Collatz Conjecture: A Coordinate System Based Approach

Abstract

The Collatz conjecture states any number, N_0, after successive computations will always yield one, initiating the recursive sequence of 1→4→2→1→4 because 1 equals itself via $\frac{(3n+1)}{2}$ for n = 1. Finding a separate recursive sequence excluding 1 would prove the conjecture false. Analysis of the Collatz conjecture with a dual matric system (beta tables and gamma tables) revealed only one such recursive sequence is possible. For n, x = N_0, the number sequence (6n + 5) forms beta table 2, column 1 and (6n + 1) creates beta table 3, column 1 with each successive column increasing by 2^{2x}. The formula $\frac{(n-1)}{3}$ was used on each beta entry to produce the gamma tables defined by the following sequences, for n = N_0: (4n + 3) = m forms gamma table 2, column 1 and (8n + 1) = m forms gamma table 3, column 1 where each successive column increases by 4m + 1. This reveals the $\frac{(3n+1)}{2}$ quotients of all odd numbers connect as follows: $(4n+3) \rightarrow (6n+5)$ and $(8n+1) \rightarrow (6n+1)$, accounting for all possible unique connections between odd numbers via $\frac{(3n+1)}{2}$. The differences between connecting sequences are $(4n+3)-(6n+5)=2m$ and $(8n+1)-(6n+1)= -2m$ where the difference only equals zero for n = 0 as follows: $(8 \times 0+1)-(6 \times 0+1)= 0$, indicating the recursive loop as found with $\frac{(3n+1)}{2}$ for n = 1. Using the matric system, I herein demonstrate the uniqueness of the only known recursive loop and prove it is the only one of its kind thus solving a major part of the Collatz Conjecture.

Definitions of terms

Collatz Conjecture – For any number N_0 put through repeated iterations of the Collatz conjecture equations and accompanying rules, 1 will eventually be reached through sufficient cycles and begin a recursive loop of 4 → 2 → 1 → 4→ 2 → 1… If false, then a number(s) exists which will exclude one from the cycle or a cycle will increase indefinitely with no recursive loop.

Collatz Conjecture Equations – the set of two equations and their respective rules governing the Collatz conjecture:

1. Equation and rule 1: (3n + 1) is used for any odd number (2n + 1)
2. Equation and rule 2: $\frac{n}{2}$ is used for any even number 2n.

Collatz Number – the sequence of numbers being the $\frac{(n-1)}{3}$ integer derivatives of the 2^x sequence whereby 2^x (1, 2, 4, 8, 16, 32, 64…) ➔ (1, Ø, 5, Ø, 21, Ø, 85…). $1(2^x) = 2^x$ → so for n=0 $(2n+1)(2^x) = 2^x$. Implicates the recursive sequence.

Matric Systems – Two sets of N and N_0 arranged into two interrelated matric systems derived from the logic of 3n + 1 and $\frac{n}{2}$: consists of the three primary tables, termed beta $\beta_x\underline{X}$, and its three derivative tables, gamma, $\gamma_x\underline{X}$. Represent an ordered way to complete the Collatz cycle for a particular number, n. Gamma tables are used for odd entries; beta tables are used for even entries to yield an odd entry. See tables 1 through 7.

Where $\beta_x\underline{X}^{z-n}$ and $\gamma_x\underline{X}^{z-n}$ indicate position in the corresponding table as follows:

β or γ: beta or gamma table; subscript x: table number; X: column number; superscript z-n: column position and row number. Together, these indicate the exact position in a table. E.g., 17 in beta table is ($\boldsymbol{\beta_2 0^{3-1}}$); i.e., 17 is located in beta table 2, column zero, 3rd row down and first row entry; and 17 gamma is ($\boldsymbol{\gamma_3 2^{3-2}}$) gamma table 3, column 2 row 3, 2nd row entry. See table 1.

Beta Table System – Separation of N_0 into three distinct tables based on $(2m + 1)(2^{2x})$ where m = the sequence of numbers 3n, 3n + 2, and 3n + 1, respectively, for β_1X, β_2X, β_3X. E.g., 5(2) = 10; 10(2) = 20; 20(2) = 40… Used to derive the γ_xX tables. A mathematical device highlighting the connections between two specific odd numbers as they cycle through the Collatz equations. See tables 2 through 4.

- The odd and even numbers are derived as follows (n, x = N_0) :
  - $(6n + 3)2^x$
  - $(6n + 1)2^x$
  - $(6n + 5)2^x$

Beta sequence – Comprised of a row in the beta table system. Each successive row entry increases by 2n from the previous row entry starting with the primary beta.

Primary Beta (row head) – Unique entries down the first column of the beta grid, containing the entire odd number sequence of 2n+1 among the three β_x tables. Reached through successive $\frac{n}{2}$ divisions of a 2n entry along a row. $\beta_1 0$, $\beta_2 0$, $\beta_3 0$. Also considered a critical quotient.

- $\beta_1 0 = 6n + 3$, n = N_0

- $\beta_2 0 = 6n + 5$, $n = N_0$
- $\beta_3 0 = 6n + 1$, $n = N_0$

Lesser Beta – Unique entries following the primary beta of which they are a product: $\beta_x 0(2^z)$ where super z represents row position. They are a doubling of the previous entry of the same row.

Gamma Table System – Derivation of N_0 from N into three corresponding distinct tables derived from, respectively, β_1X, β_2X, and β_3X via $\frac{(n-1)}{3}$ for each entry. N_0 is produced by all n of β_xX who are of the form 3 remainder 1. Demonstrates all 3 remainder one numbers can reproduce N_0. See tables 1 through 7. The gamma tables are defined by the gamma equations.

> Note: Derivatives of the β_1X, β_2X, and β_3X tables having either 3 remainder 0 (3r. 0) or 3 remainder 2 (3r. 2) are non-integers and are not considered in this paper; they are symbolized by Ø in tables 2 through 4.

Gamma sequence – Comprised of a row in the gamma table system. Each successive row entry increases by 4n + 1 from the primary gamma.

Zeroth Gamma – The first column, $\gamma_3 0$, of the gamma 3 table. Derived from $\beta_3 0$ [the integer sequence of (6n + 1)], it yields the number sequence 2n.

Primary Gamma (row head) – Unique entries down the first column of the gamma tables derived from the corresponding even β_xX entries. The first whole number $\frac{(n-1)}{3}$ derivative of a β_xX column (located in a $\gamma_2 1$, $\gamma_3 2$table). The γ_1Ø table has no integer derivatives. Also considered a critical quotient.

