Introduction to Hestenes' Use of Geometric Algebra in Treating Constant-Acceleration Motion

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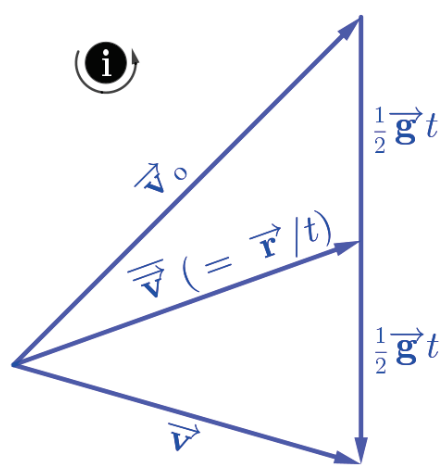
Abstract

As an aid to teachers and students who wish to apply Geometric Algebra to high-school-level physics, we provide the first installment in a guide to Hestenes's treatment of constant-acceleration motion. Specifically, we present a more-detailed version of Hestenes' solution to the problem of finding the time and distance at which a projectile will cross a given line of sight. We begin by reviewing the GA ideas that we will use, and finish by verifying the solution via a GeoGebra worksheet.



The hodograph, which illustrates the vector equation $\overline{\vec{\mathbf{v}}} = \frac{\vec{\mathbf{r}}}{t} = \vec{\mathbf{v}}_0 + \frac{1}{2}\vec{\mathbf{g}}t$. Here, $\overline{\vec{\mathbf{v}}}$ is the vector of average velocity for the time interval t, and "$\mathbf{i}$" is the right-handed unit bivector of the plane that is parallel to ("contains") the four vectors that are shown.

1 Introduction

In this document, I hope to provide some of the "judicious guidance" that David Hestenes says is necessary for students to get through his book *New Foundations for Classical Mechanics*.[1]

The document is intended to be understandable by students and teachers who are familiar with the basics of GA.

The examples that are usually presented when teaching constant-acceleration motion concern the trajectories of projectiles. That is the language that will be used here, but the analyses, equations, and solutions hold for any situations in which the acceleration is constant.

2 What We Will See in this Document

- The differential equations of constant-acceleration motion
- The hodograph
- Finding the time and distance at which a projectile will cross a given line of sight
- Validation of the solution via a GeoGebra construction

3 Ideas that We Will Use

Please note that I don't use the terms "division of vectors" or "division of bivectors". Instead, I use the multiplicative inverses that those "divisions" actually represent.

- Transformations (equivalents) of outer products of vectors
  - For any two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, $\vec{\mathbf{a}} \wedge \vec{\mathbf{b}} = \left[\left(\vec{\mathbf{a}} \mathbf{i} \right) \cdot \vec{\mathbf{b}} \right] \mathbf{i}$. This equivalent is especially useful when constructing GeoGebra worksheets to check solutions.
  - For any four coplanar vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$, and $\vec{\mathbf{d}}$, $\left(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}} \right) \left(\vec{\mathbf{c}} \wedge \vec{\mathbf{d}} \right) = \left\{ \left[\left(\vec{\mathbf{a}} \mathbf{i} \right) \cdot \vec{\mathbf{b}} \right] \mathbf{i} \right\} \left\{ \left[\left(\vec{\mathbf{c}} \mathbf{i} \right) \cdot \vec{\mathbf{d}} \right] \mathbf{i} \right\}$ $= - \left[\left(\vec{\mathbf{a}} \mathbf{i} \right) \cdot \vec{\mathbf{b}} \right] \left[\left(\vec{\mathbf{c}} \mathbf{i} \right) \cdot \vec{\mathbf{d}} \right]$.

