On Area Element in Polar Coordinates

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Abstract

A simple and elementary derivation for formula for area element in polar coordinates is given.

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1 Introduction

For converting a double integral from Cartesian coordinates to polar coordinates we have to convert area element $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$ to polar coordinates. Most derivations of the formula are either based on geometry or Jacobian (or tensor product)[1, 2, 3, 4, 5, 6, 7].

In this note, a self-contained and a simple derivation without using geometry or Jacobian is given. The conversion, is also required for example, for computing integral

$$\int_0^\infty e^{-x^2}\mathrm{d}x$$

1.1 Preliminaries

If P is a point in the plane with coordinates (x_P, y_P) then the distance of P from origin, say r will be the length of the segment OP, and $r^2 = (OP)^2 = x_P^2 + y_P^2$. Let us choose $x = r\cos\theta$, then as $\cos^2\theta + \sin^2\theta = 1$, y can be taken as $y = r\sin\theta$.

Coordinates (r, θ) are called polar coordinates.

Thus, to summarise, if polar coordinates are (r, θ), then Cartesian coordinates are

$$x = r\cos\theta$$

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$$y = r\sin\theta$$

If we are using some other Cartesian coordinate system, with the same origin, in which the coordinates of P are (x'_P, y'_P) then also $(OP)^2 = x'^2_P + y'^2_P$.

Let us assume that the second coordinate frame is at an angle φ with respect to the first. Then $\theta' = \theta+\varphi$, and $x' = r\cos(\theta+\varphi)$ and $y' = r\sin(\theta+\varphi)$.

Thus, $x' = r\cos(\theta+\varphi) = r\cos\theta\cos\varphi - r\sin\theta\sin\varphi = x\cos\varphi - y\sin\varphi$. Here we used $x = r\cos\theta$, and $y = r\sin\theta$.

Similarly, $y' = r\sin(\theta+\varphi) = r\sin\theta\cos\varphi + r\cos\theta\sin\varphi = y\cos\varphi + x\sin\varphi = x\sin\varphi + y\cos\varphi$.

Or in matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix $\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ is called the rotation matrix.

2 Area Element

Differentiating

$$x = r\cos\theta$$
$$y = r\sin\theta$$

We get

$$\mathrm{d}x = \mathrm{d}r\cos\theta - r\sin\theta\mathrm{d}\theta$$
$$\mathrm{d}y = \mathrm{d}r\sin\theta + r\cos\theta\mathrm{d}\theta$$

In matrix form, the equation can be written as

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \mathrm{d}r \\ \mathrm{d}\theta \end{pmatrix}$$

Or equivalently,

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathrm{d}r \\ r\mathrm{d}\theta \end{pmatrix}$$

As the square matrix is a 2-d rotation matrix, and as areas do not change under rotation the area element $\mathrm{d}A = (\mathrm{d}r)(r\mathrm{d}\theta)$.

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