Root Finding Problem

Edgar Valdebenito

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Abstract: In this note, we consider the alternative form of the rootfinding problem known as the fixed-point problem.

Keywords: rootfinding, fixed-point iteration, number Pi .

I. Introduction: root finding problem

Recall that

$$2\int_0^{\pi}\left(\sin\left(x-\sqrt{\pi^2-x^2}\right)\right)^2 dx = \pi \tag{1}$$

Define

$$f(y) = 2\int_0^{y}\left(\sin\left(x-\sqrt{y^2-x^2}\right)\right)^2 dx \tag{2}$$

The fixed-point problem is to find a value q, called a fixed point, such that

$$f(q) = q \tag{3}$$

Given $y_1 = 3$, define

$$y_{n+1} = f(y_n) \quad n = 1, 2, 3, \ldots \tag{4}$$

The idea is to generate a sequence of values that one hopes will converges to the correct result, and stop when we are satisfied that we are close enough to the limit.

II. Fixed-point iteration

$$y_1 = 3\,,\ \ y_{n+1} = f(y_n) \quad n = 1, 2, 3, \ldots \tag{5}$$

n	y_n
1	3
2	3.1397077490994629364057777233059473798139974321591021592416756848266555770293222
3	3.1415926491252556944794381440657649099212399776512705648543836226501752887271487
4	3.1415926535897932384626433239544438121837693370155904765056022068985170791187880
5	3.1415926535897932384626433832795028841971693993751058209749445923078164062860698

$$y_n \longrightarrow \pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2\,n+1} \tag{6}$$

$$|y_5-\pi| = 1.39\ldots\cdot 10^{-76} \tag{7}$$

III. Convergence

Fixed point iteration for a differentiable $f(x)$ converges to a fixed point q if the initial error is sufficiently small and $|f'(q)| < 1$. The iteration diverges for all initial values if $|f'(q)| > 1$.

IV. Error sequence

If $\epsilon_n = y_n - q$, $n = 1, 2, 3,$ ... is the error sequence we have

$$\epsilon_{n+1} = f^{(1)}(q)\,\epsilon_n + \frac{1}{2}\,f^{(2)}(q)\,\epsilon_n^2 + \frac{1}{6}\,f^{(3)}(q)\,\epsilon_n^3 + \ldots \tag{8}$$

for

$$f(y) = 2\int_0^y \left(\sin\left(x - \sqrt{y^2 - x^2}\,\right)\right)^2 dx \tag{9}$$

we have ($q = \pi$)

$$f^{(1)}(\pi) = f^{(2)}(\pi) = 0\,,\;\; f^{(3)}(\pi) = 4 \tag{10}$$

$$\epsilon_{n+1} \approx \frac{2}{3}\,\epsilon_n^3\,,\;\; n = 1, 2, 3,\;\ldots \tag{11}$$

V. End note

$$2\int_0^\pi \left(\cos\left(x - \sqrt{\pi^2 - x^2}\,\right)\right)^2 dx = \pi \tag{12}$$

$$8\int_0^\pi \left(\cos\left(x - \sqrt{\pi^2 - x^2}\,\right)\right)^4 dx = 3\,\pi \tag{13}$$

$$8\int_0^\pi \left(\sin\left(x - \sqrt{\pi^2 - x^2}\,\right)\right)^4 dx = 3\,\pi \tag{14}$$

$$8\int_0^\pi \left(\cos\left(x - \sqrt{\pi^2 - x^2}\,\right)\right)^2 \left(\sin\left(x - \sqrt{\pi^2 - x^2}\,\right)\right)^2 dx = \pi \tag{15}$$

VI. References

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