Massive Photon Interactions in Longitudinal Displacement Current

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***Abstract*— We show the scalar field defined in the vectoral form of Maxwell's equations is found to break the Lorentz Gauge in the case of massive photons. Using the biquaternion form of Maxwell's equations, the differential forms of the scalar field is shown to have relation to photon mass. We show both the scalar field gradient and photon mass are related to the displacement current using the Proca equations, and propose a means to determine the spatial geometric structure of the massive photon based on its intrinsic capacitance.**

***Index Terms*— Biquaternion Electrodynamics, Proca Equations, Photon Mass, Displacement Current, Longitudinal Wave Propagation, Quantum Optics**

The current upper bound for photon mass is given as $9.52 \times 10^{-46}\ kg$ [2]. The significance of non-zero photon mass is in the reformulation of Maxwell equations, Quantum Electrodynamics (QED), and Quantum Field theory (QFT). As we will show, nonzero photon mass suggests the Lorentz gauge to no longer hold. The current form of Maxwell equations have been formulated by Heaviside [1], using vector notation [3] as

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{1}$$

$$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \tag{2}$$

$$\nabla \cdot E = \frac{\rho_v}{\epsilon_0} \tag{3}$$

$$\nabla \cdot B = 0 \tag{4}$$

The scalar potential in electrodynamics is interpreted as the volt and the vector potential as the magnetic vector potential, these are defined by Maxwell as electric potential and electromagnetic momentum respectively in [4], in modern notation in terms of E and B as

$$E = -\nabla\phi - \frac{\partial A}{\partial t} \tag{4}$$

$$B = \nabla \times A \tag{5}$$

We now apply these definitions of E and B in terms of the scalar and vector potentials to (2) and (3) in which $\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$ is added to (7)

$$\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla^2 A + \nabla(\nabla \cdot A + \frac{1}{c^2}\frac{\partial\phi}{\partial t}) = \mu_0 J \tag{6}$$

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi - \frac{\partial}{\partial t}(\nabla \cdot A + \frac{1}{c^2}\frac{\partial\phi}{\partial t}) = \frac{\rho_v}{\epsilon_0} \tag{7}$$

In the Lorentz gauge $(\nabla \cdot A + \frac{1}{c^2}\frac{\partial\phi}{\partial t}) = 0$. We shall assign S to be equal to this scalar field

$$S = \nabla \cdot A + \frac{1}{c^2}\frac{\partial\phi}{\partial t} \tag{8}$$

S satisfies the homogeneous wave equation given the Lorentz gauge. Performing the divergence of (6) and the scaled $(\frac{1}{c^2})$ time differential of (7) and adding the results, we determine the continuity of charge in which $\nabla \cdot J + \frac{\partial \rho_v}{\partial t} = 0$, related to this homogeneous wave equation of S as

$$-\nabla^2 S + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} S = \nabla \cdot J + \frac{\partial \rho_v}{\partial t} \tag{9}$$

It has been shown in the Aharonov-Bohm effect, the physical significance of scalar and vector potential [5]. In the case of massive photons, the scalar field becomes related to photon mass, scalar and vector potentials. The Proca equations (10,11,12,13) [6] describe the original Maxwell equations (1,2,3,4) including the additional photon mass term μ_γ

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{10}$$

$$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t} - \mu_\gamma^2 A \tag{11}$$

$$\nabla \cdot E = \frac{\rho_v}{\epsilon_0} - \mu_\gamma^2 \phi \tag{12}$$

$$\nabla \cdot B = 0 \tag{13}$$

The photon mass appears in the equations describing Ampere's circuital law and Gauss' law. This hints at the longitudinal nature of the magnetic vector potential. One aspect of the Proca equations is, a zero photon mass results in the original Maxwell equations of classical electrodynamics. In our interest to determine the relation between S and μ_γ, we will turn to the biquaternion form of electromagnetics in order to determine the form of ∇A, in which A is the biquaternion containing the scalar and vector potentials [8]. The biquaternion defines a four dimensional Minkowski space as

$$X = i_t ct + \vec{i} \cdot \vec{x}$$

In which $\vec{i} = (i, j, k)$ and $(i_t t, \vec{x})$ represent the complex scalar and complex vector quantities and $i_t i_t = ii = jj =$