[1] "Though my book has been a continual best seller in the series for well over a decade, it is still unknown to most teachers of mechanics in the U.S. To be suitable for the series, I had to design it as a multipurpose book, including a general introduction to GA and material of interest to researchers, as well as problem sets for students. It is not what I would have written to be a mechanics textbook alone. Most students need judicious guidance by the instructor to get through it." ([2])

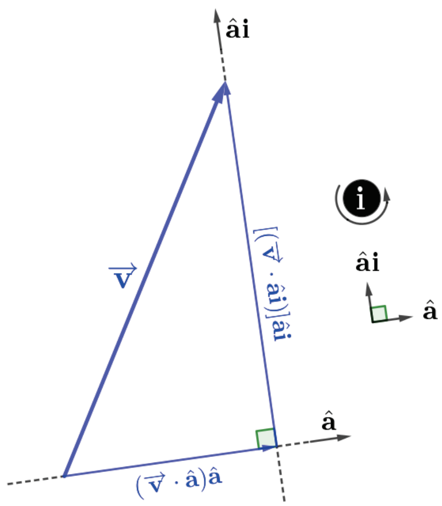


Figure 1: An example of expressing a vector $\overrightarrow{\mathbf{v}}$ as the sum of its projections upon two mutually perpendicular unit vectors: $\overrightarrow{\mathbf{v}} = \left(\overrightarrow{\mathbf{v}} \cdot \hat{\mathbf{a}}\right)\hat{\mathbf{a}} + \left(\overrightarrow{\mathbf{v}} \cdot (\hat{\mathbf{a}}\mathbf{i})\right]\hat{\mathbf{a}}\mathbf{i}$.

  - If we express the vectors $\overrightarrow{\mathbf{a}}$ and $\overrightarrow{\mathbf{b}}$ as $\overrightarrow{\mathbf{a}} = a_1\overrightarrow{\mathbf{e}}_1 + a_2\overrightarrow{\mathbf{e}}_2$ and $\overrightarrow{\mathbf{b}} = b_1\overrightarrow{\mathbf{e}}_1 + b_2\overrightarrow{\mathbf{e}}_2$ where $\left\{\overrightarrow{\mathbf{e}}_1, \overrightarrow{\mathbf{e}}_2\right\}$, are orthornormal unit vectors, then $\overrightarrow{\mathbf{a}} \wedge \overrightarrow{\mathbf{b}} = (a_1b_2 - a_2b_1)\,\overrightarrow{\mathbf{e}}_1\overrightarrow{\mathbf{e}}_2$.

- The "norm" of a bivector $\overrightarrow{\mathbf{a}} \wedge \overrightarrow{\mathbf{b}}$ (written here as $|\overrightarrow{\mathbf{a}} \wedge \overrightarrow{\mathbf{b}}|$) is a positive scalar. It can be calculated in several ways:

  - $|\overrightarrow{\mathbf{a}} \wedge \overrightarrow{\mathbf{b}}| = |\left(\overrightarrow{\mathbf{a}}\mathbf{i}\right) \cdot \overrightarrow{\mathbf{b}}|$ (the absolute value of the scalar $\left(\overrightarrow{\mathbf{a}}\mathbf{i}\right) \cdot \overrightarrow{\mathbf{b}}$).
  - If we express the vectors $\overrightarrow{\mathbf{a}}$ and $\overrightarrow{\mathbf{b}}$ as $\overrightarrow{\mathbf{a}} = a_1\overrightarrow{\mathbf{e}}_1 + a_2\overrightarrow{\mathbf{e}}_2$ and $\overrightarrow{\mathbf{b}} = b_1\overrightarrow{\mathbf{e}}_1 + b_2\overrightarrow{\mathbf{e}}_2$, then $|\overrightarrow{\mathbf{a}} \wedge \overrightarrow{\mathbf{b}}| = |a_1b_2 - a_2b_1|$.
  - For any two unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, $|\hat{\mathbf{a}} \wedge \hat{\mathbf{b}}| = |\sin\theta|$, where θ is the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

- Our last set of ideas concerns the writing of any vector $\overrightarrow{\mathbf{v}}$ as the sum of its components with respect to a pair of mutually perpendicular unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ that are coplanar with $\overrightarrow{\mathbf{v}}$.

