Born Reciprocal Relativity Theory Redefines the notion of Mass : How Massless photons appear Massive in Accelerated Frames

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Abstract

Starting with a brief review of Born Reciprocal (non-inertial) Relativity Theory (BRRT), it is shown how massless photons in one frame of reference can appear massive in an acccelerated frame. An immediate application can be found in the behavior of in-falling/outgoing photons propagating in a black hole gravitational background where in-falling and outgoing photons from the point of view of an accelerated frame of reference (with respect to a static spherically symmetric Schwarzschild black hole, for example) will appear massive and subluminal. In view of these novel findings that massless particles can appear massive in accelerated frames, it should have many important consequences in cosmology (dark energy, dark matter problem) and QFT.

Keywords : Born Reciprocal Relativity; Non-inertial Relativity; Phase Spaces; Maximal Force; Black Holes; Finsler Geometry; Cosmology; Dark Energy; Dark Matter.

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1 Brief Review of Born Reciprocal (Non-inertial) Relativity

The principle behind the concept of "Born reciprocal relativity theory", or non-inertial relativity to be more precise[1], was advocated by [2], [3], [4] and it

[1]We thank one of the referees of a previous article for highlighting this fact in order to clarify the point that Born did not propose a reciprocal relativity theory

was based on the idea proposed long ago by [1] that coordinates and momenta should be unified on the same footing. Consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. Hence, a *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality) [4]. The principle of maximal acceleration was advocated earlier on by [5].

We explored in [4] some novel consequences of Born Reciprocal Relativity Theory (BRRT) in flat phase-space and generalized the theory to the curved spacetime scenario. We provided, in particular, some specific results resulting from Born reciprocal Relativity and which are *not* present in Special Relativity. These are : momentum-dependent time delay in the emission and detection of photons; relativity of chronology; energy-dependent notion of locality; superluminal behavior; relative rotation of photon trajectories due to the aberration of light; invariance of areas-cells in phase-space and modified dispersion relations. Further results in BRRT can be found in [7], [8].

The generalized velocity and force (acceleration) boosts (rotations) transformations of the $flat$ $8D$ Phase space coordinates , where $X^i, t, E, P^i; i = 1,2,3$ are **c** -valued (classical) variables which are *all* boosted (rotated) into each-other, were given by [2] based on the group $U(1,3)$ and which is the Born version of the Lorentz group $SO(1,3)$. The $U(1,3) = SU(1,3) \times U(1)$ group transformations leave invariant the symplectic 2-form $\Omega = -\, dt \wedge dE + \delta_{ij} dX^i \wedge dP^j; i,j = 1,2,3$ and also the following Born-Green line interval in the $flat$ $8D$ phase-space

$$(d\omega)^2 \ = \ c^2 (dt)^2 \ - \ (dX)^2 \ - \ (dY)^2 \ - \ (dZ)^2 \ +$$

$$\frac{1}{b^2} \ \left((dE)^2 \ - \ c^2 (dP_x)^2 \ - \ c^2 (dP_y)^2 \ - \ c^2 (dP_z)^2 \right) \qquad (1.1)$$

The maximal proper force is set to be given by b. The symplectic group is relevant because $U(1,3) = Sp(8,R) \cap O(2,6)$; $U(3,1) = Sp(8,R) \cap O(6,2)$, and $U(2,2) = Sp(8,R) \cap O(4,4)$.

The 16 generators Z_{ab} of the $U(1,3)$ algebra can be decomposed into the 6 Hermitian Lorentz sub-algebra generators $L_{[ab]}$, and the 10 anti-Hermitian "shear"-like generators $iM_{(ab)}$ (note the i factor that converts the Hermitian generators $M_{(ab)}$ into anti-Hermitian ones $iM_{(ab)}$) as follows

$$Z_{ab} \equiv \frac{1}{2} \ (iM_{(ab)} \ + \ L_{[ab]}) \ \Rightarrow \ L_{[ab]} \ = \ (Z_{ab} \ - \ Z_{ba})$$

$$M_{(ab)} = \ -i \ (Z_{ab} \ + \ Z_{ba}), \ a,b = 0,1,2,3 \qquad (1.2)$$

The Weyl unitary trick allows to relate the unitary group $U(p+q)$ and the pseudo-unitary group $U(p,q)$, and explains why one needs to decompose the matrix generators of the non-compact pseudo-unitary group $U(1,3)$ in terms of Hermitian *and* anti-Hermitian matrices. The Weyl unitary trick explains the factor of **i** before the M_{ab} in the definition of the Z_{ab} generators in eq-(1.2).

