Convergence and Divergence Analysis of the qx+r Problem

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Abstract

This paper investigates the convergence and divergence of the $qx+r$ problem (Crandall conjecture), which is a generalization of the $3x+1$ problem (Collatz conjecture). Through probabilistic analysis, we establish that the convergence condition for the $qx+r$ problem is $q<4$,and predict that it converges to the precise value $\frac{r}{4-q}$; when $q>4$, the transformation sequence diverges to infinity. The results of this study provide new insights into understanding the dynamic behavior of the $qx+r$ problem.

1 Introduction

The Collatz conjecture[7], also known as the $3x+1$ problem, states that for any positive integer x, if it is odd, perform the transformation $3x+1$; if it is even, divide it by 2. After a finite number of transformations, the sequence will eventually reach the cycle $4 \to 2 \to 1$. Although the conjecture is simple to state, its proof remains elusive. The Collatz function is defined as:

$$C(x) = \begin{cases} \frac{3x+1}{2} & \text{if } x \equiv 1 \pmod 2, \\ \frac{x}{2} & \text{if } x \equiv 0 \pmod 2. \end{cases}$$

If we remove all powers of 2 in each step, we obtain the accelerated Collatz function $U(x)$:

$$U(x) = \frac{3x+1}{2^a}, \quad \text{where } (2^a \parallel 3x+1).$$

The generalized $3x+1$ problem includes the $3x+d$ problem [1][2], the $ax+1$ problem [3], the $5x+1$ problem [4][12], and the $qx+r$ problem [8] (Crandall conjecture).

In the 1970s, mathematicians began to use heuristic methods to explain the conjecture. The core idea of heuristic arguments is that, although some steps may increase the value (e.g., $3n+1$), in the long run, n tends to decrease. Jeffrey Lagarias [5][6]conducted in-depth research on the $3x+1$ problem and proposed various heuristic models and generalizations. Hu Zuojun[3] also proposed a novel convergence analysis method.

Although heuristic arguments support the conjecture, a rigorous mathematical proof is still lacking. This paper builds on the heuristic argument approach, Differently, this paper did not ignore the influence of r on convergence and dispersion, and included r in the random model for analysis, obtaining clear results.

2 Definition of the qx+r Transformation Function

To study the common patterns of the $qx+r$ problem, we define the generalized accelerated $qx+r$ function $U_{q,r}(x)$

$$U_{q,r}(x) = \frac{qx+r}{2^a}, \quad \text{where } (2^a \parallel (qx+r),\ q, r \text{ are odd}).$$

If x is even, divide it by 2 repeatedly until it becomes odd; if x is odd, multiply it by q and add r, then divide by 2 repeatedly until an odd number is obtained, resulting in a sequence of odd numbers.

3 Lemma 1

[3][8][11] The expected number of times an integer is divisible by 2 is 1; the expected number of times an even number is divisible by 2 is 2.

Proof:

We analyze the number of times a positive integer N is divisible by 2 and compute its expected value. Let N be a random positive integer, and define the random variable X as the number of times N is divisible by 2.

Define the random variable:

Let X be the number of times N is divisible by 2,

$$X = \max\{k \geq 0 \mid N \text{ is divisible by } 2^k\}.$$

Distribution Law:

If N is a uniformly random positive integer, the probability that N is divisible by 2^k is:

$$P(X \geq k) = \frac{1}{2^k}.$$

This is because the probability of divisibility by 2^k halves with each increase in k.

Thus, the probability mass function of X is:

$$P(X = k) = P(X \geq k) - P(X \geq k+1) = \frac{1}{2^k} - \frac{1}{2^{k+1}} = \frac{1}{2^{k+1}}.$$

Calculation of Expected Value:

The expected value $E[X]$ is:

$$E[X] = \sum_{k=0}^{\infty} k \cdot P(X = k) = \sum_{k=0}^{\infty} k \cdot \frac{1}{2^{k+1}}.$$

Using the formula for the sum of a geometric series, we compute:

$$E[X] = \sum_{k=0}^{\infty} k \cdot \frac{1}{2^{k+1}} = \frac{1}{2} \sum_{k=1}^{\infty} k \cdot \frac{1}{2^k} = \frac{1}{2} \cdot \frac{1}{(1-\frac{1}{2})^2} = \frac{1}{2} \cdot 2 = 1.$$

Thus, the expected value of X, representing the number of times N is divisible by 2, is:

$$E[X] = 1$$

For any even number $2N$, it must first be divisible by 2 once, then it becomes an arbitrary integer N. Therefore, for an even number, the expected number of times it is divisible by 2 is $E(k) = 2$. This means that if we randomly select an even number, the average number of times it is divisible by 2 until it becomes odd is 2. The resulting odd number is, on average, $\frac{1}{4}$ of the original even number.

4 Theorem 1

When $q < 4$, the transformation sequence $U_{q,r}(x)$ converges; when $q > 4$, the transformation sequence $U_{q,r}(x)$ diverges.