  - The component of $\overrightarrow{\mathbf{v}}$ in the direction of $\hat{\mathbf{a}}$ is $\left(\overrightarrow{\mathbf{v}} \cdot \hat{\mathbf{a}}\right)\hat{\mathbf{a}}$.
  - The component of $\overrightarrow{\mathbf{v}}$ in the direction of $\hat{\mathbf{b}}$ is $\left(\overrightarrow{\mathbf{v}} \cdot \hat{\mathbf{b}}\right)\hat{\mathbf{b}}$.
  - Thus, the vector $\overrightarrow{\mathbf{v}}$ can be written as $\overrightarrow{\mathbf{v}} = \left(\overrightarrow{\mathbf{v}} \cdot \hat{\mathbf{a}}\right)\hat{\mathbf{a}} + \left(\overrightarrow{\mathbf{v}} \cdot \hat{\mathbf{b}}\right)\hat{\mathbf{b}}$.
  - A suitable pair of mutually perpendicular unit vectors for this purpose is $\{\hat{\mathbf{a}}, \hat{\mathbf{a}}\mathbf{i}\}$. Thus, $\overrightarrow{\mathbf{v}} = \left(\overrightarrow{\mathbf{v}} \cdot \hat{\mathbf{a}}\right)\hat{\mathbf{a}} + \left[\overrightarrow{\mathbf{v}} \cdot (\hat{\mathbf{a}}\mathbf{i})\right]\hat{\mathbf{a}}\mathbf{i}$ (Fig. 1).

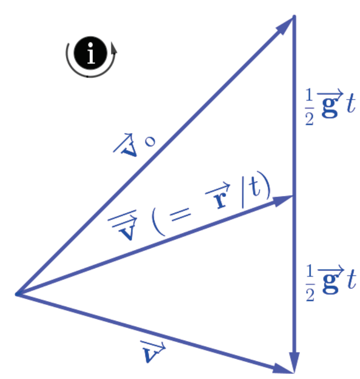


Figure 2: The hodograph, which illustrates the vector equation $\overline{\overrightarrow{\mathbf{v}}} = \frac{\overrightarrow{\mathbf{r}}}{t} = \overrightarrow{\mathbf{v}}_0 + \frac{1}{2}\overrightarrow{\mathbf{g}}t$. "$\mathbf{i}$" is the right-handed unit bivector of the plane that is parallel to ("contains") the four vectors that are shown.

4 The Differential Equations of Motion for Constant Acceleration, and Their Solutions

The differential equation for constant-acceleration motion is $\ddot{\overrightarrow{\mathbf{x}}} = \dot{\overrightarrow{\mathbf{v}}} = \overrightarrow{\mathbf{g}}$, which has the solution $\dot{\overrightarrow{\mathbf{x}}} = \overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{v}}_0 + \overrightarrow{\mathbf{g}}t$, and thus

$$\overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{x}} - \overrightarrow{\mathbf{x}}_0 = \overrightarrow{\mathbf{v}}_0 t + \tfrac{1}{2}\overrightarrow{\mathbf{g}}t^2, \tag{4.1}$$

in which we can divide both sides by t to obtain

$$\overline{\overrightarrow{\mathbf{v}}} = \frac{\overrightarrow{\mathbf{r}}}{t} = \overrightarrow{\mathbf{v}}_0 + \tfrac{1}{2}\overrightarrow{\mathbf{g}}t. \tag{4.2}$$

5 The Hodograph

The hodograph represents the vector formulation $\overline{\overrightarrow{\mathbf{v}}} = \frac{\overrightarrow{\mathbf{r}}}{t} = \overrightarrow{\mathbf{v}}_0 + \frac{1}{2}\overrightarrow{\mathbf{g}}t$ (Eq. (4.2)).

The hodograph isn't a replacement for the conventional vector equation. Instead, it's an additional representation, or tool. Application of GA to that representation may, in many cases, provide formulations whose geometric interpretations give us productive insights. In addition, the hodograph provides good opportunities for learning solution techniques for GA equations.