Lesser Gamma – Unique whole number entries along a gamma row following the primary gamma of which they are a product. Increase from the previous gamma by (4n + 1). Refer to the m, n grid system for further exposition. They connect to the same primary beta as their primary gamma. E.g., 53 is a lesser gamma to primary gamma 3; 3 is a critical quotient twin to 5 as follows: 3→10→5

Gamma Equations –

$$(6n+5)\frac{(2[2^{2m}-1])}{3}+(4n+3)=\gamma_2 X \text{ and } (6n+1)\frac{(4[2^{2m}-1])}{3}+(8n+1)=\gamma_3 X$$

Generalized as: $y(2x_1) + z = \gamma_2X$ and $a(4x_2) + b = \gamma_3X$ with γ_2 and γ_3 representing the odd numbers N produced therein (where x_1 and x_2 are multiples of the Collatz Numbers {1, 5, 21, 85, 341…}). Used in the formation of the m, n grid system.

m, n grid - A grid system derived from the gamma tables where the gamma Equations are used to encode each entry of the respective table into a coordinate system based on the variables m and n. See table 5 through 7.

Primary Column – Column 1 (β_1X) of the beta tables: $\beta_1 0$ $\beta_2 0$ $\beta_3 0$. Column 2 in gamma table 2 ($\gamma_2 1$) and column 3 in gamma table 3 ($\gamma_3 2$). Gamma table 1 has no integer results and hence no primary Column and is denoted as γ_1Ø.

$\beta_1 0$ $\beta_2 0$ $\beta_3 0$ & γ_1Ø $\gamma_2 1$ $\gamma_3 2$

Critical Quotient – Primary Column entries; they are the critical juncture connecting beta and gamma grids via $\frac{(n-1)}{3}$ and 3n + 1. Consist of primary critical quotients and secondary critical quotients. Each critical quotient has a twin. See table 8.

Critical quotients consist of five sequences of odd numbers which correspond to the primary column of each table:

- $\beta_1 0$ (6n + 3)
- $\beta_2 0$ (6n + 5) and $\gamma_2 1$ (4n + 3)
- $\beta_3 0$ (6n + 1) and $\gamma_3 2$ (8n+1)

Primary Critical Quotient – Entries of the form (4n + 3) and (8n + 1). Located along the critical quotient columns which appear both as a $\beta_x X$ entry and as a $\gamma_x X$ entry. Also defined as $\gamma_2 1$ and $\gamma_3 2$ column entries.

Secondary Critical Quotient – Appear only once in the primary columns as a $\beta_x 0$ critical quotient entry. They are lesser gamma entries being of the form 3(8n – 1), (6n + 5) and (6n + 1); respectively of the $\beta_1 0$, $\beta_2 0$, and $\beta_3 0$. They are derived from those entries not found as $\beta_1 0$, $\gamma_2 1$ and $\gamma_3 2$ column entries (which will not be of the form (4n + 3) or (8n + 1)). Appear only once. Synonymous with Lesser critical quotient.

Primary Doubling – The first doubling of $\beta_x 0$ (the odd numbers) to produce a whole number integer in the gamma grid; for $\beta_2 0$ the first doubling produces a whole number gamma product $\gamma_2 1$; for $\beta_3 0$ the second doubling produces a whole number gamma product, $\gamma_3 2$. $\beta_1 0$ produces no integer gamma entries. See also critical quotients.

Twin – For γ_x and β_x tables, these are $\gamma_x X$ entries and the corresponding primary $\beta_x 0$ odd number entries. A $\gamma_x X$ entry, after 3n+1, divides to a primary $\beta_x 0$ via successive $\frac{n}{2}$ divisions. Directionally determined from $\gamma_x \rightarrow \beta_x$, not vice versa. They are located on the same row in their respective primary columns. They are equivalent in exactly one instance: in the case of 1. Key to understanding the lone recursive loop. Listed in the order of β_x, γ_x some examples are: 1, 1; 5, 3; 7, 9; 11, 7; 17, 11; 13, 17.

Identical Twin – The primary β_x entries ($\beta_x 0$) and their exact γ_x table equivalent (e.g.,1, 1; 17, 17; 21, 21). Directionally from $\beta_x 0 \rightarrow \gamma_x X$. Used when a $\beta_x 0$ has been determined in order to continue the Collatz cycle. Unlike the Twin, identical twins exist in separate matric systems at different locations and in that sense are non-equivalent: e.g., $\frac{34}{2}$ = 17 ($\beta_2 0^{3\text{-}1}$) is ID twin to 17 ($\gamma_3 2^{3\text{-}3}$); 17(3) + 1 = 52; $\frac{52}{2}$ = 26, $\frac{26}{2}$ = 13 ($\beta_3 0^{3\text{-}1}$) which is ID twin to 13 ($\gamma_2 3^{1\text{-}4}$). They may technically be the same number but for understanding the Collatz Cycles it is necessary to explain them as existing in separate number systems.

Conjoined/Siamese Twin – An integer whose primary beta and primary gammas are one and the same. One is the only possible case; it is conjoined to itself in the two tables via the recursive loop. One appears conjoined to itself in the graph version at the apex of a triangle. Unlike identical twins, one may exist in the same three-dimensional grid system, same location. …3(1) + 1 ➔ 1 🡐 $\frac{2}{2} = \frac{4}{2}$…

Critical Difference – The difference between twin numbers. Essentially the difference between the a, b and x, y factors of the gamma equations.

- $(6n + 5) - (4n + 3) = 2n; n = N_0$
- $(6n + 1) - (8n + 1) = -2n; n = N_0$

See Table 8.

Triangular-faced Trapezoidal prism – The geometric shape resulting from plotting the interconnected beta and gamma tables onto a three-dimensional grid system. The short base is formed by the difference between a primary beta and it's twin primary gamma. The legs are formed by the lesser beta and gamma pairs. The long base is technically infinite though the distance of a chosen beta/gamma lesser pair will determine base length. No difference exists between 1 and it therefore forms the triangular apex of this shape.

4n + 1 and 4n +3 – Prime number generators, interestingly but maybe inconsequential, constituents behind the logic of the matric system structure as seen in the gamma equations.

Introduction

Mathematicians are warned against pursuing its solution given its seemingly simple rules and to instead focus on something that can actually be solved [1 -3, 5]. Prompting the quip from prolific peripatetic mathematician Paul Erdös "Mathematics is not yet ready for such problems" [1] it seems we are nearer but paradoxically no closer to solving it. They have been termed Hailstone numbers and "Wondrous Numbers" [4] and have as many names as there are ways of teasing apart its patterns [6]. It has been featured on "Numberphile" with an exceptional primer to this vexing problem given by Prof. David Eisenbud [2].

