A New Approach to the Collatz Conjecture: Proof of the Absence of Cycles

Song Kwang-sun

March 4, 2025

Abstract

The Collatz conjecture posits that for every natural number, a specific iterative rule leads to 1 or forms a cycle. This paper introduces a simplified Collatz function, inverse Collatz, and double inverse Collatz to prove that no cycles exist beyond the known $1 \to 4 \to 2 \to 1$. By analyzing number generation through parameters a and k, we demonstrate the logical impossibility of additional cycles.

1 Introduction

Proposed by Lothar Collatz in 1937, the Collatz conjecture remains unsolved. It postulates that for any natural number $n \in \mathbb{N}$, the following rules lead to 1 or a periodic cycle: - If n is even, divide by 2, - If n is odd, compute $3n + 1$.

This study focuses not on proving the conjecture but on demonstrating that no cycles exist beyond $1 \to 4 \to 2 \to 1$. To this end, we introduce a simplified Collatz function and reverse approaches (inverse Collatz and double inverse Collatz).

2 Definitions

2.1 Original Collatz Function

The original Collatz function is defined as:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ 3n+1 & \text{if } n \text{ is odd.} \end{cases}$$

2.2 Simplified Collatz Function

The simplified Collatz function f' combines steps for odd numbers:

$$f'(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{3n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

For example, $f'(5) = \frac{3 \cdot 5+1}{2} = 8$.

2.3 Inverse Collatz

Inverse Collatz finds n such that $f'(n) = m$:

$$n = (m+1) \cdot \left(\frac{3}{2}\right)^k - 1, \quad k \in \mathbb{N},$$

where n must be an integer. We simplify this to $m_k = a \cdot 3^k - 1$, with $a, k \in \mathbb{N}$.

2.4 Double Inverse Collatz

Double inverse Collatz generates numbers:

$$m = a \cdot 2^k - 1, \quad a, k \in \mathbb{N}.$$

For example, $a = 1$, $k = 3$: $m = 1 \cdot 2^3 - 1 = 7$.

2.5 Set N

The set $N = \{1, 2, 3, 4, \dots\}$ includes all natural numbers, and we analyze those that do not form cycles.

3 Lemmas

3.1 Lemma 1: Generation of All Odd Numbers

Lemma 1: Every odd number m can be expressed as $m = a \cdot 2^k - 1$, and in particular, all odd numbers are generated when $k = 1$.

Proof: For any odd m, $m + 1$ is even. Setting $k = 1$:

$$m + 1 = a \cdot 2^1 = 2a, \quad a \in \mathbb{N},$$

$$m = 2a - 1.$$

For $a = 1, 2, 3, \dots$, $m = 1, 3, 5, \dots$, generating all odd numbers. For example, $m = 15 = 2 \cdot 8 - 1$.

3.2 Lemma 2: Reduction of k

Lemma 2: In double inverse Collatz, cases with $k \geq 2$ reduce to $k = 1$.

Proof:

$$m = a \cdot 2^k - 1,$$

Setting $a' = 2a$:

$$m' = 2a \cdot 2^k - 1 = a \cdot 2^{k+1} - 1.$$

Thus, larger k can be reduced to $k = 1$ by increasing a. For example, $m = 15 = 1 \cdot 2^4 - 1 = 8 \cdot 2^1 - 1$.

4 Main Theorem

4.1 Main Theorem 1: Absence of Cycles

Main Theorem 1: In the sequence generated by the Collatz function f', no cycles exist beyond $1 \to 4 \to 2 \to 1$.

Proof: We use induction and reverse analysis. Assume that for $N = \{1, 2, \ldots, n\}$, no cycles exist, and examine $n + 1$.

1. **If $n + 1$ is even**:

$$f'(n+1) = \frac{n+1}{2},$$

which is less than or equal to n, hence in N, and converges to 1 without forming a cycle.

2. **If $n + 1$ is odd**: By Lemma 1, $n + 1 = a \cdot 2^k - 1$.

$$f'(n+1) = \frac{3(n+1)+1}{2} = \frac{3(a \cdot 2^k - 1) + 1}{2} = 3a \cdot 2^{k-1} - 1.$$

Repeated application reduces k, reaching $3a - 1$ when $k = 0$. **Cycle Assumption**: If $n + 1$ were in a cycle, inverse Collatz $m_k = a \cdot 3^k - 1$ implies: - For $k = 1$: $m_1 = a \cdot 3 - 1$, - For $k = 2$: $m_2 = a \cdot 9 - 1$. If $a = 3b$, then:

$$m_2 = 3b \cdot 9 - 1 = b \cdot 27 - 1,$$

showing exponential growth with increasing k. **Contradiction**: A cycle must be finite, but increasing a and k leads to unbounded growth, preventing cycle formation. Conversely, applying f' reduces k, decreasing m_k. Example: $m_2 = 80 = 9 \cdot 3^2 - 1$, $f'(80) = 40$, $f'(40) = 20$, eventually reaching 1.

3. **Impossibility of Divergence**: For $m_k = a \cdot 3^k - 1$ to diverge, k must increase indefinitely, but f' reduces k, making divergence impossible.

Thus, $n + 1$ cannot form a cycle beyond $1 \to 4 \to 2 \to 1$.

5 Counterexample Search

For inverse Collatz $m_k = a \cdot 3^k - 1$ to form a cycle, k would need to increase while maintaining a fixed period. However: - For $a = 3$, $k = 3$: $m = 3 \cdot 3^3 - 1 = 80$, - $f'(80) = 40$, $f'(40) = 20$, $f'(20) = 10$, $f'(10) = 5$, $f'(5) = 8$, reaching 1. - For $a = 9$, $k = 2$: $m = 80$, same as above. - For $a = 27$, $k = 1$: $m = 80$. - In double inverse Collatz, $m = a \cdot 2^k - 1$, e.g., $a = 6$, $k = 2$, follows the same path. **Impossibility of Chain Counterexamples**: If $m = 80$ were in a cycle, multiples of a and k would imply infinitely many counterexamples, but small numbers (e.g., 5) already converge to 1, leading to a contradiction.

Conclusion: No counterexamples exist.

6 Conclusion

This paper proves, using a simplified Collatz function, inverse Collatz, and double inverse Collatz, that no cycles exist beyond $1 \to 4 \to 2 \to 1$. This does not confirm the Collatz conjecture but logically establishes the absence of additional cycles. If counterexamples are suspected, specific cases can be provided for further clarification.