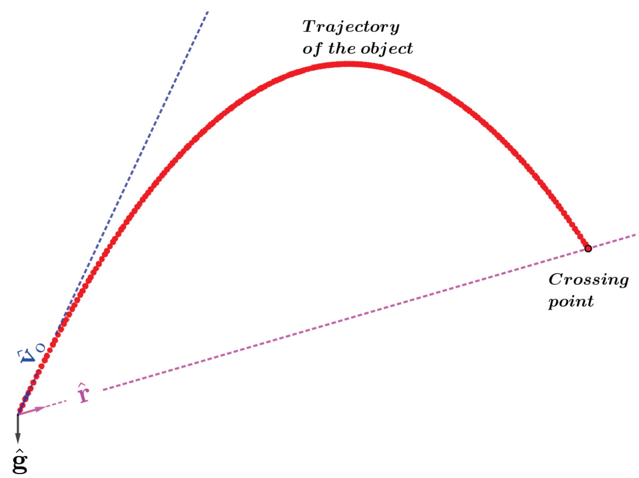


Figure 3: Relationship between the object's trajectory and the vectors $\hat{\mathbf{v}}_o$, $\hat{\mathbf{r}}$, and $\hat{\mathbf{g}}$.

6 Solving Our Projectile-Motion Problem Starting from the Hodograph Equation

Because this introduction is intended for HS-level students, I will provide more details than usual. But first, a word of encouragement . . .

> Don't let the odd-looking equations and expressions scare you! We will see how they can be translated easily into other forms that are convenient. For example, we have seen that one equivalent of $|\hat{\mathbf{r}} \wedge \hat{\mathbf{v}}_o|$ is "the absolute value of the sine of the angle between $\hat{\mathbf{r}}$ and $\hat{\mathbf{v}}_o$".

Our approach will differ from the usual ones in another way. To keep textbooks down to an affordable length, authors (when they present important equations) tend to present the most efficient derivations possible. However, those equations were almost never found via these efficient routes. Instead, someone in the past had an insight, then "followed her nose", thus arriving at a useful result that she (or others) later found a way to derive more efficiently. Because that process is more or less the way in which good problem-solvers (including students) often work, our approach here will have a similar "exploratory" spirit.

6.1 Time of Flight t and range r for a Given Direction $\hat{\mathbf{r}}$

Here, r is the range. $\hat{\mathbf{r}}$ is the direction that interests us (Fig. 3).

6.1.1 The Time of Flight t (Eq. 2.6 in Hestenes's *NFCM*, [?], p. 128)

We will derive Hestenes's Eq. (2.6): $t = \frac{2v_o}{g}\left[\frac{|\hat{\mathbf{r}}\wedge\hat{\mathbf{v}}_o|}{|\hat{\mathbf{g}}\wedge\hat{\mathbf{r}}|}\right]$, starting from Hestenes's Eq. (2.5): $\overline{\overrightarrow{\mathbf{v}}} = \frac{\overrightarrow{\mathbf{r}}}{t} = \overrightarrow{\mathbf{v}}_0 + \frac{1}{2}\overrightarrow{\mathbf{g}}t$. In keeping with this document's "exploratory" spirit, let's see what we might accomplish by taking the outer product of both sides. (Recall that $\overline{\overrightarrow{\mathbf{v}}} = \overrightarrow{\mathbf{r}}/t$.)