A simple number line is created by choosing any integer and following the simple equations of 3n +1 for n odd and $\frac{n}{2}$ for n even producing the classic Hailstone pattern so familiar to those in the know. I am, of course, alluding to the Collatz conjecture and its baffling effect on entangling the number line into an ungovernable mess of numbers. This conjecture, also called the 3n + 1, $\frac{n}{2}$ problem, is a deeply explored though largely unsolved problem. This simple mathematical curiosity has evaded a proper solution to determine if all entries will return to one. It has been demonstrated that it is almost true for almost all numbers [7] and many proofs can be found doing a simple Google Scholar search. But still, it evades a definitive proof and the question remains: does a sequence exist that does not contain 1? If there does exist such a sequence it is said such a sequence is either recursive to the exclusion of 1 or it will increase without bound [8].

The classic hailstone pattern produced by playing with a given number is a snapshot of what is happening with the set of integers when a single integer is put through the Collatz equations. When several hailstone sequences were compared together in my studies using both MS Excel and paper, key patterns of interconnection became evident. Through analysis of these hailstone sequences, these key patterns eventually led me to their re-ordering based on the divisibility of positive integers by three. In turn, this eventually led to the transformation of these hailstones into a two-part matric system.

By laying out two sets of integers in table form, it became evident the up and down hailstone signatures came from an orderly and completely predictable arrangement of both separate but linked positive integer table sets. This matric system method is outlined in this paper and it forms the basis to further understanding the effect of the deceptively simple Collatz equations on the number line.

The main purpose of this paper is to show a simple, direct proof that only one recursive sequence exists within the framework of the Collatz system by demonstrating another number cannot beget a separate recursive sequence in the same manner as found with one. By teasing apart the logic behind the Collatz equations, I will demonstrate how 3n + 1 and $\frac{n}{2}$ cannot produce a secondary recursive sequence by using two sets of tables derived from the Collatz equations. One table system consists of N arranged into 3 parts. The second table system is a derivative of the first table system consisting of N_0. Within this matric framework are key patterns which have been discovered (herein termed critical quotients) and used to show a proof that only one recursive sequence exists. These critical quotients include the entire set of odd numbers connected in such a way that precludes the possibility of another such recursive sequence. By demonstrating these critical quotients diverge from each other by $|2n|$, I will demonstrate that a separate recursive sequence of the form found in the 4→2→1→4 sequence is not possible.

Materials and Methods

All calculations performed using calculator (Texas Instruments, TI-30xa/TI-36x), Microsoft Excel (MS Excel), pencil, pen, marker (Sharpie) and paper. Unfortunately, chalk and chalkboard were not available for these analyses.

Results

Table formation

Detailed below is the summary of my findings used to derive the tables from the Collatz conjecture. They explain the logic behind the tables and the mathematical reasoning used to arrive at their formation. These patterns become evident by studying the tables. They can be used in conjunction with tables 1 through 8 to illustrate both how the tables were formed and the patterns therein. I developed vocabulary words pertaining to concepts of the matric system to more easily and concisely discuss my findings found in the definition of terms sections.

Patterns evident in hailstone sequences were initially used to devise the beta table and then to derive the gamma table. I first constructed the tables, analyzed their patterns, formulized them, then further analyzed these formulas in conjunction with the tables. Eventually arriving at the key finding of the critical quotients and the divergent differences between twin critical quotients. The results below represent the summary of the mathematical reasoning I used to find the logic behind the tables and how I arrived at my conclusion.

Part 1: Groundwork for gamma table formation. What does 3n + 1 yield?

- 3n + 1 = 6n + 4 for n = 2m + 1 for m = N_0
    - i.e., odd numbers yield all even numbers of form 3 r. 1 from 3n + 1
        - So, the 3n + 1 Collatz equation yields only a subset of 2n, those of form 6n + 4
    - 3n + 1 ≠ 3r.0 and 3n + 1 ≠ 3r.2
        - 3(2n+1) +1 = 6n+4 → 6n + 4 = $2(3n + 2)$ → $\frac{[(6n+4)-1]}{3}$ = $2n+1$
        - This portion is key for the logic of the gamma table formation
- 3n + 1 = 6n + 1 for n = 2m, m = N
    - i.e., even numbers yield only a portion of N from 3n + 1
    - Therefore 3n + 1 only yields a subset of N
- **Conclusion 1**: 3n + 1 yields only a subset of N
    - 3n + 1 = 6n + 1 and 6n + 4, for n = N
        - 3n +1 = 1 for n = 0 of number set N_0
            - We get the additional number 1, technically not a multiple of 3
            - But this is an additional number not produced from n = N

Part 2: Key to understanding gamma table formation. What if we reverse 3n + 1 for N?

- In reverse using $\frac{(n-1)}{3}$ for n = N, we find that N_0 is produced from N such that N = N_0 for $\frac{(n-1)}{3}$
- From $\frac{(n-1)}{3}$ we get both 2n and 2n + 1 for n = N
    - $\frac{(n-1)}{3}$ = N for n = 3m+1 and m =N → $\frac{[(3m+1)-1]}{3}$
    - $\frac{(n-1)}{3}$= 2n for n = 6m+1, m = N → $\frac{[(6m+1)-1]}{3}$
    - $\frac{(n-1)}{3}$ = 2n+1 for n = 6m +4, m = N → $\frac{[(6m+4)-1]}{3}$
        - This is the basis for gamma table formation.
        - $\frac{(n-1)}{3}$ for even numbers of form 6m + 4 yield whole number odd integers
        - $\frac{(n-1)}{3}$ for odd numbers of form 6m + 1 yield whole number even integers
    - Zero only appears as the $\frac{(n-1)}{3}$ root of 1
        - Just as 3n + 1 = 1 for n = 0, $\frac{(n-1)}{3}$ = 1 for n = 1
        - This will play a role in understanding uniqueness of the recursive loop sequence and the non-uniqueness of 1 in the matric system
- **Conclusion 2**: $\frac{(n-1)}{3}$ will yield the entire set of N under the following conditions:
    - n = 3m + 1 to yield N
    - n = 6m + 1 to yield 2n
    - n = 6m + 4 to yield 2n + 1

The gamma table system is set up as follows:

- $\gamma_2 X$: $(4n+3)$ for column one and $(4n+3)\times 4+1$ for each successive row entry
- $\gamma_3 X$: $(8n+1)$ for column one and $(8n+1)\times 4+1$ for each successive row entry

Part 3: Key to understanding beta table formation. How do even numbers derive from 2n + 1?