$kk = ijk = -1$, and i_t is the imaginary unit describing the Wick rotation of time [9] and ∇ is defined as

$$\nabla = (\frac{i_t}{c}\frac{\partial}{\partial t} + \vec{i}\cdot\vec{\nabla})$$

Which performs a time derivative of the scalar component and differentiation from the quaternion product of nabla with of the vector component of the biquaternion, and $\vec{\nabla}$ is the vector calculus equivalent. Next we define A as the biquaternion containing the scalar and vector potential $(i_t\frac{\phi}{c}, \vec{A})$

$$A = i_t\frac{\phi}{c} + \vec{i}\cdot\vec{A} \tag{14}$$

Then calculate ∇A

$$\begin{aligned}\nabla A &= \left(\frac{i_t}{c}\frac{\partial}{\partial t} + \vec{i}\cdot\vec{\nabla}\right)\left(i_t\frac{\phi}{c} + \vec{i}\cdot\vec{A}\right)\\ &= -\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \frac{i_t}{c}\frac{\partial(\vec{i}\cdot\vec{A})}{\partial t} + \vec{i}\cdot\frac{i_t}{c}\vec{\nabla}\phi + (\vec{i}\cdot\vec{\nabla})(\vec{i}\cdot\vec{A})\end{aligned}$$

In which $(\vec{i}\cdot\vec{\nabla})(\vec{i}\cdot A)$ is simplified using the quaternion product as $-\vec{\nabla}\cdot A + \vec{i}\cdot(\vec{\nabla}\times A)$. Grouping the scalar and vector components and simplifying:

$$\nabla A = -(\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \vec{\nabla}\cdot A) + \vec{i}\cdot((\vec{\nabla}\times A) + \frac{i_t}{c}(\frac{\partial(A)}{\partial t} + \vec{\nabla}\phi)) \tag{15}$$

Which exemplifies the choice of biquaternions, the scalar field S defined in the vector representation of Maxwell's equations is found as the scalar component of the differentiated scalar and vector potentials in biquaternion form. More over, we apply the definition of E and B in terms of potentials from (4) and (5) with the definition of S in (8) to write (15) in terms of E, B and S

$$\nabla A = -S + \vec{i}\cdot(B - \frac{i_t}{c}E) \tag{16}$$

We will now reproduce the scalar and vector potential form of Maxwell equations (6) and (7) through the quaternion product of the negative conjugate of nabla $-\nabla^* = -\frac{i_t}{c}\frac{\partial}{\partial t} + \vec{i}\cdot\vec{\nabla}$ with $\nabla A + S$

$$\begin{aligned}(-\nabla^*)(\nabla A + S) &= (-\frac{i_t}{c}\frac{\partial}{\partial t} + \vec{i}\cdot\vec{\nabla})(\vec{i}\cdot(B - \frac{i_t}{c}E))\\ &= -\frac{i_t}{c}\frac{\partial}{\partial t}(\vec{i}\cdot B) - \frac{1}{c^2}\frac{\partial}{\partial t}(\vec{i}\cdot E) - (\vec{\nabla}\cdot(B - \frac{i_t}{c}E))\\ &\quad + \vec{i}\cdot(\vec{\nabla}\times(B - \frac{i_t}{c}E))\\ &= -\vec{\nabla}\cdot B + \frac{i_t}{c}\vec{\nabla}\cdot E\\ &\quad + \vec{i}\cdot(\vec{\nabla}\times B - \frac{1}{c^2}\frac{\partial}{\partial t}E - \frac{i_t}{c}(\frac{\partial}{\partial t}B + \vec{\nabla}\times E))\end{aligned}$$

By applying the relations (4, 3) we find the scalar portion of this biquaternion is analogous to (7), and the vector portion is analogous to (6) by applying (1) and (2), in other words

$$(-\nabla^*)(\nabla A + S) = \frac{i_t}{c}\frac{\rho_v}{\epsilon_0} + \vec{i}\cdot(\mu_0 J) \tag{17}$$

