Diophantine equation on sixth degree -

-type (6-4-4) with four terms on both sides

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Abstract

There are numerical solutions available on Wolfram world of mathematics website (ref. # 1) for the equation, (a^6+b^6+c^6+d^6)=(e^6+f^6+g^6+h^6). In this paper the author has arrived at numerical solution by algebra instead of elliptical theory. There are methods for the (6-4-4) equation in (ref #6) providing numerical solutions but parametric solutions are not shown. Also on the internet the author has not come across a similar method (given in this paper) for the above mentioned equation.

Consider the below equation:

$$(a^6+b^6+c^6+d^6)=(e^6+f^6+g^6+h^6) \;---(1)$$

We have the Identity:

$u^6+v^6=(x^6-3uvx^2(2x^2-3ab)-2(uv)^3)$ -------- (2)

Where, x = (u+v)

In equation (1) we take:

[$m=a+b=e+f, n=c+d=g+h,\; ab=u,\; cd=v,\; ef=p,\; gh=q$] ----- (3)

equation (1) after transposing we have:

$(a^6+b^6)-\;(e^6+f^6)=\;(g^6+h^6)-\;(c^6+d^6)$ -------------------------------- (4)

substituting from (3) into (4) we get:

$$a^6+b^6=m^6-3um^2(2m^2-3u)-2(u)^3$$

$$e^6+f^6=m^6-3vp(2m^2-3p)-2(p)^3$$

$$g^6+h^6=n^6-3qn^2(2n^2-3q)-2(q)^3$$

$$c^6+d^6=n^6-3vn^2(2n^2-3v)-2(v)^3$$

substituting above four equations in eqn (4) & simplifying we get:

$$(p-u)(6m^4 - 9m^2(p+u) + 2(p^2+pu+u^2) =$$

$$(v-q)(\ 6n^4 - 9n^2(v+q) + 2(v^2+vq+q^2) \text{ ------------ (5)}$$

In-order to solve eqn. (5) we apply the condition: $(p-u) = (v-q)$ or

$(p+q) = (u+v)$ --------------------- (6)

hence we get from eqn. (5):

$6m^4 - 9m^2(p+u) + 2(p^2+pu+u^2) = 6n^4 - 9n^2(v+q) + 2(v^2+vq+q^2)$ --- (7)

The author noticed that the terms, [$9m^2(p+u)$] & [$9n^2(v+q)$] $\mathit{can\ be\ eliminated\ by\ taking}$:

$2n^2 = -(p+u)$ & $2m^2 = -(v+q)$ ----------------- (8)

hence we get:

$6m^4 + 2(p^2+pu+u^2) = 6n^4 + 2(v^2+vq+q^2)$ or,

$3m^4 + (p^2+pu+u^2) = 3n^4 + (v^2+vq+q^2)$ --------------- (9)

since we have: (p-u)=(v-q) we get:

$$(v^2+vq+q^2) - (p^2+pu+u^2) = 3(vq-pu)$$

substituing above in (7) we get:

$(m^4 - n^4) = (vq - pu)$ ------------------------------ (10)

hence we have the three above conditions to satisfy:

namely equations, (6), (8) & (10)

From eqn (8) we have: $-(p+u) = 2n^2$ & $-(v+q) = 2m^2$

hence, $2(m^2+n^2) = -[(p+u)+(v+q)] = -[(p+q)+(u+v)]$

since, from eqn (6) we have: $(p+q) = (u+v)$, we get:

$$2(m^2+n^2) = [(p+u)+(v+q)] = [(p+q)+(u+v)] = -2(u+v)$$

hence, $(m^2+n^2) = -(u+v)$ ---------------- (11)

we square both sides of eqn. (11) & we get:

$m^4 = (u+v)^2 - 2m^2n^2 - n^4$ ---- (12)

we also have from eqn (10):

$$(m^4 - n^4) = (vq - pu)$$

substituting for (m^4), from eqn (12) in above we get a quadratic in the variable "u" :

$u^2 + u(2v+p) + (v^2 - vq - 2n^2(m^2+n^2)) = 0$ --------------- (13)

In-order to get integer solutions to eqn (1), the determinant in equation (13),

needs to be a square:

hence the determinant, (w^2), is given by:

$w^2 \ = \ 4v(p+q) + p^2 + 8n^2(m^2+n^2)$ $---------$ (14)

for eqn (14), the integer solution is:

$$(m,n,v,p,q) \ = \ (167,148,-42485,-36500,-13293)$$

substituting above in eqn (13) we get: $u = -7308$

hence, $(m,n,u,v,p,q) \ = \ (\ 167,148,-7308,-42485,-36500,-13293)$ $------$ (15)

we also have the equality:

$$m = (a+b) = (e+f) \quad \&,$$

$$n = (c+d) = (g+h)$$ ----------------------- (16)

& $\quad (m,n) = (167, 148)$

Also we have:

$u = ab,\ v = cd,\ p = ef, \quad q = gh$ -------------------- (17)

hence from (15),(16) & (17) we get:

$$m = a+b = 167, \qquad u = (a*b) = -(7308)$$

solving above for (a,b), we get, $(a,b) = \ (203,-36)$ ----------- (18)

Next:

$$n = c+d = 148, v = (c*d) = -(42485)$$

$solving\ above\ for\ (c,d), we\ get: (c,d) = (293,-145)$ -------------------- (19)

Next:

$$m = e + f = 167, p = (e * f) = -(36500)$$

$solving\ above\ for\ (e, f), we\ get: (e, f) = (292, -125)$ $-----------------$ (20)

Next:

$$n = g + h = 148, q = (g * h) = -(13293)$$

$solving\ above\ for\ (g, h), we\ get: (g, h) = (211, -63)$ -- (21)

hence from, eqn. (18),(19),(20),(21) we get:

$$(a, b, c, d) = (203, -36, 293, -145) \quad \&$$

$$(e, f, g, h) = (292, -125, 211, -63)$$

Therefore:

$$(a, b, c, d)^6 = (e, f, g, h)^6 \qquad or,$$

$$(203, -36, 293, -145)^6 = (292, -125, 211, -63)^6 \qquad ----- (22)$$

above equation (22), can also be written as:

$$(203, -36, 293, -145)^k = (292, -125, 211, -63)^k \text{ -------------------- } (23)$$

Note:

The author noticed that equation (23) is also valid for the degree's, (k=1,2,4,6).

Conclusion:

Since the degree six, is an even power, the negative signs in the numerical solution of equation (22) can be removed & the equation can be written as below:

$$(\ 203, 36, 293, 145\ )^6 = (\ 292, 125, 211, 63\ )^6$$

Also others can attempt to see if a parametric solution is possible for equation (1), Which is shown above.

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