- $\frac{2m}{2}$ = 2n + 1 for m = 2n + 1, n = N_0
    - $\frac{[2(2m+1)]}{2}$ = $\frac{(4m+2)}{2}$ = 2n + 1 for m = n, n = N_0
    - So 2^x(2n + 1) returns to 2n + 1 when divided by 2
- **Conclusion 3**: $\frac{n}{2}$ will yield 2n + 1 after sufficient divisions of 2
    - The set of odd numbers can be used to produce all even numbers
    - $\frac{n}{2}$ solved for n = N_0 subset (2n) yields the N_0 subset (2n + 1)
    - $\frac{n}{2}$ = 2n + 1 or some multiple thereof
    - Key point: (2n + 1)2^x = N for n, x = N_0
        - E.g., (2 x 0 + 1)2^0 = 1; (2 x 0 + 1)2^1 = 2…
    - $\beta_1 0$ = 6n + 3, n = N_0
    - $\beta_2 0$ = 6n + 5, n = N_0
    - $\beta_3 0$ = 6n + 1, n = N_0
- From this reasoning the beta tables are constructed

The beta matric system is set up as follows for n, x = N_0:

- $\beta_1 X$: $(6n+3) = m$ for column one and$(m)(2^{2x})$ for each successive row entry
- $\beta_2 X$: $(6n+5) = m$ for column one and $(m)(2^{2x})$ for each successive row entry
- $\beta_3 X$: $(6n+1) = m$ for column one and $(m)(2^{2x})$ for each successive row entry

Section Conclusion

As a result, we have modified two sets of numbers: one is N for the $\frac{n}{2}$ or 2n number set used to form the beta tables; the other is N_0 for the 3n + 1 or $\frac{(n-1)}{3}$ set used to form the gamma tables. Both derived from the Collatz conjecture equations.

Using this mathematical reasoning we derived a two-part matric system of N and N_0: β_x and γ_x. Each consists of three tables where β_x is a reordering of N and γ_x as a derivative of β_x via $\frac{(n-1)}{3}$ is a reordering of N_0.

The logic of the beta matric system begins by breaking N into a simple table where 2n+1 forms column 1, and each row is built by multiplying the preceding number by 2, forming rows of $(2n+1)2^x$. This creates a table of N where 2n+1 develops column 1 and successive columns build N subset (2n) via $(2n+1)2^x = 2n$. This table is then separated into 3 tables (β_1, β_2, β_3) where column 1 of each table is an odd number (2n+1) of a particular form. (See Tables 3 – 5)

1. Table 1 column 1 are 3 remainder 0 odd numbers which have the form 6n+3 called β_1
2. Table 2 column 1 are 3 remainder 2 odd numbers which have the form 6n+5 called β_2
3. Table 3 column 1 are 3 remainder 1 odd numbers which have the form 6n+1 called β_3
    a. Successive columns for each β_x are doublings (2^x for x = N) of the previous entries.

What happens if we take the $\frac{(n-1)}{3}$ root of the beta table?

These 3 tables are used to create 3 additional tables by solving each β_xX entry for $\frac{(n-1)}{3}$, producing the γ_1, γ_2, and γ_3 tables. From this, $N_0$is produced; but in a unique and revealing pattern.

The γ tables produced arrange the set of integers N_0 according to a fixed pattern through the γ equations. Additionally, each whole number entry along a row increases by 4n+1; e.g., 3(4) +1 =13; 13(4) + 1 = 53; 53(4) + 1 = 213. Key observations of the gamma tables are:

- γ_1 produces no N as it consists of all 3 remainder 0 integers.
- γ_2 produces only 2n+1 quotient entries.
- γ_3 produces both the entire 2n series as entries in column 0 and the remaining entries are 2n+1 in all other consequential columns for which whole numbers are produced.

Since each γ row is produced by its corresponding β row via $\frac{(n-1)}{3}$, they have a logic to them deriving from its β row whereby they can be independently produced using another set of equations herein termed gamma equations. These equations naturally contain elements of the beta table logic: increases by 2^x and $\frac{(n-1)}{3}$ are found in $\frac{(2^{2m}-1)}{3}$ odd number elements (6n + 5 and 6n + 1):

$$(6n+5)\frac{(2[2^{2m}-1])}{3}+(4n+3) = \gamma_2 X \text{ and } (6n+1)\frac{(4[2^{2m}-1])}{3}+(8n+1) = \gamma_3 X$$

For m, n = N_0

These say that for any given gamma row, successive row entries increase according to a 2n multiple of a given Collatz Number (i.e. $\frac{(2^{2m}-1)}{3}$) factor (2^1 for β_x; 2^2 for γ_x), the multiplicative factor of 6n+5 (β_2) or 6n+1 (β_3) with the additive factor of 4n+3 (γ_2) or 8n+1 (γ_3), respectively. Further, these can be arranged in a grid system termed the m, n grids (See Tables 6 to 8) according to the gamma equations. Gamma equations are explicated further below.

The β_x / γ_x tables and associated gamma equations demonstrate the interconnection of the two sets of positive integers according to the logic of the Collatz conjecture and will aid in explaining why only one recursive sequence exists in this framework.

Summary of the beta and gamma table logic

The tables work according to the logic and reasoning of the preceding section parts 1 to 3. In summary, beta tables show that:

- Every 2n divides by ($\frac{n}{2}$) (after successive repetitions) to a 2n+1
  - Therefore every (2n+1) is the generator of a unique (2n) number line, which when taken as a whole include all N.
- Taking the $\frac{(n-1)}{3}$ root of each entry reproduces all integers, N_0. Only integers with form 3 remainder one yield integer results.
  - Therefore $\frac{(n-1)}{3}$ for n = 3n + 1 = N; $\frac{[(3n+1)-1)]}{3} = \frac{(3n)}{3}$ = n
    - This is the basis of the gamma tables

From these tables we see that beta tables are simply a particular series of 2n + 1 and their 2^x multiples. Also, we can notice how each element of the gamma equations is evident within these tables. The gamma equations, independently from the beta tables, reconstruct the gamma tables into a coordinate system, each entry of the table having two coordinates, the variables m and n (Tables 6 to 8). These two equations compute all odd integers in unique fashion and are derived from the $\frac{(n-1)}{3}$ quotients of the beta tables.

$$(6n+5)\frac{(2[2^{2m}-1])}{3}+(4n+3) = \gamma_2 X \text{ and } (6n+1)\frac{(4[2^{2m}-1])}{3}+(8n+1) = \gamma_3 X\text{, For m, n} = N_0$$

Derivation of the γ equation variables are as follows:

γ_2: $((6n+5)\frac{(2[2^{2m}-1])}{3}+(4n+3) = \gamma_2 X$, For m, n = N_0

- G eneralized form of the equation: yx + z = γ_2
  - y = 5, 11, 17, 23, 29... → 3m+2 for m = 2n+1 → 3(2n+1) + 2 = **6n+5**
  - z = 3, 7, 11, 15, 19… → 2m+1 for m = 2n+1 → 2(2n+1) + 1 = **4n+3**
  - x = 2[0, 1, 5, 21, 85...]