As mentioned earlier, S is known as the Lorentz Gauge and is set to zero. We now assume the Lorentz Gauge no longer holds, and S is non-zero. If S is not zero, gauge transformation invariance of the scalar and vector potentials indicates the biquaternion potential will be invariant. Equation (17) is equivalently calculated using the d'Alembert operator where $\Box = -\nabla\nabla^* = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}\cdot\vec{\nabla}$. With non-zero S, we recalculate (17) as $-\nabla^*\nabla A$ where ∇A is defined in (16).

$$\begin{aligned}(-\nabla^*)(\nabla A) &= (-\frac{i_t}{c}\frac{\partial}{\partial t} + \vec{i}\cdot\vec{\nabla})(-S + \vec{i}\cdot(B - \frac{i_t}{c}E))\\ &= -\vec{\nabla}\cdot B + \frac{i_t}{c}\vec{\nabla}\cdot E\\ &\quad + \vec{i}\cdot(\vec{\nabla}\times B - \frac{1}{c^2}\frac{\partial}{\partial t}E - \frac{i_t}{c}(\frac{\partial}{\partial t}B + \vec{\nabla}\times E)\\ &\quad + \frac{i_t}{c}\frac{\partial}{\partial t}S - \vec{i}\cdot(\vec{\nabla}S)\end{aligned}$$

Such that $\frac{i_t}{c}\frac{\partial}{\partial t}S - \vec{i}\cdot(\vec{\nabla}S)$ is the additional term in the expansion. We may now choose to group these terms with their electromagnetic vector and scalar counterparts, using (3) and (2), given (1) and (2) and the equality of (17) unchanged

$$\vec{\nabla}\cdot E + \frac{\partial}{\partial t}S = \frac{\rho_v}{\epsilon_0} \tag{18}$$

$$\vec{\nabla}\times B - \frac{1}{c^2}\frac{\partial}{\partial t}E - \vec{\nabla}S = \mu_0 J \tag{19}$$

Examination of (18) and (19) with their Proca-Maxwell counterparts (12) and (11) respectively show the following relation

$$\frac{\partial}{\partial t}S = \mu_\gamma^2\phi \tag{20}$$

$$\vec{\nabla}S = -\mu_\gamma^2 A \tag{21}$$

Of the scalar field S with the photon mass. These equations (20, 21) show the Lorentz gauge as invalid in the case of non-zero photon mass, that is $S \neq 0$ [7]. Using this relation in the homogeneous wave equation (9) shows the following

$$-\nabla^2 S = -\vec{\nabla}\cdot\vec{\nabla}S = \vec{\nabla}\cdot\mu_\gamma^2 A$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}S = \frac{1}{c^2}\frac{\partial}{\partial t}\mu_\gamma^2\phi$$

Which in the wave equation shows

$$\vec{\nabla}\cdot\mu_\gamma^2 A + \frac{1}{c^2}\frac{\partial}{\partial t}\mu_\gamma^2\phi = \nabla\cdot J + \frac{\partial\rho_v}{\partial t} = 0 \tag{22}$$

Where J and p_v are zero in free space, describing a homogeneous wave equation in terms of variation of photon mass and potential. This of course assumes a particular velocity as c, which is not determined as in the scalar field case. Additional experimentation is necessary to validate the claims presented in this equation.