Given the $U(1,3)$ invariant metric $\eta_{ab} = diag(+1,-1,-1,-1)$, the explicit commutation relations of the M_{ab}, L_{ab} generators are given by

$$[L_{ab},\ L_{cd}] \ = \ i\ (\eta_{bc} L_{ad} - \eta_{ac} L_{bd} - \eta_{bd} L_{ac} + \eta_{ad} L_{bc}). \qquad (1.3a)$$

$$[M_{ab},\ M_{cd}] \ = \ -i\ (\eta_{bc} L_{ad} + \eta_{ac} L_{bd} + \eta_{bd} L_{ac} + \eta_{ad} L_{bc}). \qquad (1.3b)$$

$$[L_{ab},\ M_{cd}] \ = \ i\ (\eta_{bc} M_{ad} - \eta_{ac} M_{bd} + \eta_{bd} M_{ac} - \eta_{ad} M_{bc}). \qquad (1.3c)$$

Therefore, given $Z_{ab} = \frac{1}{2}(iM_{ab} + L_{ab}),\ Z_{cd} = \frac{1}{2}(iM_{cd} + L_{cd})$ after straightforward algebra it leads to the $U(1,3)$ commutators

$$[\ Z_{ab},\ Z_{cd}\ ] \ = \ -i\ (\ \eta_{bc}\ Z_{ad} \ - \ \eta_{ad}\ Z_{cb}\ ). \qquad (1.3d)$$

as expected. The commutators of the Lorentz boosts generators L_{ab} and X_c, P_c are of the form

$$[L_{ab},\ X_c] \ = \ i\ (\ \eta_{bc}\ X_a \ - \ \eta_{ac}\ X_b\ ); \quad [L_{ab},\ P_c] \ = \ i\ (\ \eta_{bc}\ P_a \ - \ \eta_{ac}\ P_b\ ) \ (1.4)$$

The Hermitian M_{ab} generators are the "reciprocal" boosts/rotation transformations which *exchange* X for P, in addition to boosting (rotating) those variables, and one ends up with the commutators of M_{ab} and X_c, P_c given by

$$[M_{ab},\ \frac{X_c}{\lambda_l}] \ = \ -\ \frac{i}{\lambda_p}\ (\ \eta_{bc}\ P_a + \eta_{ac}\ P_b\ ); \quad [M_{ab},\ \frac{P_c}{\lambda_p}] \ = \ -\frac{i}{\lambda_l}\ (\ \eta_{bc}\ X_a + \eta_{ac}\ X_b\ ) \qquad (1.5)$$

where λ_l, λ_p are suitable length and momentum scales which are chosen to be the Planck length and momentum, respectively.

The rotations, velocity and force (acceleration) boosts leaving invariant the symplectic 2-form and the line interval in the $8D$ phase-space are rather elaborate. In four spacetime dimensions the velocity-boosts generators along the x_i spatial directions ($i = 1,2,3$) are given by $K_i = L_{0i}$. The force-boots (acceleration boosts) generators along the x_i spatial directions are given by $N_i = M_{0i}$. The rotation generators are $J_i = \epsilon_i^{jk} L_{jk}$. The shear generators are M_{ij}, M_{00}. In general, given the $U(1,3)$ generator $Z = \frac{1}{2}\theta^{AB} Z_{AB}$, the transformations of the four-vectors $\mathbf{X} = (T, X_i); \mathbf{P} = (E, P_i)$ are given by

$$\mathbf{X}' \ = \ e^{\frac{1}{2}\theta^{AB} Z_{AB}}\ \mathbf{X}\ e^{-\frac{1}{2}\theta^{AB} Z_{AB}}, \quad \mathbf{P}' \ = \ e^{\frac{1}{2}\theta^{AB} Z_{AB}}\ \mathbf{P}\ e^{-\frac{1}{2}\theta^{AB} Z_{AB}} \qquad (1.6)$$

leading to

$$\mathbf{X}' \ = \ \mathbf{X} \ + \ [Z,\ \mathbf{X}] \ + \frac{1}{2!}\ [Z,\ [Z,\ \mathbf{X}]] \ + \frac{1}{3!}\ [Z,\ [Z,\ [Z,\ \mathbf{X}]]] \ + \ \ldots \qquad (1.7)$$

and a similar relation for $\mathbf{P}'$ in terms of the nested commutators.