γ_3: $(6n+1)\frac{(4[2^{2m}-1])}{3}+(8n+1) = \gamma_3 X$, For m, n = N_0

- Generalized form of the equation: ax + b = γ_3
  - a = 7, 13, 19, 25… → 3m+1 for m = 2n → 3(2n) + 1 = **6n+1**
  - b = 1, 9, 17, 25… → 2m+1 for m = 4n → 2(2(2n)) + 1 = **8n+1**
  - x = 4[0, 1, 5, 21, 85…]

Astoundingly, the gamma tables can be derived by at least three different methods:

1. Taking the $\frac{(n-1)}{3}$ derivative for each even number as arranged in the beta tables
2. From the following formulae (n = N_0):
   a. 8n + 1 = a
      i. 4a + 1 = b; 4b + 1 = c; 4c + 1 = d…
   b. 4n + 3 = z
      i. 4z + 1 = y; 4y + 1 = x; 4x + 1 = w…
3. From the following equations (m, n = N_0):
   a. $(6n+5)\frac{(2[2^{2m}-1])}{3} + (4n+3) = \gamma_2 X$
   b. $(6n+1)\frac{(4[2^{2m}-1])}{3} + (8n+1) = \gamma_3 X$

Critical quotients: Is there a revealing connection between the beta and gamma tables?

Critical quotients consist of one full set of odd numbers (beta primary columns) and one partial set of odd numbers (gamma primary columns). Also, they show the critical juncture where a gamma entry connects to a beta entry; or, how an odd number connects with another odd number via $\frac{(3n+1)}{2}$.

Critical quotients consist of five separate sequences of odd numbers which correspond to the primary column of each table: the primary column in the beta tables ($\beta_1 0$, $\beta_2 0$, $\beta_3 0$) consisting of all odd numbers; the primary column in the gamma grids ($\gamma_2 1$, $\gamma_3 2$), consisting of odd numbers of the form (4n + 3) and (8n + 1).

- $\beta_1 0$ (6n + 3)
  - Governs $\beta_1 0$ column increase
- $\beta_2 0$ (6n + 5) and $\gamma_2 1$ (4n + 3)
  - The equations for the $\beta_2 0$ and $\gamma_2 1$ critical quotient lines. (6n + 5) – (4n + 3) = 2(n + 1) such that the difference between twin rows increases by 2n (n = N_0) starting with row one.
    - (6n + 5) governs $\beta_2 0$ column increase
    - (4n + 3) governs $\gamma_2 1$ column increase
    - The difference of the additive factors for n = 0, is 2: 5 – 3 = 2
      - This continues with successive rows (See Table 8)
- $\beta_3 0$ (6n + 1) and $\gamma_3 2$ (8n+1)
  - The equations for the $\beta_3 0$ and $\gamma_3 2$ critical quotient lines. (6n + 1) – (8n + 1) = -2n such that the difference between twin rows increases by 2n (n = N) starting with row one.
    - (6n + 1) governs $\beta_3 0$ column increase
    - (8n + 1) governs $\gamma_3 2$ column increase
    - The difference of the additive factors for n = 0, is 0: 1 – 1 = 0 for n = 0
      - Then the difference is 2 for n = 1: 7 – 9 =-2
      - This continues with successive rows (See Table 8)

Of unknown significance: The critical quotient sequence not including beta table one primary column contain a full set of odd numbers except those of the following form:

- 3(8n - 1) – odd numbers from $\beta_1 0$
- So together, the following sequences include all odd numbers except those of the form $3(8n-1)$
  - $(4n+3)$
  - $(6n+5)$
  - $(8n+1)$
  - $(6n+1)$

The critical quotients reveal the following pertinent findings by the proceeding steps:

The beta matric system is set up as follows for n, x = N_0:

- $\beta_1 X$: $(6n+3)$ for column one and $(6n+3)(2^{2x})$ for each successive row entry

- $\beta_2 X$: $(6n + 5)$ for column one and $(6n + 5)(2^{2x})$ for each successive row entry
- $\beta_3 X$: $(6n + 1)$ for column one and $(6n + 1)(2^{2x})$ for each successive row entry

The gamma matric system consists of all odd numbers arranged as follows:

- $\gamma_2 X$: $(4n + 3)$ for column one and $(4n + 3) \times 4 + 1$ for each successive row entry
- $\gamma_3 X$: $(8n + 1)$ for column one and $(8n + 1) \times 4 + 1$ for each successive row entry

Each row of each gamma table is equal to an even product of the corresponding beta table via 3n + 1.

- $(4n + 3) \times 3 + 1 = (6n + 5)\ (2^{2x})$
- $(8n + 1) \times 3 + 1 = (6n + 1)(2^{2x})$

This revealed the connection between any two odd numbers via $\frac{(3n+1)}{2}$ is always unequal except in the case of one where the recursive loop occurs.

- $(4n + 3) \neq (6n + 5)$
- $(8n + 1) \neq (6n + 1)$ except for n = 0

And by using the following

- $(4n + 3) \times 3 + 1 = (6n + 5)$
- $(8n + 1) \times 3 + 1 = (6n + 1)$

Arranged in the matric system I show their differences are as follows:

- $(6n + 5) - (4n + 3) = 2m$
- $(6n + 1) - (8n + 1) = -2m$
    - Except for n = 0

In sum, the critical quotients contain the following significant information:

- $(4n + 3) \neq (6n + 5)$
- $(6n + 5) - (4n + 3) = 2m$
- $(8n + 1) \neq (6n + 1)$
    - except for n = 0
- $(6n + 1) - (8n + 1) = -2m$

With the following conclusion:

- The recursive loop occurs because the difference between one derived using two different formulae equals zero.

The two different formulae arose from using the matric system which elucidated the connections between odd numbers via $\frac{(3n+1)}{2}$. From which I found certain key patterns leading to defining the critical quotients and concluding their differences as proof only one recursive loop of the only known type exists.

Results Summary: Why does this indicate a lone recursive loop?

The connection between beta and gamma is direct, as gamma is derived from beta. However, this does not fully reveal the connection between the odd numbers of each table, or their connection via the Collatz equations. It does not detail the full amount of information contained in the gamma table nor how the odd numbers connect to the beta odd numbers; there is more data, and it is revealed in the patterns in the gamma table.

