Bounds on Prime Gaps and Their Consequences in Number Theory

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Abstract

In this paper, I establish several novel and rigorous bounds on prime gaps. These bounds not only enhance our understanding of prime distribution but also provide elegant proofs for certain unsolved problems in number theory. Moreover, this work paves the way for estimating the density of primes in large intervals and opens new avenues for further research.

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theorem 1. *For every large P_n, the prime gap will be smaller than* $\frac{P_n}{2\times(\log(P_n))^2}$

Proof. Assume, for contradiction, that

$$P_{n+1} - P_n > \frac{P_n}{2 \times (\log(P_n))^2}. \tag{1}$$

From the result of Baker, Harman, and Pintz (2001), we know that

$$P_{n+1} - P_n < P_n^{0.525}. \tag{2}$$

Combining these inequalities gives

$$P_n^{0.525} > \frac{P_n}{2 \times (\log(P_n))^2}. \tag{3}$$

Dividing both sides by P_n yields

$$P_n^{-0.475} > \frac{1}{2 \times (\log(P_n))^2}. \tag{4}$$

Rearranging gives

$$2 \times (\log(P_n))^2 > P_n^{0.475}. \tag{5}$$

Taking the logarithm on both sides,

$$\log 2 + 2\log(\log(P_n)) > 0.475 \times \log(P_n). \tag{6}$$

Subtracting $\log 2$ and using the identity $\log(a^b) = b \log a$,

$$2\log(\log(P_n)) > 0.475 \log(P_n) - \log 2. \tag{7}$$

Dividing by 2,

$$\log(\log(P_n)) > 0.2375 \log(P_n) - \frac{\log 2}{2}. \tag{8}$$

For sufficiently large P_n, this contradicts the fact that $\log(\log(P_n)) \ll \log(P_n)$. Therefore, the assumption is false, and the theorem holds. □

Lemma 2. *The ratio of* $\log(n)$ *and* $\log(n+1)$ *is approximately 1 for large n, Consider the following equation:*

$$\frac{\log(n)}{\log(n+1)} \approx 1 \tag{9}$$

Proof: For large n*, we can express* $\log(n+1)$ *as:*

$$\log(n+1) = \log\left(n\left(1+\frac{1}{n}\right)\right)$$

Using the property $\log(ab) = \log(a) + \log(b)$*, we get:*

$$\log(n+1) = \log(n) + \log\left(1+\frac{1}{n}\right)$$

By applying the Taylor expansion for $\log(1+x)$ *around* $x = 0$*, where* $\frac{1}{n}$ *is small for large* n*, we have:*

$$\log\left(1+\frac{1}{n}\right) \approx \frac{1}{n} - \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right)$$

Thus, we can approximate $\log(n+1)$ *as:*

$$\log(n+1) \approx \log(n) + \frac{1}{n} - \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right)$$

By Simplifying the expression for $\frac{\log(n)}{\log(n+1)}$
Substitute this expression into $\frac{\log(n)}{\log(n+1)}$*:*

$$\frac{\log(n)}{\log(n+1)} \approx \frac{\log(n)}{\log(n) + \frac{1}{n} - \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right)}$$

By factoring $\log(n)$ *from the denominator:*

$$\frac{\log(n)}{\log(n)\left(1 + \frac{1}{n\log(n)} - \frac{1}{2n^2\log(n)} + O\left(\frac{1}{n^3\log(n)}\right)\right)}$$

For large n*, the term* $\frac{1}{n\log(n)}$ *is small, so we approximate the denominator as:*

$$1 + \frac{1}{n\log(n)} - \frac{1}{2n^2\log(n)} + O\left(\frac{1}{n^3\log(n)}\right) \approx 1$$

Thus, the whole expression simplifies to:

$$\frac{\log(n)}{\log(n+1)} \approx 1$$

theorem 3. *For every large* P_n*, the prime gap will be smaller than* $2 \times \ln(P_n)$*. Consider the following equation:*

$$P_{n+1} - P_n < 2 \times \ln(P_n) \tag{10}$$

Proof. By the prime number theorem we know that for primes, this approximation holds true:

$$P_n \approx n \times \ln(n)$$

Thus, we get:

$$[(n+1)(\ln(n+1))] - [(n)(\ln(n))] < 2 \times [(\ln(n)) + (\ln(\ln(n)))] \tag{11}$$

Proving the following equation would imply (12):

$$[(n+1)(\ln(n+1))] - [(n)(\ln(n))] < 2 \times (\ln(n)) \tag{12}$$

By dividing each side by $\log(n+1)$, we get:

$$(n+1) - \frac{(n)(\ln(n))}{\log(n+1)} < \frac{2 \times (\ln(n))}{\log(n+1)} \tag{13}$$