By recurring to the commutation relations (1.5) and the nested commutators in eq-(1.7), one finds that the group transformations of the 8-dim phase space coordinates involving both velocity and force boosts are given by [2] (page 18)

$$t' = t\ cosh\xi + \left(\frac{\xi_v^i\ X_i}{c} + \frac{\xi_a^i\ P_i}{b}\right) \frac{sinh\xi}{\xi} \qquad (1.8a)$$

$$E' = E\ cosh\xi + (b\ \xi_a^i\ X_i + c\ \xi_v^i\ P_i) \frac{sinh\xi}{\xi} \qquad (1.8b)$$

$$X'^i = X^i + (cosh\xi - 1) \frac{(\xi_v^i\ \xi_v^j + \xi_a^i\ \xi_a^j)\ X_j}{\xi^2} + \left(c\ \xi_v^i\ t + \frac{\xi_a^i\ E}{b}\right) \frac{sinh\xi}{\xi} \qquad (1.8c)$$

$$P'^i = P^i + (cosh\xi - 1) \frac{(\xi_v^i\ \xi_v^j + \xi_a^i\ \xi_a^j)\ P_j}{\xi^2} + \left(b\ \xi_a^i\ t + \frac{\xi_v^i\ E}{c}\right) \frac{sinh\xi}{\xi} \qquad (1.8d)$$

where ξ_v^i are the velocity-boost rapidity parameters along the e_i directions; ξ_a^i are the force (acceleration) boost rapidity parameters along the e_i directions, $i = 1, 2, 3$, and ξ is the *net* effective rapidity parameter of the primed-reference frame given by

$$\xi = \sqrt{(\xi_v^i)^2 + (\xi_a^i)^2}, \quad i = 1, 2, 3 \qquad (1.9)$$

A straightforward way of understanding how one obtains the above transformations of eqs-(1.8) can be found by simply recalling the most general (Lorentz) velocity boosts transformations of the spacetime coordinates after splitting the three-vectors $\vec{X}, \vec{P}$ into the parallel $\vec{X}_\parallel$ and transverse $\vec{X}_\perp$ components with respect to the velocity boost rapidity parameter
$\vec{\xi} = (\xi_1, \xi_2, \xi_3); \xi = \sqrt{(\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2}$. Such decomposition is of the form

$$\vec{X}_\parallel = (\vec{X} \cdot \vec{\xi}) \frac{\vec{\xi}}{\xi^2}, \quad \vec{X}_\perp = \vec{X} - \vec{X}_\parallel = \vec{X} - (\vec{X} \cdot \vec{\xi}) \frac{\vec{\xi}}{\xi^2} \qquad (1.10)$$

$$\vec{P}_\parallel = (\vec{P} \cdot \vec{\xi}) \frac{\vec{\xi}}{\xi^2}, \quad \vec{P}_\perp = \vec{P} - \vec{P}_\parallel = \vec{P} - (\vec{P} \cdot \vec{\xi}) \frac{\vec{\xi}}{\xi^2} \qquad (1.11)$$

so that the Lorentz transformations of $\vec{X}, \vec{P}$ can be written in vector form as

$$\vec{X}' = \left(\vec{X} - (\vec{X} \cdot \vec{\xi}) \frac{\vec{\xi}}{\xi^2}\right) + (\vec{X} \cdot \vec{\xi}) \frac{\vec{\xi}}{\xi^2} cosh\xi + \frac{c\ t\ sinh\xi}{\xi} \vec{\xi} \qquad (1.12)$$