The primary gamma table columns are of two equations and when given to 3n + 1 yield an even number which divides to the beta odd quotient. Because succeeding gamma odds of a row are even products of the primary gamma by $2^2 + 1$, they yield by 3n + 1 an even number of the same beta odd root. Thus, the lesser gammas are equivalent to the primary gamma, and both will yield the same primary beta.

This is seen in how the primary beta's increase as follows:

- From the following formulae ($n = N_0$):
  - 8n + 1 = a
    - 4a + 1 = b; 4b + 1 = c; 4c + 1 = d…
  - 4n + 3 = z
    - 4z + 1 = y; 4y + 1 = x; 4x + 1 = w…

More information is contained in the gamma tables in how they can be constructed independent of the beta table. It also sets up the gamma table as a coordinate system where each entry has position of m_x and n_y for x, $y = N_0$. Each table can be constructed using one equation for each:

- From the following equations:
  - $(6n+5)\frac{(2[2^{2m}-1])}{3} + (4n+3) = \gamma_2 X$
  - $(6n+1)\frac{(4[2^{2m}-1])}{3} + (8n+1) = \gamma_3 X$

Together, these equations contain further information showing how primary beta and gamma odds connect to an odd number different than itself, except for one which connects to itself. These connections, the critical quotients, differ from each other in increasing magnitude by |2m|. As such, we clearly show that a recursive loop of the type for which one initiates is not possible for any pair of odd numbers.

Being the only of its kind, I do not disprove the possibility there may be another loop which would include more than one sequence of numbers, or multiple rows as laid out in the matric system. This seems unlikely since all numbers expand from one and its 2^{2x} multiples. And if they are produced from one, it seems logical all numbers can return to it via this peculiar set of Collatz equations and its simple rules.

The matric system shows any given entry can be shown to expand out to infinity. But all numbers originate from one. It seems unlikely any given number could expand continually without eventually returning to its source on the $1(2^{2x})$ number line. That is, since all numbers originate from this Collatz line, all will of course return to it precluding the possibility for a runaway sequence.

Discussion

Critical quotients and the recursive sequence

The Collatz Conjecture follows the course of any given number through a series of calculations and in every calculated case always arrives at one. I propose, through a series of proofs, this will always be the case. After playing with the CC I arrived at a certain way of ordering the number line N_0. This in turn led to the table system I developed described in this paper.

The solution hinged on separating the number line into three parts: 3 remainder zero, 3 remainder one, and 3 remainder 2. Once I did this, I was able to make quick progress in developing the table system and eventually determining the equations governing the gamma tables, or the $\frac{(n-1)}{3}$ derivative of each odd numbers 2^N expansion.

Theses equations, call them gamma equations, show that for every even number whose value less one is divisible by three will be an active player in the Collatz conjecture. Stemming from every single odd number is an infinite sequence of odd numbers connecting to it via its $\frac{(n-1)}{3}$ derivative of its 2^N expansion.

This means that an infinite sequence of odd numbers will $\frac{(3n+1)}{2}$ to exactly one odd number in a completely predictable manner thanks to the table system and the gamma equations. And this is true for every odd number. Each odd number gives rise to an infinite sequence of odd numbers each in turn giving rise to their own infinite odd number sequences ad infinitum. Even more amazingly, they all start from the 1^{2N} sequence thereby connecting all numbers into the framework of N_0.

Each odd number is linked together to a single odd number sequence unequal to itself, except for one, via the $\frac{(3n+1)}{2}$ equation. The connection between each odd number unfolds in two series of diverging sequences, the divergence of which is 2n and -2n. So each odd number down the number line is linked to infinitely many other odd numbers in such a way that excludes the possibility of recursion as occurs for one and only one. This proves recursion will not occur in the Collatz Conjecture as it does for one.

Put another way, no odd number produces another odd number in its $\frac{(n-1)}{3}$ derivatives which equal itself: it is impossible.

in two separate diverging sequences (see divergent sequences table) proving a recursive sequence is not possible except for the case of one.

Analysis of the Collatz conjecture using the beta-gamma matric system illustrates the case for a lone recursive sequence. Using the two-part matric system revealed an underlying logic fundamental to understanding how numbers connect and cycle through the Collatz equations. Collatz conjecture equations operate on N such that all numbers connect based on how odd numbers produce even numbers (beta tables). By studying the tables, it became evident the subset of even numbers of form 3 remainder 1 reproduced N_0 via $\frac{(n-1)}{3}$ (gamma tables).

The ordering of numbers in the beta-gamma matric system revealed an underlying pattern of key subsets of odd numbers, in the form of primary gamma numbers and their gamma sequences, connecting to unique primary beta numbers from which they ultimately derived. That is the connection between critical quotients, or twins. The connections between critical quotients, I discovered, are an abbreviated version of how numbers cycle through the Collatz equations, or how beta and gamma grids work. They show how one odd number connects to only one other odd number as twin entries along primary columns (See Table 8). In other words, an odd number from the gamma table is connected to an even multiple (of form 3r.1) of an odd number from the corresponding beta table row.

Put succinctly: the primary doubling of a primary beta derives to a primary gamma divergent by 2n. *This means a primary beta is derived from a number not equal to itself (e.g., 5 is derived from 3 and only 1 is derived from 1).* One is the singular case where a primary beta equals its primary gamma (the so-called Siamese twin): one, as a primary beta, doubles from 2 to 4 and 4 has a gamma derivative $\frac{(n-1)}{3}$ of one.

The critical quotients also revealed the foundational element for which a recursive sequence exists in this specific arrangement of N: when a twin is equal to itself. The possibility of a single recursive sequence became evident in the divergence of the critical quotients along the primary columns of the interconnected tables (Table 8). It became apparent the primary columns represent diverging series of odd numbers such that successive twins diverge from each other by a factor of 2n (beta-gamma table two) or -2n (beta-gamma table three). These series start diverging from beta-gamma table three, row one, where the difference between one and one is zero ($\beta_3 0^{1-1} - \gamma_3 2^{1-3} = 0$) which is the only location where this can happen.

This is the reason only one recursive sequence is possible. If another set of twin critical quotients were equal, they too would cause a recursive sequence. But that definitively cannot be the case because they diverge by a factor of |2n|. These matric systems account for all N_0, arranged according to the gamma equations and seen in the m, n grid (see table 8), such that no exception to the rule is possible because all numbers will operate according to the rules of the table. Primary gamma numbers link to a specific primary beta number; lesser gamma numbers link to the same primary beta as its parent gamma; primary and lesser gamma numbers include all odd numbers. Precluding the possibility for a rogue number to throw in a monkey wrench and cause a recursive loop.