Rearranging (19) shows, the contribution of the negative gradient $-\vec{\nabla}S$ in the displacement current, which is equal to $\mu_\gamma^2 A$. Furthermore, (20) and (21) show the time variation of the scalar field resulting in non-zero photon mass, scalar and vector potentials. In other words, time variation of the quantum medium results in the production of photon mass and electromagnetism. To further understand the relation presented, we integrate (21), applying Stokes' theorem and equate this with (20)

$$S = -\mu_\gamma^2 \int B \cdot d\vec{S} = -\phi_m \mu_\gamma^2$$

$$-\frac{\partial \phi_m}{\partial t} = \phi$$

Which is Faraday's law of induction. It can be shown a time integral of (20) produces the same result . In the condition where photon mass is zero, the Lorentz gauge holds and electromotive force is no longer derived from the scalar field as shown. When the vector and scalar potentials are taken to be the principal quantities in electromagnetism, this leads one to hold that photon mass therefore exists, however negligible its quantity be in macroscopic measurements. Let us take equation (19) in free space, such that $J = 0$, rearranging terms

$$-\frac{1}{c^2}\frac{\partial}{\partial t}E = -\vec{\nabla}S + \vec{\nabla} \times B = \mu_\gamma^2 A + \vec{\nabla} \times B \tag{23}$$

Where at the center of the displacement current, in which $B = 0$, analogous to the condition at the center of a cylindrical conductor, should we assume the negative sign to be the result from the magnetic vector potential, we determine the mass to be in proportion to the time variation of the electric field, and equivalently, the gradient in the scalar field. We will now determine the known properties of particles in the equivalent class as the photon, to discern its relation to the scalar and vector potentials. In particle physics, photons are considered to be Spin-1 bosons with zero mass, and only transverse states of polarization. Massive photons therefore have qualities similar to other massive spin-1 particles. This includes the longitudinal polarization. In the context of the displacement current, the electric field is longitudinal, in the direction of power flow in the circuit. We assume the real photon, that is to say, not the virtual photon, as the mediator of this energy transfer, and given a finite mass, interaction involving the transverse and longitudinal polarization state and mass of the photon. The direction of momentum in the photon is related to the Poynting vector, $p = \frac{\langle E \times H \rangle}{c}$ which means by definition the longitudinal electric field is transverse to the photon, and electromagnetic energy transfer is in the photon's longitudinal direction. Therefore, in this model, time variation of the transverse electric field with respect to the photon momentum direction results in time variation of photon mass, as indicated in (23). The energy of the photon is given from the relation

$$E = h\nu \tag{24}$$

In which h describes the quanta of energy relating the photon's energy with frequency. The time varying electric field constitutes both a longitudinal displacement current and transverse electromagnetic field of equal frequency. Due to the relativistic energy relations being based on momentum, we must turn to electromagnetic quantities for further calculation. The photon energy is shared between the electric and magnetic fields, where the electric field's energy density is based on its capacitance at that location in space [10]. In other words, $E_c = \frac{h\nu}{2}$ The capacitance of two parallel plates of area A separated by distance d with permittivity ϵ_0 is defined as

$$C = \frac{\epsilon_0 A}{d} \tag{25}$$

Next we consider the spatial geometry for the photon to store this energy capacitively. Using the relation $E_c = \frac{h\nu}{2} = \frac{1}{2}CV^2$ with the energy determined in (24) and the voltage determined from the scalar potential $\Delta\phi = \phi(x+d) - \phi(x)$ we get the following relation

$$\frac{h\nu}{\Delta\phi^2 \epsilon_0} = \frac{A}{d} \tag{26}$$

Which relates the spatial geometry of the massive photon to the energy, potential and permittivity to capacitively store its electrical energy. This relates only the ratio of area to distance, therefore even at the smallest scales, capacitance of a photon with volume $\approx Ad$ is possible. Since this ratio depends on the energy and scalar potential in the system, we imagine the massive photon volume varying with scalar potential and frequency, where frequency determines the energy and with potential, the required volume and geometry. The possibility of time varying permittivity with photon volume is indicated as well. With the mass and energy related with $E = mc^2$ we may assume a massive photon changes both its volume and mass as energy is capacitively stored. Similar statements can be made with regards to inductance, we focus on capacitance due to the magnitude of the electric field compared to the magnetic flux density, related with $E = cB$, in which the quantity of energy is equally shared across both fields. The geometric relation in determining the intrinsic capacitance and inductance provides the quantitative means to determine the underlying structure of the massive photon, and additionally provide insight into the underlying geometric structure of the mass, the variations of which, create the principal quantities through which the classical electrodynamic relations are derived.

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