geq 2} a_{n-2} \frac{x^n}{(n-1)!} \\ &= x\sum_{n\geq 2} a_{n-1} \frac{x^{n-1}}{(n-1)!} + x\sum_{n\geq 2} a_{n-2} \frac{x^{n-1}}{(n-1)!} \\ &= x[F(x) - a_0] + x\sum_{n\geq 2} a_{n-2} \frac{x^{n-1}}{(n-1)!} \end{aligned}$$

$$\begin{aligned} F'(x) - a_1 &= F(x) - a_0 + xF'(x) + \sum_{n\geq 2} a_{n-2} \frac{x^{n-1}}{(n-1)!} + x\sum_{n\geq 2} a_{n-2} \frac{(n-1)x^{n-2}}{(n-1)!} \\ &= F(x) - a_0 + xF'(x) + \sum_{n\geq 2} a_{n-2} \frac{x^{n-1}}{(n-1)!} + x\sum_{n\geq 2} a_{n-2} \frac{x^{n-2}}{(n-2)!} \\ &= F(x) - a_0 + xF'(x) + \sum_{n\geq 2} a_{n-2} \frac{x^{n-1}}{(n-1)!} + xF(x) \end{aligned}$$

$$\begin{aligned} F''(x) &= F'(x) + F'(x) + xF''(x) + \sum_{n\geq 2} a_{n-2} \frac{(n-1)x^{n-2}}{(n-1)!} + F(x) + xF'(x) \\ &= F'(x) + F'(x) + xF''(x) + \sum_{n\geq 2} a_{n-2} \frac{x^{n-2}}{(n-2)!} + F(x) + xF'(x) \\ &= F'(x) + F'(x) + xF''(x) + F(x) + F(x) + xF'(x) \\ &= xF''(x) + (2+x)F'(x) + 2F(x) \end{aligned}$$

Thus

$$F(x) = \frac{A}{(1-x)^2} + B\frac{xe^{-x}}{(1-x)^2}$$

and

$$F'(x) = \frac{2A}{(1-x)^3} + B\frac{(x^2+1)e^{-x}}{(1-x)^3}.$$

Since $F(0) = a_0$, $A = 1$. Since $F'(0) = a_1$ and $A = 1$, $B = 0$. Thus

$$\begin{aligned} F(x) &= \frac{1}{(1-x)^2} \\ &= \left(\sum_{n\geq 0} x^n\right)^2 \\ &= \sum_{n\geq 0}(n+1)x^n \\ &= \sum_{n\geq 0} n!(n+1)\frac{x^n}{n!} \\ &= \sum_{n\geq 0}(n+1)!\frac{x^n}{n!}. \end{aligned}$$

To conclude $a_n = (n+1)!$.

It is given that

$$a_{n+2} = 2a_{n+1}a_n$$

where $a_0 = a_1 = 1$. Find closed formula for this recurrence relation using generating functions.

Let $b_n = \log_2 a_n$. Then $b_0 = \log_2 a_0 = 0$ and $b_1 = 0$.

$$b_{n+2} = \log_2 a_{n+2} = \log_2 (2a_{n+1}a_n) = \log_2 2 + \log_2 a_{n+1} + \log a_n = 1 + b_{n+1} + b_n.$$

Let $B(x) = \sum_{n\geq 0} b_n x^n$. Consider

$$\sum_{n\geq 0} b_{n+2}x^n = \sum_{n\geq 0} x^n + \sum_{n\geq 0} b_{n+1}x^n + \sum_{n\geq 0} b_n x^n$$

$$\begin{aligned} \frac{B(x) - b_0 - b_1 x}{x^2} &= \frac{1}{1-x} + \frac{B(x) - b_0}{x} + B(x) \\ \frac{B(x)}{x^2} &= \frac{1}{1-x} + \frac{B(x)}{x} + B(x) \\ B(x)\left(1 + \frac{1}{x} - \frac{1}{x^2}\right) &= -\frac{1}{1-x} \end{aligned}$$

Let $\alpha = \frac{-1+\sqrt{5}}{2}$ and $\beta = \frac{-1-\sqrt{5}}{2}$.