To sum it up, primary beta-gamma connections are divergent by |2n|; in the case of 1, the difference is zero which defines a recursive loop. This divergence is the key component to understanding why recursive loops occur; it is because a number must come to equal itself to be recursive. The primary beta-gamma columns show all possible primary connections between odd numbers, i.e., where one odd number connects to another odd number via $\frac{(3n+1)}{2}$. Lesser connections, those between a beta and a lesser gamma are essentially the same connection as the connection between the beta and its primary gamma. The only difference is that the lesser gamma connects to a greater 2^x multiple of the primary beta: e.g., gamma 3 connects to beta 5 via 10; gamma 13 connects to beta 5 via 40. We see the only difference is that the lesser gamma must go through x many more cycles to return to the same beta.

The logic of it proceeds as follows:

- 4n + 3 and 8n + 1 make up the primary columns of the gamma tables

        - All odd numbers not of form 4n + 3 and 8n + 1 are lesser gamma's
    - Lesser gammas have the same primary beta connection as their primary gamma
        - Therefore, the difference between a lesser gamma and its primary beta will always be greater than zero
    - Because the difference between all primary gammas and primary betas will always be greater than zero (excluding the case of one), and
    - Because lesser gammas are always greater than their primary gamma (their row head of which they are a factor (by 4n + 1)), and
    - Because lesser gammas connect to the same exact primary beta as the first element in that gamma row
        - There is no possibility of another beta-gamma connection being equal and yielding itself via $\frac{(3n+1)}{2}$
    - This eliminates the possibility for a recursive loop other than one
    - CAVEAT: This does not prove all numbers will indeed return to one given enough cycles.
        - But it certainly demonstrates a number will not return to itself

According to the logic of the Collatz conjecture and the mathematical reasoning employed to arrive at this conclusion there cannot be an odd number that is not a critical quotient which will cause a separate recursive loop, Q.E.D. If this were the case, I would be proving the Collatz conjecture false. But I am saying it is not false due to the impossibility of a hidden recursive sequence using the Collatz conjecture rules, lying hidden somewhere in structure of numbers too numerous to count in the known lifetime of the universe.

But, this begs the question: can one sequence be linked to another sequence/s and then come back to itself so causing a recursive loop that way? That question is not explored here, and it may not even be a legitimate question. The case for a recursive loop as happens with one, comes from one dividing to an odd number equal to itself: in a manner of speaking a one separate from, but equal to, itself. But I demonstrated another odd number cannot equal itself in the same manner as one equals itself therefore proving a secondary recursive loop created by the same logic that causes the only known recursive loop, impossible.

In conclusion, the Collatz conjecture equations are a vexing and curiously contrived set of simple equations which probe a uniquely ordered set of positive integers, N_0. By using the matric systems and understanding key findings of the critical quotients we see how all numbers are interrelated and dependent upon each other, much as the normal number line of 1, 2, 3, 4… are dependent on each other by n + 1. Only here, it is much less obvious and much more complicated; each ordering of N and N_0 in the tables has a different relationship. All because we are using different rules: those of the Collatz conjecture. The Collatz conjecture equations can be viewed as a means to probe how all numbers derive from one, instead of in the sequential order normally associated N, they connect through the two Collatz equations and their rules.

It is my hope that this resolution is satisfactory and sparks interest in further analyses of the Collatz conjecture as presented in this paper. There seem endless curious, enigmatic, and sometimes vexing patterns contained in these number sequences so arranged. Not least of which is the shape which these number tables form. These tables give rise to a geometric shape; a trapezoidal prism with two triangular faces. The triangular faces arise from the difference between primary betas and gammas while the trapezoidal shape arises from the row expansions of each primary beta and gamma. Each trapezoid is another entry illustrating the 2n, positive and negative, divergence of successive primary beta and gamma primaries whose trapezoidal sides are the lesser beta/gamma entries seen as an infinite series of growing trapezoids connected to the growing triangular face.

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Declaration of Interest

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Tables

Beta Table 3				Gamma Table 3			
$\beta_3 0^{1-1}$	$\beta_3 0^{1-2}$	$\beta_3 0^{1-3}$	$\beta_3 0^{1-4}$	$\gamma_3 0^{1-1}$	$\gamma_3 1^{1-2}$	$\gamma_3 2^{1-3}$	$\gamma_3 3^{1-4}$
$\beta_3 0^{2-1}$	$\beta_3 0^{2-2}$	$\beta_3 0^{2-3}$	$\beta_3 0^{2-4}$	$\gamma_3 0^{2-1}$	$\gamma_3 0^{2-2}$	$\gamma_3 0^{2-3}$	$\gamma_3 0^{2-4}$
$\beta_3 0^{3-1}$	$\beta_3 0^{3-2}$	$\beta_3 0^{3-3}$	$\beta_3 0^{3-4}$	$\gamma_3 0^{3-1}$	$\gamma_3 0^{3-2}$	$\gamma_3 0^{3-3}$	$\gamma_3 0^{3-4}$
$\beta_3 0^{4-1}$	$\beta_3 0^{4-2}$	$\beta_3 0^{4-3}$	$\beta_3 0^{4-4}$	$\gamma_3 0^{4-1}$	$\gamma_3 0^{4-2}$	$\gamma_3 0^{4-3}$	$\gamma_3 0^{4-4}$
$\beta_3 0^{5-1}$	$\beta_3 0^{5-2}$	$\beta_3 0^{5-3}$	$\beta_3 0^{5-4}$	$\gamma_3 0^{5-1}$	$\gamma_3 0^{5-2}$	$\gamma_3 0^{5-3}$	$\gamma_3 0^{5-4}$

Table 1: Representation of the order of beta table 3 and its derivative gamma table 3.

