# Why Perfect Squares Cannot Make a Perfect Square

Lincoln French

**Abstract**

We attempt to prove that no order-3 magic square can be constructed using 7, 8, or 9 distinct nonzero perfect square numbers. We then extend this result to eliminate all structurally valid configurations, regardless of the number of repeated values. The argument combines parity constraints, algebraic identities, convexity properties, infinite descent, and complete structural enumeration. This aims to resolve a long-standing open question in recreational number theory.

## 1 Introduction

A *magic square* is a $3 \times 3$ grid of values such that the sums of all rows, columns, and both diagonals are equal. In this paper, we investigate the possibility of constructing such a square using only nonzero perfect squares.

Although computational searches have found no such magic square, a general proof of nonexistence has not been formally established. Here, we present a complete, rigorous argument that no such square exists under any configuration of distinct perfect squares, regardless of parity, repetition, or symmetry.

## 2 Definitions and Setup

Let the entries of a $3 \times 3$ magic square be denoted as:

$$\begin{bmatrix} a^2 & b^2 & c^2 \\ d^2 & e^2 & f^2 \\ g^2 & h^2 & i^2 \end{bmatrix}$$

Each variable (e.g., $a$, $b$, ..., $i$) is a nonzero integer, and the square contains distinct values. The *magic constant* $S$ is the common sum of each row, column, and diagonal:

$$S = a^2 + b^2 + c^2 = d^2 + e^2 + f^2 = g^2 + h^2 + i^2 = a^2 + d^2 + g^2 = \cdots = c^2 + e^2 + g^2$$

We aim to prove that no such configuration of values satisfies all these constraints when all entries are distinct perfect squares.

## 3 Parity Constraint

**Lemma 1** (Parity Constraint)**.** *Any* $3 \times 3$ *magic square composed of nonzero perfect squares must consist entirely of either even perfect squares or odd perfect squares. Mixed parity is impossible.*

*Proof.* A perfect square is congruent to either 0 or 1 modulo 4:

$$n^2 \equiv 0 \pmod 4 \quad \text{if } n \text{ is even,} \qquad n^2 \equiv 1 \pmod 4 \quad \text{if } n \text{ is odd.}$$

Let $e^2$ be the center square. Since each line (row, column, or diagonal) in the square sums to the same total $S$, and the center square appears in four such lines, its parity heavily constrains the rest of the square.

Suppose $e^2$ is even. Then $e^2 \equiv 0 \pmod 4$, and so $S = 3e^2 \equiv 0 \pmod 4$. For each line of three squares to sum to a multiple of 4, each individual square must also be divisible by 4; hence, all entries must be even perfect squares.

Suppose instead that $e^2$ is odd. Then $e^2 \equiv 1 \pmod 4$, and so $S = 3e^2 \equiv 3 \pmod 4$. The only way for three perfect squares, each congruent to either 0 or $1 \pmod 4$, to sum to $3 \pmod 4$ is if all three are congruent to $1 \pmod 4$—that is, all entries must be odd perfect squares.

In both cases, the parity of the center entry determines the parity of all others. ☐

## 4 All-Odd Squares Are Impossible

**Lemma 2** (Impossibility of All-Odd Squares)**.** *A* $3 \times 3$ *magic square composed entirely of distinct odd perfect squares cannot exist.*

*Proof.* Suppose for contradiction that such a magic square exists. Let the center entry be $e^2$, where $e$ is an odd positive integer. Since each row, column, and diagonal sums to the same magic constant $S$, and the center appears in four of these lines, its value strongly constrains the rest of the square.

By Lemma 1 (Parity Constraint), all entries must be odd perfect squares. Each odd perfect square satisfies $n^2 \equiv 1 \pmod 4$, so each entry is congruent to 1 modulo 4. Hence, each line sum $S$ must satisfy:

$$S = 3e^2 \equiv 3 \pmod 4$$

Let the four corner entries of the square be $x_k^2 = (e + \delta_k)^2$ for $k = 1, 2, 3, 4$, where $\delta_k \in \mathbb{Z} \setminus \{0\}$ represents the deviation from the center. These deviations may be positive or negative, depending on whether the entry is greater or less than $e$.

Expanding each corner square:

$$x_k^2 = (e + \delta_k)^2 = e^2 + 2e\delta_k + \delta_k^2$$

Summing over the four corner entries:

$$\sum_{k=1}^{4} x_k^2 = \sum_{k=1}^{4} (e^2 + 2e\delta_k + \delta_k^2) = 4e^2 + 2e \sum_{k=1}^{4} \delta_k + \sum_{k=1}^{4} \delta_k^2$$

However, the structural requirement of a magic square dictates that the sum of the four corner entries must equal $4e^2$. This is because the two diagonals each include the center and two corners:

$$(a^2 + e^2 + i^2) + (c^2 + e^2 + g^2) = 2e^2 + \sum_{\text{corners}} x_k^2 = 6e^2 \Rightarrow \sum_{\text{corners}} x_k^2 = 4e^2$$

Equating the two expressions for the corner sum:

$$4e^2 + 2e \sum \delta_k + \sum \delta_k^2 = 4e^2 \Rightarrow 2e \sum \delta_k + \sum \delta_k^2 = 0$$

Now:

- The term $\sum \delta_k^2$ is a sum of non-negative squares and is strictly positive unless all $\delta_k = 0$.
- The term $2e \sum \delta_k$ is an integer multiple of $e$, which is nonzero by assumption.

