Artificial Prime Numbers: A Relative Perspective on Primality

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Abstract

This article explores the notion of artificial prime numbers in the context of specific sets of integers. A formal definition is presented, properties are discussed, and comparisons are made with the classical notion of primality. Additionally, potential applications of these numbers in number theory and cryptography are analyzed.

Keywords: Prime numbers, Andrica conjecture, artificial prime numbers, number theory, Goldbach conjecture, Bertrand's postulate, cryptography

1. Introduction

Prime numbers have fascinated mathematicians throughout history due to their fundamental role in number theory. They are defined as the integers greater than 1 that have exactly two divisors: 1 and the number itself. This property makes them the "building blocks" of the integers, as any positive integer can be expressed uniquely as a product of primes, known as prime factorization. This characteristic is not only crucial in arithmetic but also has applications in various fields, including cryptography, where the difficulty of factoring large composite numbers into their underlying primes ensures the security of many systems.

However, the concept of primality does not have to be limited to prime numbers in the general context. It can be explored in more restricted contexts, such as specific sets of numbers. This is where the notion of artificial prime numbers arises.

Artificial prime numbers are defined in relation to a given set of positive integers, where a number is considered prime if it cannot be divided by any other element of the set, except by itself. This idea allows for a deeper analysis of the relationships between numbers in a particular context and offers a new perspective on primality, opening the door to the exploration of more complex numerical structures.

In the next section, we will examine the formal definition of artificial prime numbers, their properties, and examples that illustrate their behavior in comparison to the classical notion of primality. This approach will not only enrich our understanding of numbers but also provide tools to investigate number theory in broader and more varied contexts.

2. Definition of Artificial Prime Number

Let S be a non-empty set of positive integers such that $S \subseteq \mathbb{Z}^{+}$ y $\forall x \in S, x > 1$. A number $q \in S$ is called an artificial prime number if and only if the following condition holds:

$$\forall d \in S,\ (d \neq q) \implies (d \nmid q)$$

Where $d \nmid q$ means that d does not divide q. In other words, q is an artificial prime number if there is no element d in the set S that divides q, excluding q itself.

2.1. Justification of the definition

- **Division in the set:** By considering only the elements of a set S, the condition that q cannot be divided by any other number d in S (other than q itself) establishes a form of "primality" within that context. This means that, although q may be divisible by other numbers in general (outside of set S), within S it is "prime" because it has no divisors in S other than itself

- **Restriction to the set:** The key lies in the restriction to the elements of S. In this sense, the definition is useful and valid, as it allows us to identify a type of "prime number" that is relevant to the specific set we are considering.

- **Analogy with classical primality:** In number theory, a prime number has no positive divisors other than 1 and itself. Here, instead of 1, we are using the restriction to the elements of the set S. This maintains the structure of the definition of primality, adapting it to a more limited context.

2.2. Relative Primality: The definition introduces a concept of "primality" relative to the set S. This means that the "primality" of q is evaluated in relation to the numbers present in S. A number may not be prime in the classical sense, but it can be considered prime within the context of the set S.

2.3. Properties of Artificial Prime Numbers

1. **Relative Indivisibility:**

   - A number q is an artificial prime if there is no other number $d \in S$ (different from q) that divides q. This implies that q is "indivisible" within the context of the set S.

2. Set Dependency:

   - Artificial primality is dependent on the set S. A number that may be artificially prime in one set may not be in another.

3. Possibility of Composite Numbers:

   - Unlike classical prime numbers, artificial primes can be composite numbers. That is, a composite number can be considered an artificial prime if it has no divisors in the set S other than itself.

4. Sieve of Eratosthenes for Artificial Primes in $\boldsymbol{S}$:

   - In a set S that contains an artificial prime number q, any number $b > q$ that is a multiple of this number cannot be an artificial prime in that same set. This establishes a hierarchical relationship among the numbers.

5. Order Relations:

   - If we consider an ordered set of integers, artificial primes can be identified based on their position relative to other elements. However, their distribution does not necessarily follow the distribution of prime numbers among the integers.

6. Difficulty of Identification:

   - In large sets or those without an obvious pattern, identifying artificial primes can be computationally difficult, similar to identifying prime numbers in general.

2.4. Definition of Artificial Coprimes

Given a set S of positive integers, two numbers a and b in S are said to be artificial coprimes if and only if there is no number $d \neq 1$ in S such that d divides both a and b.

Any number that is coprime in the traditional sense is also considered an artificial coprime within the context of a set S of positive integers. This property highlights the connection between classical coprimality and artificial coprimality, with the latter being defined more specifically within a set context.

On the other hand, artificial coprimality supports and complements the definition of artificial prime numbers by providing a broader framework for understanding divisibility relations within a set S. Both concepts allow for the exploration of interesting and useful properties in mathematics, illustrating how they interact with each other in the context of divisibility.