$\beta_1 0$	$\beta_1 1$	$\beta_1 2$	$\beta_1 3$	$\beta_1 4$	$\gamma_1 0$	$\gamma_1 1$	$\gamma_1 2$	$\gamma_1 3$	$\gamma_1 4$
6n+3	$\beta_1 0 \times 2$	$\beta_1 1 \times 2$	$\beta_1 2 \times 2$	$\beta_1 3 \times 2$					
3	6	12	24	48	Ø	Ø	Ø	Ø	Ø
9	18	36	72	144	Ø	Ø	Ø	Ø	Ø
15	30	60	120	240	Ø	Ø	Ø	Ø	Ø
21	42	84	168	336	Ø	Ø	Ø	Ø	Ø
27	54	108	216	432	Ø	Ø	Ø	Ø	Ø

*Table 2: Beta 1 and derivative gamma 1 table. Beta table 1 has no derivative integers quotients in the gamma table 1 as it consists of all numbers evenly divisible by 3. Where Ø is a non-integer quotient of a β*x $\frac{(n-1)}{3}$ *derivative.*

$$(6n+5)\frac{(2[2^{2m}-1])}{3} + (4n+3) = \gamma_2 X$$

$\beta_2 0$	$\beta_2 1$	$\beta_2 2$	$\beta_2 3$	$\beta_2 4$	$\gamma_2 0$	$\gamma_2 1$	$\gamma_2 2$	$\gamma_2 3$	$\gamma_2 4$
6n+5	$\beta_2 0 \times 2$	$\beta_2 1 \times 2$	$\beta_2 2 \times 2$	$\beta_2 3 \times 2$		$\frac{(\beta_2 1 - 1)}{3}$		$\frac{(\beta_2 3 - 1)}{3}$	
5	10	20	40	80	Ø	3	Ø	13	Ø
11	22	44	88	176	Ø	7	Ø	29	Ø
17	34	68	136	272	Ø	11	Ø	45	Ø
23	46	92	184	368	Ø	15	Ø	61	Ø
29	58	116	232	464	Ø	19	Ø	77	Ø

Table 3: Beta 2 and derivative gamma 2 tables. Notice how the primary beta column 0 ($\beta_2 0$*) increases by 6n + 5, the multiplicative variable of the gamma 2 equation. And that gamma 2, column 1 (*$\gamma_2 1$*) increases by 4n + 3, the additive variable of the gamma 2 equation. Together, these primary columns form what are termed the critical quotients.* $\beta_2 0$ *and* $\gamma_2 1$ *are the primary columns containing the critical quotients; entries from the same row of* $\beta_3 0$ *and* $\gamma_3 2$ *are twins. Where Ø is a non-integer quotient of a β*x $\frac{(n-1)}{3}$ *derivative.*

$$(6n+1)\frac{(4[2^{2m}-1])}{3} + (8n+1) = \gamma_3 X$$

$\beta_3 0$	$\beta_3 1$	$\beta_3 2$	$\beta_3 3$	$\beta_3 4$	$\gamma_3 0$	$\gamma_3 1$	$\gamma_3 2$	$\gamma_3 3$	$\gamma_3 4$
6n+1	$\beta_3 0 \times 2$	$\beta_3 1 \times 2$	$\beta_3 2 \times 2$	$\beta_3 3 \times 2$	$\frac{(\beta_3 0 - 1)}{3}$		$\frac{(\beta_3 2 - 1)}{3}$		$\frac{(\beta_3 4 - 1)}{3}$
1	2	4	8	16	0	Ø	1	Ø	5
7	14	28	56	112	2	Ø	9	Ø	37
13	26	52	104	208	4	Ø	17	Ø	69
19	38	76	152	304	6	Ø	25	Ø	101
25	50	100	200	400	8	Ø	33	Ø	133

Table 4: Beta 3 and derivative gamma 3 tables. Notice how beta 2 column 0 ($\beta_3 0$*) increases by 6n + 1, the multiplicative variable of the gamma 3 equation. And that gamma 3, column 2 (*$\gamma_3 2$*) increases by 8n + 1, the additive variable of the gamma 3 equation.* $\beta_3 0$ *and* $\gamma_3 2$ *are the primary columns containing the critical quotients; entries from the same row of* $\beta_3 0$ *and* $\gamma_3 2$ *are twins. Where Ø is a non-integer quotient of a β*x $\frac{(n-1)}{3}$ *derivative.*

$$(6n+5)\frac{(2[2^{2m}-1])}{3}+(4n+3)=\gamma_2X$$

Row Equation	m, n	m =0	n = 1	n = 2	n = 3	n = 4
(5x) + 3	n = 0	3	13	53	213	853
(11x) + 7	m = 1	7	29	117	469	1877
(17x) + 11	m = 2	11	45	181	725	2901
(23x) + 15	m = 3	15	61	245	981	3925

Table 5: Gamma 2 table. ***.*** *Showing equations on the leftmost column that can also be used to derive each row; in sequence, x=0, 2, 10, 42, 170… Also showing how the gamma 2 equation is used for each respective entry for m and n. See corresponding m, n grid below (Table 7).*

$$(6n+1)\frac{(4[2^{2m}-1])}{3}+(8n+1)=\gamma_3X$$

Row Equation	m, n	m =0	n = 1	n = 2	n = 3	n = 4
(1x) + 1	n = 0	1	5	21	85	341
(7x) + 9	m = 1	9	37	149	597	2389
(13x) + 17	m = 2	17	69	277	1109	4437
(19x) + 25	m = 3	25	101	405	1621	6485

Table 6: Gamma 3 table. Showing equations on the leftmost column that can also be used to derive each row; in sequence, x=0, 4, 20, 84, 340… Also showing how the gamma 3 equation is used for each respective entry for variables m and n. See corresponding m, n grid below (Table 7).

$$(6n+5)\frac{(2[2^{2m}-1])}{3}+(4n+3)=\gamma_2X \text{ or } (6n+1)\frac{(4[2^{2m}-1])}{3}+(8n+1)=\gamma_3X$$

m, n	m, n	n =0	n = 1	n = 2	n = 3	n = 4
m = 0	0,0	1,0	2,0	3,0	4,0	5,0
m = 1	0,1	1,1	2,1	3,1	4,1	5,1
m = 2	0,2	1,2	2,2	3,2	4,2	5,2
m = 3	0,3	1,3	2,3	3,3	4,3	5,3

Table 7: The m, n grid: A grid work basis applicable to both γ_x tables where each entry from the γ_x table is replaced by a coordinate number set, m and n. These tables rely only on the integer results from the beta tables ignoring the non-integer numbers. By so doing, each result can be expressed as a point on a grid system. Each entry consists of m and n which are the variables in the gamma equations. This demonstrates how the equations are used to compose the gamma tables independent of the beta extraction using $\frac{(n-1)}{3}$.

$B_2 0$	$Y_2 1$	$(B_2 0 - Y_2 1)$	$B_3 0$	$Y_3 2$	$(B_3 0 – Y_3 2)$
5	3	2	1	1	0
11	7	4	7	9	-2
17	11	6	13	17	-4
23	15	8	19	25	-6
29	19	10	25	33	-8
35	23	12	31	41	-10
6n + 5	4n + 3	2n	6n + 1	8n + 1	-2n

Table 8: Critical quotient differences. The difference between the critical quotients demonstrates the divergence of the critical quotient s and illustrates that a recursive loop cannot occur which does not include one.