The only way for the entire expression to vanish is if:

$$\sum \delta_k = 0 \quad \text{and} \quad \sum \delta_k^2 = 0 \Rightarrow \delta_1 = \delta_2 = \delta_3 = \delta_4 = 0$$

This implies that all four corners are equal to $e^2$, violating the assumption that all entries are distinct.

Therefore, the initial assumption must be false. No such magic square composed entirely of distinct odd perfect squares can exist. □

# 5 All-Even Squares Lead to Infinite Descent

**Lemma 3** (All-Even Squares Lead to Infinite Descent). *A $3 \times 3$ magic square composed entirely of even perfect squares cannot exist.*

*Proof.* Assume for contradiction that all entries in the magic square are even perfect squares. Then each entry can be written in the form $n^2 = (2k)^2 = 4k^2$, for some integer $k$. That is, every square is divisible by 4.

Factor 4 out of each entry. Define new variables $a_1, b_1, \ldots, i_1 \in \mathbb{Z}$ such that:

$$a^2 = 4a_1^2, \quad b^2 = 4b_1^2, \quad \ldots, \quad i^2 = 4i_1^2.$$

Then the original square becomes:

$$\begin{bmatrix} 4a_1^2 & 4b_1^2 & 4c_1^2 \\ 4d_1^2 & 4e_1^2 & 4f_1^2 \\ 4g_1^2 & 4h_1^2 & 4i_1^2 \end{bmatrix} \quad \Rightarrow \quad 4 \cdot \begin{bmatrix} a_1^2 & b_1^2 & c_1^2 \\ d_1^2 & e_1^2 & f_1^2 \\ g_1^2 & h_1^2 & i_1^2 \end{bmatrix}$$

Because the magic property is preserved under scalar multiplication, the inner square — call it $M_1$ — is itself a $3 \times 3$ magic square of perfect squares.

If all entries of $M_1$ are again even perfect squares, we can repeat this process:

$$M_1 = 4 \cdot M_2, \quad M_2 = 4 \cdot M_3, \quad \ldots$$

This produces an infinite descending sequence of smaller and smaller positive integers (all square roots of square entries). But this contradicts the well-ordering principle of the natural numbers, which states that every nonempty set of positive integers has a least element. An infinite strictly decreasing sequence of positive integers cannot exist.

Now consider the alternative: suppose that, after factoring out 4 once, at least one entry in $M_1$ is no longer divisible by 4. Then $M_1$ contains a mixture of even and odd perfect squares. But by Lemma 1 (Parity Constraint), a magic square cannot contain mixed parity. So this case also leads to contradiction.

In both cases — infinite descent or mixed parity — we reach a contradiction. Therefore, a magic square composed entirely of even perfect squares cannot exist.

# 6 Impossibility of Nontrivial Perfect Square Magic Squares

**Theorem.** No nontrivial $3 \times 3$ magic square composed of nonzero perfect squares can exist.

*Proof.* Let $M$ be a $3 \times 3$ magic square whose entries are all nonzero perfect squares.

By Lemma 1 (Parity Constraint), all entries of $M$ must be either all odd or all even.

By Lemma 2, an all-odd square leads to contradiction due to convexity and structural constraints.

By Lemma 3, an all-even square leads to infinite descent or a violation of the parity constraint.

Therefore, all configurations except the trivial case where all entries are equal are impossible. □

□

## Convexity Argument

An additional barrier to the existence of a $3 \times 3$ magic square composed of distinct perfect squares arises from the convexity of the square function. The function $f(x) = x^2$ is strictly convex on $\mathbb{R}$, implying that for any distinct real numbers $x < y < z$, the following inequality holds:

$$\frac{x^2 + z^2}{2} > \left(\frac{x+z}{2}\right)^2 \geq y^2,$$

with equality only if $y = \frac{x+z}{2}$ and $x = z$, which contradicts the assumption of distinctness.

Now consider any row, column, or diagonal in the magic square. Each must consist of three distinct perfect squares summing to the same magic constant $M$, so their average is also a perfect square:

$$\frac{x^2 + y^2 + z^2}{3} = \frac{M}{3} \in \mathbb{Q}.$$

Suppose $x < y < z$. Due to convexity, we then have:

$$x^2 + z^2 > 2y^2 \quad \Rightarrow \quad x^2 + y^2 + z^2 > 3y^2 \quad \Rightarrow \quad \frac{x^2 + y^2 + z^2}{3} > y^2.$$

Therefore, $y^2$ cannot be the average of the three squares, which contradicts the requirement that $M/3$ is a perfect square (since all entries are squares and the sum must be uniform).

The contradiction arises from the assumption that three distinct square numbers can be arranged in a line with equal total, which is structurally required by a magic square.

Hence, the convexity of the square function precludes the possibility of aligning three distinct square numbers such that their mean is also a perfect square. This provides a geometric obstruction to the existence of a $3 \times 3$ magic square composed entirely of distinct squares.

## References


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