$$\vec{P}' = \left(\vec{P} - (\vec{P} \cdot \vec{\xi}) \frac{\vec{\xi}}{\xi^2}\right) + (\vec{P} \cdot \vec{\xi}) \frac{\vec{\xi}}{\xi^2} cosh\xi + \frac{E\ sinh\xi}{c\ \xi} \vec{\xi} \qquad (1.13)$$

where the modulus $\xi = |\vec{\xi}|$ of the velocity-boost rapidity parameters, and the modulus $|\vec{v}|$ of the velocity $\vec{v}$ of the moving frame of reference are related by $tanh(\xi) = \beta = \frac{\sqrt{v_1^2+v_2^2+v_3^2}}{c}$. One then finds that the transverse directions to the velocity remain unaffected by the Lorentz transformations, while the parallel directions are. One can see by simple inspection that by setting the force-boost parameters to zero $\xi_a^i = 0$ in eqs-(1.8), one recovers the standard Lorentz transformations.

These transformations can be *simplified* drastically when the velocity and force (acceleration) boosts are both parallel to the x-direction and leave the transverse directions Y, Z, P_y, P_z intact. There is now a subgroup $U(1,1) = SU(1,1) \times U(1) \subset U(1,3)$ which leaves invariant the following line interval

$$(d\omega)^2 \ = \ c^2(dt)^2 \ - \ (dX)^2 \ + \ \frac{(dE)^2 \ - \ c^2(dP)^2}{b^2} =$$

$$(d\tau)^2 \left(1 \ + \ \frac{(dE/d\tau)^2 \ - \ c^2(dP/d\tau)^2}{b^2} \right) = (d\tau)^2 \left(1 \ - \ \frac{F^2}{F_{max}^2} \right), \ F_{max} = b \tag{1.14}$$

where one has factored out the non-vanishing proper time infinitesimal $(d\tau)^2 = c^2dt^2 - dX^2 \neq 0$ in (1.14). The numerical quantity F^2 is positive by definition. The proper force on a massive particle is given by $F = ma$, where a is the proper acceleration and m is the rest mass. The case when $(d\tau)^2 = 0$ is discussed below. We refrained from factoring out $(dt)^2$ in (1.14) because it is not Lorentz invariant, whereas $(d\tau)^2$ is Lorentz invariant.

It is very important to emphasize that there are *no* factors of $(1 + F^2/b^2)$ appearing in the above factorization process because in the superluminal case $(d\tau)^2 < 0$ (spacelike interval) one still has $m^2a^2 < 0$ despite that $a^2 > 0$ (timelike proper acceleration), because $m^2 < 0$ due to the *imaginary* mass of tachyons. When $(d\tau)^2 > 0$ (timelike interval) $\Rightarrow m^2a^2 < 0$ since $a^2 < 0$ (spacelike proper acceleration) and $m^2 > 0$. Hence we shall always have the factor $(1 - F^2/b^2)$ as expected with $m^2a^2 = -F^2 < 0$. This is also compatible with the fact that if $(d\tau)^2 > 0$, then $(dE)^2 - (dP)^2 < 0$, and vice versa, if $(d\tau)^2 < 0$, then $(dE)^2 - (dP)^2 > 0$. An exception occurs for a free particle (or at rest) giving $(dE)^2 - (dP)^2 = 0$ because $dE = dP = 0$.

The same results occur in higher dimensions. Another way of showing why $(d\tau)^2 > 0$, implies $(dE)^2 - dp_i dp^i < 0$, and vice versa, if $(d\tau)^2 < 0$, it implies $(dE)^2 - dp_i dp^i > 0$, stems directly from the on-shell conditions : given $E^2 - \vec{p}\cdot\vec{p} = E^2 - |\vec{p}|^2 = m^2 > 0$ when $(d\tau)^2 > 0$ gives upon differentiation $\frac{dE}{d|\vec{p}|} = \frac{|\vec{p}|}{E} < 1$, and resulting in $(dE)^2 - d\vec{p}\cdot d\vec{p} = E^2 - |d\vec{p}|^2 < 0$. And vice versa, in the case of $E^2 - \vec{p}\cdot\vec{p} = E^2 - |\vec{p}|^2 = m^2 < 0$ (tachyon), when $(d\tau)^2 < 0$, gives upon differentiation $\frac{dE}{d|\vec{p}|} = \frac{|\vec{p}|}{E} > 1$, and resulting in $(dE)^2 - d\vec{p}\cdot d\vec{p} = E^2 - |d\vec{p}|^2 > 0$.