3. Potential Emerging Lines of Research Using Artificial Prime Numbers

The relationship between artificial prime numbers and classical prime numbers, through theorems and conjectures, offers a fertile ground for research in number theory. While many conjectures focus on classical primes, exploring how these ideas can be extended or adapted to the context of artificial primes may lead to new perspectives and mathematical discoveries. Below are some conjectures related to classical prime numbers and their generalization within the framework of artificial prime numbers.

3.1.1. Bertrand's Postulate

For every integer $n > 1$, there exists at least one prime number (p) such that:

$$n < p < 2n$$

3.1.2. Generalization

Given a set $S_A = \{a_n \in \mathbb{Z}^+ : a_n = a_1 + (n-1)k, n \in \mathbb{N}\}$. For every $a_1 > 1$, , there exists at least one artificial prime number (q) such that:

$$a_n < q < 2(a_n + k - 1)$$

Note that when $k = 1$, we obtain Bertrand's Postulate.

3.1.3. Goldbach's Conjecture

Every even number greater than 2 can be expressed as the sum of two prime numbers.

3.1.4. Generalization

Given a set $S_B = \{a_n \in \mathbb{Z}^+ : a_n = 2 + (n-1)k, n \in \mathbb{N}\}$. If k is odd, then every even number greater than 2 can be expressed as the sum of at most $(k + 1)$ artificial prime numbers (q).

Note that when $k = 1$, we obtain the well-known Goldbach Conjecture.

3.1.5. Legendre's Conjecture

For every $n \in \mathbb{N}$, there always exists a prime number (p) such that:

$$n^2 < p < (n+1)^2$$

3.1.6. Generalization

Given a set $S_A = \{a_n \in \mathbb{Z}^+ : a_n = a_1 + (n-1)k, n \in \mathbb{N}\}$. Si $a_1 > 1\}$, then:

$$a_n(a_n + k - 1) < q < (a_n + k)(a_n + 2k - 1)$$

Note that when $k = 1$, we obtain Legendre's Conjecture.

3.1.7. Andrica's Conjecture

If p_n is the n-th prime number, then the inequality

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1$$

holds for every n.

3.1.8. Generalization

Given a set $S_A = \{a_n \in \mathbb{Z}^+ : a_n = a_1 + (n-1)k, n \in \mathbb{N} \geq 1\}$. If q_n is the n-th artificial prime number in the set S, then the inequality:

$$\sqrt{q_{n+1}} - \sqrt{q_n} < \sqrt{k}$$

holds for every n.

Note that when $a_1 = k = 1$ we obtain the Andrica conjecture.

3.1.9 Polignac's Conjecture

For every natural number n, there are infinitely many pairs of prime numbers (p) whose difference is $2n$. If $n = 1$, we obtain the twin prime conjecture.

3.1.10. Generalization

Given a set $S_A = \{a_n \in \mathbb{Z}^+ : a_n = a_1 + (n-1)k, n \in \mathbb{N}\}$. For every natural number n, there are infinitely many pairs of artificial prime numbers (q) whose difference is $2kn$.

3.1.11. Gilbreath's Conjecture

Let $\{p_n\}$, for $n \geq 1$, Let the ordered sequence of prime numbers, and let:

$$k_n = p_{n+1} - p_n$$

For every $a \geq 1$, let

$$k_n^a = \left|k_{n+1}^{a-1} - k_n^{a-1}\right|$$

For every a, it holds that:

$$k_1^a = 1$$

3.1.12. Generalization

Given a set $S_B = \{a_n \in \mathbb{Z}^+ : a_n = 2 + (n-1)k, n \in \mathbb{N}\}$, for cases where k is odd.

Let $\{q_n\}$, for $n \geq 1$, Let the ordered sequence of artificial prime numbers in the set S_B, and let

$$k_n = q_{n+1} - q_n$$

For every $a \geq 1$, let

$$k_n^a = \left|k_{n+1}^{a-1} - k_n^{a-1}\right|$$

For every a it holds that:

$$k_1^a = k$$

4. Some Elementary Results Related to Artificial Prime Numbers

4.1. Lemma 1: If a set S_P is composed of prime numbers, then the set

$$P_A(S_P) = S_P$$

Let $S_P = \{p_1, p_2, \dots, p_k\}$ be a set of prime numbers, that is, each p_k is a prime number. Then, the set of artificial prime numbers associated with S_P, $P_A(S_P)$, is equal to S_p. That is:

$$P_A(S_P) = S_P$$

Justification: Since prime numbers are only divisible by 1 and themselves, there is no number $p_i \in S_P$ that is divisible by another $p_j \in S_P$ with $i \neq j$. Therefore, each prime number in S_P satisfies the condition of being an artificial prime number within the set.