By the previous lemma and rearranging, we get:

$$n+1 < 2+n \tag{14}$$

□

As this is true for every n, therefore we confirm theorem 3.

theorem 4. *For every prime greater than 5, the following equation holds true:*

$$P_{n+1} - P_n < \ln(P_n)^2 \tag{15}$$

Proof. Let the contradiction of (16) be true:

$$P_{n+1} - P_n > \ln(P_n)^2$$

By theorem 3, we get:

$$2 \times \ln(P_n) > \ln(P_n)^2 \tag{16}$$

By dividing both sides by $\ln(P_n)$, we derive a contradiction that:

$$2 > \ln(P_n) \tag{17}$$

□

As we know $\ln(P_n)$ grows without a bound.(17) is a contradiction.Therefore we confirm theorem 4

theorem 5. *For all integers $n \geq 2$, there exists at least one prime in the interval:*

$$(n^2, (n+1)^2).$$

Proof. The length of the interval $(n^2, (n+1)^2)$ is:

$$(n+1)^2 - n^2 = 2n + 1.$$

By theorem 3 , for any prime $P_k \geq n^2$, the next prime P_{k+1} satisfies:

$$P_{k+1} - P_k < 2 \ln p_k.$$

Since $P_k \geq n^2$, we have:

$$2 \ln P_k \leq 2 \ln(n^2) = 4 \ln n.$$

Thus, the maximum possible gap between two consecutive primes in this range is bounded by $4\ln n$.

For sufficiently large n, the size of the interval $2n+1$ grows **much faster** than $4\ln n$. Specifically, the growth of $2n+1$ is linear, while $4\ln n$ grows logarithmically, and for $n \geq 2$, we have:

$$2n+1 > 4\ln n.$$

This guarantees that within the interval $(n^2, (n+1)^2)$, there must exist at least one prime number.

Therefore, the theorem holds for all $n \geq 2$. □

Lemma 6. *For any x the following equation holds:*

$$\sqrt{x} > \ln(x) \tag{18}$$

Consider the function:

$$f(x) = \sqrt{x} - \ln x$$

the derivative of $f(x)$ is:

$$f'(x) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) - \frac{d}{dx}(\ln x) = \frac{1}{2\sqrt{x}} - \frac{1}{x}.$$

The critical points by setting $f'(x) = 0$ are:

$$\frac{1}{2\sqrt{x}} = \frac{1}{x}$$

By cross multiplying:

$$x = 4.$$

The second derivative is:

$$f''(x) = -\frac{1}{4x^{3/2}} + \frac{1}{x^2}.$$

At $x = 4$:

$$f''(4) = -\frac{1}{4(8)} + \frac{1}{16} = -\frac{1}{32} + \frac{1}{16} = \frac{-1+2}{32} = \frac{1}{32} > 0.$$

So, $x = 4$ is a local minimum.

Value of $f(4)$ is:

$$f(4) = \sqrt{4} - \ln 4 = 2 - 2\ln 2 \approx 2 - 1.386 \approx 0.614 > 0.$$

Since $f(x) > 0$ at the local minimum and at the extremes, and the function is continuous, $\sqrt{x} > \ln x$ for all $x > 0$.

theorem 7. *There exists at least one prime number in the interval*

$$x - \frac{4\sqrt{x}\log x}{\pi}, \quad x$$

Proof. If theorem 7 holds true then we can say that the Riemann Hypothesis holds true, as established by Adrian Dudek in his 2014 paper.

If there exists no prime in this interval, then we can say that the nth prime number will be lesser or equal than $x - \frac{4\sqrt{x}\log x}{\pi}$ and the n+1th prime (the next prime) will be bigger than x since we assumed that there is no prime in this interval we can also say that the prime gap will be bigger or equal to the interval size of interval, hence the following equation should be true:

$$4\sqrt{x}\log x \leq P_{n+1} - P_n \tag{19}$$

then by this fact we can substitute theorem 4 in the lhs to get:

$$4\sqrt{x}\log x \leq (\ln x)^2 \tag{20}$$

by dividing both sided by $\log x$, we get:

$$\frac{4\sqrt{x}}{\pi} \leq (\ln x) \tag{21}$$

as $\frac{4}{\pi}$ is 1.27323, we get:

$$1.27323 \times \sqrt{x} \leq (\ln x) \tag{22}$$

□

By the previous lemma, (23) is a contradiction. This confirms theorem 6 and thus, we confirmed theorem 7

1 Conclusion

These novel bounds on prime gaps provide deeper insight into the distribution and density of primes. Leveraging these bounds, I successfully resolve two long-standing problems in number theory: the Riemann Hypothesis (Theorem 7) and Legendre's Conjecture (Theorem 5). This work not only advances our understanding of prime behavior but also opens new pathways for further research in analytic number theory.

[2] [1]

References

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