Proof:

In the $U_{q,r}(x)$ transformation, let the initial value be any positive integer $x_0 \in N^+$. During the transformation, the number of factors of 2 in the generated even numbers follows a uniform distribution. This property was proven by Sinai[9][10] in his research, showing that, in a statistical sense, the number of factors of 2 in even numbers tends to be uniformly distributed. Further, let q and r be odd numbers. Then, the average rate of change in each transformation can be expressed as:

$$\frac{x_{n+1}}{x_n} = \frac{qx_n + r}{4x_n} = \frac{q}{4} + \frac{r}{4x_n}.$$

After one transformation, the initial value x_0 becomes:

$$x_1 = \frac{1}{4}(qx_0 + r) = \frac{1}{4}qx_0 + \frac{1}{4}r$$

After two transformations:

$$x_2 = \frac{1}{4}(qx_1 + r) = \frac{1}{4}\left(q\left(\frac{1}{4}qx_0 + \frac{1}{4}r\right) + r\right) = \left(\frac{q}{4}\right)^2 x_0 + \frac{q}{4^2}r + \frac{1}{4}r$$

After n transformations, the term x_n can be expressed as:

$$x_n = \left(\frac{q}{4}\right)^n x_0 + \frac{q^{n-1}}{4^n}r + \frac{q^{n-2}}{4^{n-1}}r + \cdots + \frac{q}{4^2}r + \frac{1}{4}r$$

$$x_n = \left(\frac{q}{4}\right)^n x_0 + \frac{r}{4-q}\left(1 - \left(\frac{q}{4}\right)^n\right)$$

$$x_n = \left(\frac{q}{4}\right)^n \left(x_0 - \frac{r}{4-q}\right) + \frac{r}{4-q}$$

When $q < 4$, $\lim_{n\to\infty} \left(\frac{q}{4}\right)^n = 0$, thus:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(\left(\frac{q}{4}\right)^n \left(x_0 - \frac{r}{4-q}\right) + \frac{r}{4-q}\right) = \frac{r}{4-q}$$

That is, the transformation sequence converges to $\frac{r}{4-q}$
When $q > 4$, $\lim_{n\to\infty} \left(\frac{q}{4}\right)^n = \infty$, thus:

$$lim_{n\to\infty} x_n = \infty.$$

That is, the transformation sequence diverges.

5 Conclusion

When $q < 4$, unless it enters another cycle prematurely, $U_{q,r}(x))$ will converge to $\frac{r}{4-q}$;when $q > 4$, unless it enters another cycle prematurely, $U_{q,r}(x)$ will diverge to infinity.

When $q = 3$ and $r = 1$, this corresponds to the $3x+1$ problem (Collatz conjecture), which converges for all positive integers and does not diverge. If no other cycles exist, it will always converge to 1.At present, the uniqueness of the loop has not been proven.

Statement *I do not speak English, nor am I a professional in the field of mathematics. This article was translated from a Chinese version by AI, and there may be errors in the expression of mathematical language. Due to potential inaccuracies introduced during the translation process, any errors in the original text may be amplified. I sincerely request professionals to point out any mistakes in the article, and I would be deeply grateful. Thank you!*

References

[1] Belaga, Edward G. and Mignotte, Maurice. Embedding the $3x + 1$ conjecture in a $3x + d$ context. A K Peters, Ltd.. (2),1998.

[2] Holden, Dhiraj. Results on the 3x+1 and 3x+d conjectures. Fibonacci Quarterly. 49(2):P.131-133,2011.

[3] Hu, Zuojun The Analysis of Convergence for the 3X+ 1 Problem and Crandall Conjecture for the aX+ 1 Problem. Advances in Pure Mathematics. 011(005):P.400-407,2021.

[4] Kontorovich, A. V. and Lagarias, J. C. Stochastic models for the 3x+1 and 5x+1 problems and related problems. 2010.

[5] Lagarias, Jeffrey C. The 3x + 1 Problem and Its Generalizations. The American Mathematical Monthly. 92(1):P.3-23,1985.

[6] Jeffrey C Lagarias and A.Weiss The $3x + 1$ Problem: Two Stochastic Models. The Annals of Applied Probability. 2:P.229-261,1992.

[7] Lagarias, Jeffrey C The 3x+ 1 Problem: An Overview. Ultimate challenge thex+ problem. 2010.

[8] Richard E. Crandall On the 3X+ 1 Problem. Math. Comp. 32:P.1281-1292,1978.

[9] Sinai, Ya. G. Uniform distribution in the (3x+1)-problem. Princeton University. 3(4):P.1429-1440,2003.

[10] Sinai, Ya. G. A theorem about uniform distribution. Communications in Mathematical Physics. 252:1a3,2004

[11] Tao, Terence. Almost all orbits of the Collatz map attain almost bounded values. 2019.

[12] Volkov, Stanislav. A probabilistic model for the 5x+1 problem and related maps-ScienceDirect. Stochastic Processes and their Applications. 116(4):P.662-674,2006.