$$\begin{aligned}
B(x) &= -\frac{x^2}{(1-x)(x^2+x-1)} \\
&= -\frac{1}{1-x} - \frac{1}{x^2+x-1} \\
&= -\frac{1}{1-x} - \frac{1}{\sqrt{5}}\frac{1}{(x-\alpha)} + \frac{1}{\sqrt{5}}\frac{1}{(x-\beta)} \\
&= -\frac{1}{1-x} + \frac{1}{\sqrt{5}}\frac{1}{(\alpha-x)} - \frac{1}{\sqrt{5}}\frac{1}{(\beta-x)} \\
&= -\frac{1}{1-x} + \frac{1}{\alpha\sqrt{5}}\frac{1}{(1-\frac{x}{\alpha})} - \frac{1}{\beta\sqrt{5}}\frac{1}{(1-\frac{x}{\beta})} \\
&= -\sum_{n\geq 0} x^n + \frac{1}{\alpha\sqrt{5}}\sum_{n\geq 0}\left(\frac{1}{\alpha}\right)^n x^n - \frac{1}{\beta\sqrt{5}}\sum_{n\geq 0}\left(\frac{1}{\beta}\right)^n x^n \\
&= \sum_{n\geq 0}\left(-1 + \frac{1}{\alpha\sqrt{5}}\left(\frac{1}{\alpha}\right)^n - \frac{1}{\beta\sqrt{5}}\left(\frac{1}{\beta}\right)^n\right) x^n \\
&= \sum_{n\geq 0}\left(-1 + \frac{\beta^{n+1}-\alpha^{n+1}}{\sqrt{5}(\alpha\beta)^{n+1}}\right) x^n \\
&= \sum_{n\geq 0}\left(-1 + \frac{(-1)^{n+1}(-\beta)^{n+1}-(-1)^{n+1}(-\alpha)^{n+1}}{\sqrt{5}(-1)^{n+1}}\right) x^n \\
&= \sum_{n\geq 0}\left(-1 + \frac{(-\beta)^{n+1}-(-\alpha)^{n+1}}{\sqrt{5}}\right) x^n \\
&= \sum_{n\geq 0}\left(-1 + f_{n+1}\right) x^n
\end{aligned}$$

where f_n is the n-th Fibonacci number and

$$f_n = \frac{(-\beta)^n - (-\alpha)^n}{\sqrt{5}}$$

where $-\alpha = \frac{1-\sqrt{5}}{2}$ and $-\beta = \frac{1+\sqrt{5}}{2}$. Thus $b_n = -1 + f_{n+1}$ and to conclude $a_n = 2^{-1+f_{n+1}}$.

Solve the recurrence relation

$$a_n^2 - 5a_{n-1}^2 + 4a_{n-2}^2 = 0, a_0 = 4, a_1 = 13.$$

Define $b_n = a_n^2$. Then $b_{n-1} = a_{n-1}^2$ and $b_{n-2} = a_{n-2}^2$. Moreover, $b_0 = 16$ and $b_1 = 169$. First, Solve

$$b_n - 5b_{n-1} + 4b_{n-2} = 0, b_0 = 16, b_1 = 169.$$

The characteristic equation of our new recurrence relation is $r^2 - 5r + 4 = 0$. Its roots are $r = 4$ and $r = 1$. The solution is of the form

$$b_n = \beta_1 4^n + \beta_2 1^n.$$

Since $b_0 = 16$, $16 = \beta_1 + \beta_2$. Since $b_1 = 169$, $169 = 4\beta_1 + \beta_2$. By solving for β_1 and β_2, $\beta_1 = 51$ and $\beta_2 = -35$. Thus

$$b_n = 51 \cdot 4^n - 35$$

and

$$a_n = \sqrt{51 \cdot 4^n - 35} \text{ or } a_n = -\sqrt{51 \cdot 4^n - 35}$$

If $a_n = -\sqrt{51 \cdot 4^n - 35}$, then $a_0 = -4$, a contradiction. So $a_n = \sqrt{51 \cdot 4^n - 35}$.

Evaluate

$$\sum_{k=1}^{n} \frac{f_k}{p^k}$$

where f_k is the k-th Fibonacci number and $p > 1$ is an integer.
Note that $f_k = \frac{1}{\sqrt{5}}(\alpha^k - \beta^k)$ where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. So $\alpha + \beta = 1$ and $\alpha\beta = -1$.