4.2. **Lemma 2**: If a set S_R is composed only of coprime numbers, then the set:

$$P_A(S_R) = S_R .$$

Let $S_R = \{a_1, a_2, \dots, a_n\}$ be a set of positive integers greater than 1, and suppose that $\gcd(a_i, a_j) = 1$, for all $i \neq j$. Then, the set of artificial prime numbers associated with S_R, $\mathcal{P}_A(S_R)$, is equal to S_R. that is:

$$P_A(S_R) = S_R$$

Justification: Since all the elements of S_R are coprime to each other, there is no number $a_i \in S_R$ that is divisible by another number$a_j \in S_R$ with $i \neq j$, thus fulfilling the definition of artificial prime numbers for each element of S_R.

4.3. **Lemma 3:** Let S_N be a set of positive integers greater than 1. If for every composite number $c \in S_N$, there exists at least one prime number $p \in S_N$ such that p divides c, hen the set of artificial prime numbers associated with S_N contains only the prime numbers from S_N. That is:

$$P_A(S_N) = \{p \in S_N\}$$

4.4. **Lemma 4:** In any non-empty set S_N of positive integers greater than 1, there exists a smallest number in S_N that has no divisors (artificial prime) within the set S_N other than itself.

4.5. **Lemma 5:** If S_N is the set of all positive integers greater than 1, then all artificial prime numbers in this set are prime numbers.

5. Artificial Primes: Proof of the Infinitude of Prime Numbers

Proof:

1. Let S_N be the set of all positive integers greater than 1.
2. By Lemma 4, there exists a number p_1 in S_N that is the smallest and has no divisors (artificial prime number) in S_N other than itself. This implies that p_1 is a prime number.
3. We construct a new set $S_N{}'$ that contains all the numbers in S except p_1 and its multiples..

4. Applying Lemma 4 again to $S_N{}'$, we find a new smallest number p_2 hat also has no divisors (artificial prime number) in $S_N{}'$ and by Lemma 5, we can conclude that p_2 is prime.
5. We repeat this process, constructing sets$S_N{}''$ y $S_N{}'''$that exclude p_1, p_2, and their multiples. In each of these sets, we will always find a new prime number p_n.
6. Since the process can be repeated indefinitely, each time obtaining a new prime number that is greater than all the previous ones, we conclude that there is no finite limit to the prime numbers.

6. Set Representation

Some examples are:

6.1. The set $\boldsymbol{S_N}$ of Positive Integers Greater than 1

$S_N = \{n \in \mathbb{Z}^+ : n > 1\}$, that is:

$$S_N = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ..\}$$

In this particular case, the set of artificial prime numbers associated with S_N, $P_A(S_N)$, is the set of prime numbers. That is:

$P_A(S_N) = P$, where P is the set of prime number. This is true because in the set S_N of integers greater than 1, the only numbers that have as a divisor only themselves (remember that 1 has been excluded) are the prime numbers. Any composite number in this set will have at least one prime factor in S_N that divides it.

6.2. The set $\boldsymbol{S_F}$ of Fibonacci Numbers

$$S_F \ = \{F_n : n \in \mathbb{Z}^+, n \geq 2\}$$

6.3. The set $\boldsymbol{S_X}$ of a Function

$$S_X = \{f(n) : n \in \mathbb{N}\}$$

6.4. The set $\boldsymbol{S_Z}$ of a Sequence of Positive Integers

$$S_Z = \{a_1, a_2, a_3, .., a_n : a_1 > 1\}$$

6.5. The set S_A of a Sequence of a Positive Integer Arithmetic Progression

$$S_A = \{a_n \in \mathbb{Z}^+ : a_n = a_1 + (n-1)k, n \in \mathbb{N}\}$$

As we can observe, the sets S that can be created to explore artificial prime numbers are infinite and can encompass a variety of criteria, ranging from numerical properties to specific patterns and restrictions. By analyzing these sets, solutions regarding divisibility relationships and the internal structure of numbers can be obtained. The classification of a number as an artificial prime depends on the set S in which it is being considered. A number can be an artificial prime in one set and not in another.

7. Possible Applications

The concept of artificial prime numbers has implications in several areas of mathematics, including:

7.1. **Number Theory:** Assists in the study of divisibility and factorization properties.

7.2. **Cryptography:** Provides a framework for generating keys based on the indivisibility of certain numbers in a set.

7.3. **Algorithms:** They can be used to optimize computing algorithms that depend on divisibility properties.

7.4. **Random Number Generation:** Pseudo-randomness algorithms: In some algorithms, numbers with properties similar to primes are used to generate pseudo-random sequences.

Conclusion

Artificial prime numbers offer us a new way to understand primality in a restricted context, broadening our horizon in number theory. This notion invites mathematicians and scientists to explore more deeply the interactions between numbers and to discover new properties and patterns that may not be evident in a broader framework.

Although artificial prime numbers do not replace the classical definition of primality, they provide a complementary perspective that can enrich our understanding of number theory and the relationships between numbers in specific contexts. Such explorations can lead to new ideas and approaches in mathematics. They can be useful for studying divisibility and factorization properties within a specific set, allowing for deeper analysis in number theory.

The concept can be applied in areas such as number theory, cryptography, and random number generation, where the property of divisibility plays a crucial role.

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