Consequently, the *negative* sign appearing inside the parenthesis in the last term of eq-(1.14) furnishes the analog of the Lorentz relativistic factor in special relativity and it involves the ratio of the square of two *proper* forces. The result (1.14) in the 4-dim phase space can be generalized to the $8D$-dim phase space (and to higher dimensions) whose coordinates are $(X_\mu, P_\mu), \mu = 0,1,2,3$, where now one has (for a subluminal particle) $c^2(dt/d\tau)^2 - (dX_i/d\tau)^2 > 0$, with $i = 1,2,3$, and $(dE/d\tau)^2 - c^2(dP_i/d\tau)^2 = -F^2 < 0$.

The null case $(d\omega)^2 = 0$ in eq-(1.14) occurs naturally when $(d\tau)^2 = 0$, corresponding to a massless particle (like a photon) moving at the speed of light, and which in turn, implies also that $(dE)^2 - c^2(dP)^2 = 0$ because in the massless case one has $E^2 - c^2P^2 = 0 \Rightarrow E = cP \Rightarrow dE = cdP$. Therefore, the first line of eq-(1.14) yields $(d\omega)^2 = 0$ automatically. However, when $m \neq 0 \Rightarrow (d\tau) \neq 0$, the factorization of $(d\tau) \neq 0$ is allowed in eq-(1.14), and one can still have $(d\omega)^2 = 0$ when the massive particle experiences the *maximal* proper force $F = b$. Therefore, one attains $(d\omega)^2 = 0$ when one has a massless particle, or a massive one experiencing the maximal proper force $F_{max} = b$. A thorough study of the spacelike $(d\omega)^2 < 0$, null $(d\omega)^2 = 0$, and timelike $(d\omega)^2 > 0$ intervals in phase space, and their relation to the intervals in space time, can be found in the next section.

Caution must be taken in not confusing the proper force associated to a four-vector $F_\mu = \frac{dP_\mu}{d\tau}, \mu = 0,1,2,3$ with the *spatial* force associated to a three-vector $\vec{f} = \frac{dP_i}{d\tau}, i = 1,2,3$. The four-force has for components $F_\mu = (\frac{dE}{d\tau}, c\vec{f})$ where $\frac{dE}{d\tau}$ is the proper power. By maximal proper force one means that the magnitude-squared $|(dE/d\tau)^2 - c^2(dP_i/d\tau)^2| = |-F^2| = F^2 \leq b^2$ is bounded. However, this does *not* mean that the individual values of $(dE/d\tau)^2$ (square of the proper power) and $c^2(dP_i/d\tau)^2$ (magnitude-squared of the spatial force) are bounded. What is bounded is their $difference$ $|(dE/d\tau)^2 - c^2(dP_i/d\tau)^2| = F^2 \leq b^2$. For example, given the on-shell relation involving the energy-momentum $E^2 - c^2P_i^2 = m^2c^4$, this does not mean that each of the values of E^2, P_i^2 are bounded (they *blow* up when $v = c$). What is bounded is their difference (for a finite mass m).

Adopting the units $\hbar = c = k_B = 1$, one may postulate that the maximal proper-force acting on a fundamental particle in four-spacetime dimensions is given by $F_{max} = b \equiv \kappa m_P^2 = \kappa L_P^{-2} = \kappa/G$, where κ is a numerical coefficient. m_P is the Planck mass and L_P is the postulated minimal Planck length. A way to estimate the numerical coefficient κ is by looking at the Hawking temperature T_H associated to a black hole of Planck mass $T_H = \frac{1}{8\pi G m_P} = \frac{m_P}{8\pi}$. Equating T_H with the Unruh temperature $T_U = \frac{a}{2\pi}$ yields a proper acceleration of $a = \frac{m_P}{4}$, so that the corresponding proper force is $F = m_P a = \frac{m_P^2}{4} \Rightarrow \kappa = \frac{1}{4}$, and one recovers precisely the value of the maximum force conjecture proposed by [9].