$$\begin{aligned}
\sum_{k=1}^{n} \frac{f_k}{p^k} &= \frac{1}{\sqrt{5}} \sum_{k=1}^{n} \frac{\alpha^k - \beta^k}{p^k} \\
&= \frac{1}{\sqrt{5}} \sum_{k=1}^{n} \left(\frac{\alpha}{p}\right)^k - \frac{1}{\sqrt{5}} \sum_{k=1}^{n} \left(\frac{\beta}{p}\right)^k \\
&= \frac{1}{\sqrt{5}} \frac{\frac{\alpha}{p} - \left(\frac{\alpha}{p}\right)^{n+1}}{1 - \frac{\alpha}{p}} - \frac{1}{\sqrt{5}} \frac{\frac{\beta}{p} - \left(\frac{\beta}{p}\right)^{n+1}}{1 - \frac{\beta}{p}} \\
&= \frac{1}{\sqrt{5}} \frac{\frac{\alpha}{p} - \frac{\alpha^{n+1}}{p^{n+1}}}{\frac{p-\alpha}{p}} - \frac{1}{\sqrt{5}} \frac{\frac{\beta}{p} - \frac{\beta^{n+1}}{p^{n+1}}}{\frac{p-\beta}{p}} \\
&= \frac{1}{\sqrt{5}} \frac{\left(\frac{\alpha}{p} - \frac{\alpha^{n+1}}{p^{n+1}}\right)\left(\frac{p-\beta}{p}\right)}{\left(\frac{p-\alpha}{p}\right)\left(\frac{p-\beta}{p}\right)} - \frac{1}{\sqrt{5}} \frac{\left(\frac{\beta}{p} - \frac{\beta^{n+1}}{p^{n+1}}\right)\left(\frac{p-\alpha}{p}\right)}{\left(\frac{p-\beta}{p}\right)\left(\frac{p-\alpha}{p}\right)} \\
&= \frac{1}{\sqrt{5}} \frac{\left(\frac{\alpha p^n - \alpha^{n+1}}{p^{n+1}}\right)\left(\frac{p-\beta}{p}\right)}{\frac{p^2-p-1}{p^2}} - \frac{1}{\sqrt{5}} \frac{\left(\frac{\beta p^n - \beta^{n+1}}{p^{n+1}}\right)\left(\frac{p-\alpha}{p}\right)}{\frac{p^2-p-1}{p^2}} \\
&= \frac{1}{\sqrt{5}} \frac{\left(\alpha p^n - \alpha^{n+1}\right)(p-\beta)}{p^n(p^2-p-1)} - \frac{1}{\sqrt{5}} \frac{\left(\beta p^n - \beta^{n+1}\right)(p-\alpha)}{p^n(p^2-p-1)} \\
&= \frac{1}{\sqrt{5}} \frac{\left(\alpha p^{n+1} - \alpha\beta p^n - \alpha^{n+1} p + \alpha^{n+1}\beta\right)}{p^n(p^2-p-1)} - \frac{1}{\sqrt{5}} \frac{\left(\beta p^{n+1} - \alpha\beta p^n - \beta^{n+1} p + \alpha\beta^{n+1}\right)}{p^n(p^2-p-1)} \\
&= \frac{1}{\sqrt{5}} \frac{\left(\alpha p^{n+1} - \beta p^{n+1} - \alpha^{n+1} p + \beta^{n+1} p + \alpha^{n+1}\beta - \alpha\beta^{n+1}\right)}{p^n(p^2-p-1)} \\
&= \frac{1}{\sqrt{5}} \frac{\left(\sqrt{5}p^{n+1} - \left(\alpha^{n+1} - \beta^{n+1}\right)p + \alpha^{n+1}(1-\alpha) - (1-\beta)\beta^{n+1}\right)}{p^n(p^2-p-1)} \\
&= \frac{1}{\sqrt{5}} \frac{\left(\sqrt{5}p^{n+1} - \left(\alpha^{n+1} - \beta^{n+1}\right)p + \alpha^{n+1} - \alpha^{n+2} - \beta^{n+1} + \beta^{n+2}\right)}{p^n(p^2-p-1)} \\
&= \frac{1}{\sqrt{5}} \frac{\left(\sqrt{5}p^{n+1} + \left(\alpha^{n+1} - \beta^{n+1}\right)(-p+1) - \left(\alpha^{n+2} - \beta^{n+2}\right)\right)}{p^n(p^2-p-1)} \\
&= \frac{1}{\sqrt{5}} \frac{\left(\sqrt{5}p^{n+1} + \sqrt{5}f_{n+1}(-p+1) - \sqrt{5}f_{n+2}\right)}{p^n(p^2-p-1)} \\
&= \frac{p^{n+1} + f_{n+1}(-p+1) - f_{n+2}}{p^n(p^2-p-1)}
\end{aligned}$$

References

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