$$
\begin{aligned}
\frac{\overrightarrow{\mathbf{r}}}{t} &= \overrightarrow{\mathbf{v}}_0 + \tfrac{1}{2}\overrightarrow{\mathbf{g}}t \\
\left(\frac{\overrightarrow{\mathbf{r}}}{t}\right)\wedge\overrightarrow{\mathbf{r}} &= \left(\overrightarrow{\mathbf{v}}_0 + \tfrac{1}{2}\overrightarrow{\mathbf{g}}t\right)\wedge\overrightarrow{\mathbf{r}} \\
\frac{t}{2}\left(\overrightarrow{\mathbf{g}}\wedge\overrightarrow{\mathbf{r}}\right) &= -\overrightarrow{\mathbf{v}}_0\wedge\overrightarrow{\mathbf{r}} \\
\frac{t}{2}\left(\overrightarrow{\mathbf{g}}\wedge\overrightarrow{\mathbf{r}}\right) &= \overrightarrow{\mathbf{r}}\wedge\overrightarrow{\mathbf{v}}_0 \\
t &= 2\left(\overrightarrow{\mathbf{r}}\wedge\overrightarrow{\mathbf{v}}_0\right)\left(\overrightarrow{\mathbf{g}}\wedge\overrightarrow{\mathbf{r}}\right)^{-1} \\
t &= 2\left(\overrightarrow{\mathbf{r}}\wedge\overrightarrow{\mathbf{v}}_0\right)\left[\frac{-\overrightarrow{\mathbf{g}}\wedge\overrightarrow{\mathbf{r}}}{|\overrightarrow{\mathbf{g}}\wedge\overrightarrow{\mathbf{r}}|^2}\right] \\
t &= 2\left(\overrightarrow{\mathbf{r}}\wedge\overrightarrow{\mathbf{v}}_0\right)\left[\frac{\overrightarrow{\mathbf{r}}\wedge\overrightarrow{\mathbf{g}}}{|\overrightarrow{\mathbf{g}}\wedge\overrightarrow{\mathbf{r}}|^2}\right] \qquad (6.1)
\end{aligned}
$$

For any bivector $\mathbf{B}$, $\mathbf{B}^{-1} = (-\mathbf{B})/|\mathbf{B}|^2$.

This is a good time to pause to do a "sanity check". The time t is a scalar with a positive algebraic sign, but what about the right-hand side of the result that we have just now obtained? From Section 3, we know that $\left(\overrightarrow{\mathbf{r}}\wedge\overrightarrow{\mathbf{v}}_0\right)\left(\overrightarrow{\mathbf{r}}\wedge\overrightarrow{\mathbf{g}}\right)$ is a scalar, as is $|\overrightarrow{\mathbf{r}}\wedge\overrightarrow{\mathbf{g}}|$. Therefore, yes, the right-hand side is a scalar. So far, so good.

Eq. (6.1) is a potentially useful result, but what if we know only the direction $\hat{\mathbf{r}}$ (in addition to $\overrightarrow{\mathbf{v}}_o$ and $\overrightarrow{\mathbf{g}}$), rather than $\overrightarrow{\mathbf{r}}$? If we examine Eq. (6.1), we see that both the numerator and the denominator contain the factor r^2. Thus, we will be able to cancel the r's, leaving only $\hat{\mathbf{r}}$'s:

$$
\begin{aligned}
t &= 2\left[(r\hat{\mathbf{r}})\wedge(v_o\hat{\mathbf{v}}_0)\right]\left[\frac{(r\hat{\mathbf{r}})\wedge(g\hat{\mathbf{g}})}{|(g\hat{\mathbf{g}})\wedge(r\hat{\mathbf{r}})|^2}\right] \\
&= 2v_o r^2 g\left[\hat{\mathbf{r}}\wedge\hat{\mathbf{v}}_0\right]\left[\frac{\hat{\mathbf{r}}\wedge\hat{\mathbf{g}}}{g^2r^2|\hat{\mathbf{g}}\wedge\hat{\mathbf{r}}|^2}\right] \\
&= \frac{2v_o}{g}\left[\frac{(\hat{\mathbf{r}}\wedge\hat{\mathbf{v}}_0)(\hat{\mathbf{r}}\wedge\hat{\mathbf{g}})}{|\hat{\mathbf{g}}\wedge\hat{\mathbf{r}}|^2}\right]. \qquad (6.2)
\end{aligned}
$$

This result can be transformed as suits our needs by using the equivalents that were discussed in Section 3. To transform this result into Hestenes's Eq. (2.6) (i.e., into $t = \frac{2v_o}{g}\left[\frac{|\hat{\mathbf{r}}\wedge\hat{\mathbf{v}}_o|}{|\hat{\mathbf{g}}\wedge\hat{\mathbf{r}}|}\right]$), we consider the algebraic signs of the bivectors $\hat{\mathbf{r}}\wedge\hat{\mathbf{v}}_0$ and $\hat{\mathbf{r}}\wedge\hat{\mathbf{g}}$. The "senses" of rotation of the bivectors $\hat{\mathbf{r}}\wedge\hat{\mathbf{v}}_0$ and $\hat{\mathbf{r}}\wedge\hat{\mathbf{g}}$ are contrary. That is, using the sense of $\mathbf{i}$ that is shown in Fig. 2, $\hat{\mathbf{r}}\wedge\hat{\mathbf{v}}_0 = |\hat{\mathbf{r}}\wedge\hat{\mathbf{v}}_0|\mathbf{i}$, and $\hat{\mathbf{r}}\wedge\hat{\mathbf{g}} = |\hat{\mathbf{r}}\wedge\hat{\mathbf{g}}|(-\mathbf{i})$. Therefore, $(\hat{\mathbf{r}}\wedge\hat{\mathbf{v}}_0)(\hat{\mathbf{r}}\wedge\hat{\mathbf{g}}) = |\hat{\mathbf{r}}\wedge\hat{\mathbf{v}}_0||\hat{\mathbf{r}}\wedge\hat{\mathbf{g}}|$. Using this result, Eq. (6.2) becomes

$$
\begin{aligned}
t &= \frac{2v_o}{g}\left[\frac{|\hat{\mathbf{r}}\wedge\hat{\mathbf{v}}_0||\hat{\mathbf{r}}\wedge\hat{\mathbf{g}}|}{|\hat{\mathbf{g}}\wedge\hat{\mathbf{r}}|^2}\right] \\
&= \frac{2v_o}{g}\left[\frac{|\hat{\mathbf{r}}\wedge\hat{\mathbf{v}}_0|}{|\hat{\mathbf{g}}\wedge\hat{\mathbf{r}}|}\right], \qquad (6.3)
\end{aligned}
$$

because $|\hat{\mathbf{r}}\wedge\hat{\mathbf{g}}| = |\hat{\mathbf{g}}\wedge\hat{\mathbf{r}}|$.

6.1.2 The range r (Eq. 2.8 in Hestenes's *NFCM* [1], p. 128)

This derivation won't be as detailed as the derivation of Eq. (6.3), because most of the details that we will use have already been explained at length.

Later in the derivation, we will use our result for t from Eq. (6.3). However, the first step is to take the outer product of both sides of Eq. (4.2) with $\vec{\mathbf{g}}t$.

$$
\begin{aligned}
\frac{\vec{\mathbf{r}}}{t} &= \vec{\mathbf{v}}_0 + \tfrac{1}{2}\vec{\mathbf{g}}t \\
\left\{\frac{\vec{\mathbf{r}}}{t}\right\} \wedge \left(\vec{\mathbf{g}}t\right) &= \left\{\vec{\mathbf{v}}_0 + \tfrac{1}{2}\vec{\mathbf{g}}t\right\} \wedge \left(\vec{\mathbf{g}}t\right) \\
\vec{\mathbf{r}} \wedge \vec{\mathbf{g}} &= \left(\vec{\mathbf{v}}_0 \wedge \vec{\mathbf{g}}\right) t \\
(r\hat{\mathbf{r}}) \wedge (g\hat{\mathbf{g}}) &= \left[(v_o\hat{\mathbf{v}}) \wedge (g\hat{\mathbf{g}})\right] t \\
r &= v_o \left[\left(\hat{\mathbf{v}}_o \wedge \hat{\mathbf{g}}\right)\left(\hat{\mathbf{r}} \wedge \hat{\mathbf{g}}\right)^{-1}\right] t \\
&= v_o \left[\frac{\left(\hat{\mathbf{v}}_o \wedge \hat{\mathbf{g}}\right)\left(-\hat{\mathbf{r}} \wedge \hat{\mathbf{g}}\right)}{\|\hat{\mathbf{r}} \wedge \hat{\mathbf{g}}\|^2}\right] t.
\end{aligned}
$$

At this point, we could proceed in any of several ways, but the most straightforward might be to substitute Eq. (6.3) for t, and some of our equivalents for the various outer products:

$$
\begin{aligned}
r &= v_o \left\{\frac{(\hat{\mathbf{v}}_o \wedge \hat{\mathbf{g}})(-\hat{\mathbf{r}} \wedge \hat{\mathbf{g}})}{\|\hat{\mathbf{r}} \wedge \hat{\mathbf{g}}\|^2}\right\} \underbrace{\left\{\frac{2v_o}{g}\left[\frac{\|\hat{\mathbf{r}} \wedge \hat{\mathbf{v}}_0\|}{\|\hat{\mathbf{g}} \wedge \hat{\mathbf{r}}\|}\right]\right\}}_{t} \\
&= v_o \left\{\frac{(\hat{\mathbf{v}}_o \wedge \hat{\mathbf{g}})(\hat{\mathbf{g}} \wedge \hat{\mathbf{r}})}{\|\hat{\mathbf{r}} \wedge \hat{\mathbf{g}}\|^2}\right\} \left\{\frac{2v_o}{g}\left[\frac{\|\hat{\mathbf{r}} \wedge \hat{\mathbf{v}}_0\|}{\|\hat{\mathbf{g}} \wedge \hat{\mathbf{r}}\|}\right]\right\} \\
&= \frac{2v_o^2}{g}\left\{\frac{\left[\|\hat{\mathbf{v}}_o \wedge \hat{\mathbf{g}}\|(-\mathbf{i})\right]\left[\|\hat{\mathbf{g}} \wedge \hat{\mathbf{r}}\|\mathbf{i}\right]}{\|\hat{\mathbf{r}} \wedge \hat{\mathbf{g}}\|^2}\right\}\left[\frac{\|\hat{\mathbf{r}} \wedge \hat{\mathbf{v}}_0\|}{\|\hat{\mathbf{g}} \wedge \hat{\mathbf{r}}\|}\right] \\
&= \frac{2v_o^2}{g}\left[\frac{\|\hat{\mathbf{v}}_o \wedge \hat{\mathbf{g}}\|\|\hat{\mathbf{v}}_o \wedge \hat{\mathbf{r}}\|}{\|\hat{\mathbf{r}} \wedge \hat{\mathbf{g}}\|^2}\right]. \qquad (6.4)
\end{aligned}
$$

The only change in this line is that $-\hat{\mathbf{r}} \wedge \hat{\mathbf{g}}$ is replaced with $\hat{\mathbf{g}} \wedge \hat{\mathbf{r}}$.

Hestenes gives ([1], p.128)

$$
r = \frac{2v_o^2}{g}\left[\frac{(\hat{\mathbf{g}} \wedge \hat{\mathbf{v}}_o) \cdot (\hat{\mathbf{v}}_o \wedge \hat{\mathbf{r}})}{\|\hat{\mathbf{g}} \wedge \hat{\mathbf{r}}\|^2}\right].
$$

Why is the numerator in Hestenes's result the inner product of $\hat{\mathbf{g}} \wedge \hat{\mathbf{v}}_o$ and $\hat{\mathbf{v}}_o \wedge \hat{\mathbf{r}}$, rather than the simple product of them? The use of the inner product here may seem especially puzzling because $\hat{\mathbf{g}} \wedge \hat{\mathbf{v}}_o$ and $\hat{\mathbf{v}}_o \wedge \hat{\mathbf{r}}$ are scalar multiples of the same bivector $\mathbf{i}$. Therefore, the inner product $(\hat{\mathbf{g}} \wedge \hat{\mathbf{v}}_o) \cdot (\hat{\mathbf{v}}_o \wedge \hat{\mathbf{r}})$ is in fact equal to $(\hat{\mathbf{g}} \wedge \hat{\mathbf{v}}_o)(\hat{\mathbf{v}}_o \wedge \hat{\mathbf{r}})$.

The explanation for Hestenes's use of the inner product is that he will use his result later, when he derives the maximum range for a given v_o. In that derivation, the "inner-product" form is necessary.

The reader is invited to confirm that if the " · " is omitted, Hestenes's result is the same as our Eq. (6.4).

7 GeoGebra Worksheet to Verify the Solution

Readers can test our solution via the worksheet that is available at the GeoGebra website ([4]). A screenshot of the worksheet is shown in Fig. 4.

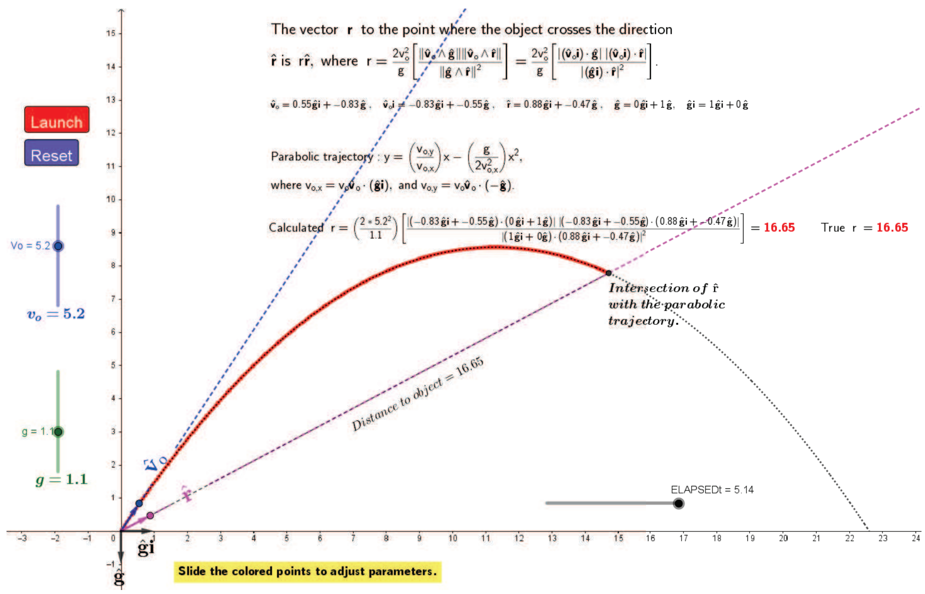


Figure 4: Screenshot of the GeoGebra worksheet for testing the solution.

GeoGebra can calculate inner ("dot") products of vectors, but its output writes the vectors in matrix form. For that reason, I used GeoGebra's built-in Latex commands to express $\hat{\mathbf{v}}_o$ and $\hat{\mathbf{r}}$ as sums of their components with respect to the pair of perpendicular unit vectors $\hat{\mathbf{g}}$ and $\hat{\mathbf{g}}\mathbf{i}$. (See Section 3).

References

[1] D. Hestenes, *New Foundations for Classical Mechanics* (Second Edition), ISBN: 0-7923-5514-8 ©2002 Kluwer Academic Publishers

[2] D. Hestenes, "Oersted Medal Lecture 2002: Reforming the mathematical language of physics". (https://www.researchgate.net/publication/243492634_Oersted_Medal_Lecture_2002_Reforming_the_mathematical_language_of_physics)

[3] A. Macdonald, *Linear and Geometric Algebra* (First Edition), CreateSpace Independent Publishing Platform (Lexington, 2012).

[4] "This Construction Tests a Geometric Algebra Calculation of Where a Projectile Intersects a Given Direction of Aimhttps://www.geogebra.org/m/dewcehxd