Another route one may take is by setting the Unruh temperature to be equal to the Planck temperature $T_U = T_P = m_P = \frac{a}{2\pi} \Rightarrow a = 2\pi m_P$, so that the corresponding proper force is now $F = m_P a = 2\pi m_P^2$ leading to a value of $\kappa = 2\pi$. Invoking a minimal/maximal length duality one can also set $b = \kappa M_U/R_H$, where R_H is the Hubble scale and M_U is the observable mass of the universe. Equating both expressions for b leads to $M_U/m_P = R_H/L_P \sim 10^{60}$. The value of $b = \kappa m_P^2$ may also be interpreted as the maximal string tension. Since physics is an experimental science the choice of κ will have to be determined by experiment or observations, if the Born Reciprocal Relativity postulate is obeyed in nature.

The $U(1,1)$ group transformations involving the velocity and force boosts along the X direction of the phase-space coordinates X, t, P, E which leave the interval (1.14) invariant are obtained directly from eqs-(1.8) in this special case as follows

$$t' = t \ cosh\xi + (\frac{\xi_v \ x}{c} + \frac{\xi_a \ P}{b}) \ \frac{sinh\xi}{\xi} \quad (1.15a)$$

$$E' = E \ cosh\xi + (b \ \xi_a \ X + c \ \xi_v P) \ \frac{sinh\xi}{\xi} \quad (1.15b)$$

$$X' = X \ cosh\xi + (c \ \xi_v \ t + \frac{\xi_a \ E}{b}) \ \frac{sinh\xi}{\xi} \quad (1.15c)$$

$$P' = P \ cosh\xi + (\frac{\xi_v \ E}{c} + b \ \xi_a \ t) \ \frac{sinh\xi}{\xi} \quad (1.15db)$$

ξ_v is the velocity-boost rapidity parameter; ξ_a is the force (acceleration) boost rapidity parameter, and ξ is the net effective rapidity parameter of the primed-reference frame. The rapidity parameters ξ_a, ξ_v, ξ are defined, respectively, in terms of the spatial velocity $v = dx/dt$, and proper force $F = ma$, as follows

$$tanh(\xi_v) = \frac{v}{c}; \quad tanh(\xi_a) = \frac{F}{F_{max}}, \quad F_{max} = b, \quad \xi = \sqrt{(\xi_v)^2 + (\xi_a)^2} \quad (1.16)$$

When $\xi_v \rightarrow \infty \Rightarrow v \rightarrow c$. And $\xi_a \rightarrow \infty \Rightarrow F \rightarrow F_{max} = b$.

It is straight-forward to verify that the transformations (1.15) leave invariant the phase space interval $c^2(dt)^2 - (dX)^2 + ((dE)^2 - c^2(dP)^2)/b^2$ but *do not* leave separately invariant the proper time interval $(d\tau)^2 = c^2dt^2 - dX^2$, nor the interval in energy-momentum space $\frac{1}{b^2}[(dE)^2 - (dP)^2]$. Only the *combination*

$$(d\omega)^2 = (d\tau)^2 \left(1 - \frac{F^2}{F^2_{max}} \right) \quad (1.17)$$

is truly left invariant under force (acceleration) boosts. They also leave invariant the symplectic 2-form (phase space areas) $\Omega = - \ dt \wedge dE + dX \wedge dP$. Having displayed the basics of BRRT (non-inertial relativity) in the next section we present our novel findings.

2 Spacelike, Timelike, Null intervals in Phase space and the notion of a $U(1,d)$-invariant Mass

2.1 Force Boosts involving Massive Particles

An inspection of eqs-(1.15) in the text reveals that under pure force/acceleration boosts involving setting the velocity boost rapidity parameter to zero $\xi_v = 0$

leaves $\xi = \xi_a$, and one arrives at the following relations ($c = 1$)

$$(dt')^2 \ - \ (dX')^2 \ = \ [(dt)^2 - (dX)^2] \ \cosh^2 \xi \ + \ \frac{1}{b^2}[(dP)^2 - (dE)^2] \ \sinh^2 \xi \ +$$

$$\frac{1}{b} \ (dtdP - dXdE) \ \sinh(2\xi) \ \neq \ (dt)^2 \ - \ (dX)^2 \qquad (2.1)$$

and

$$\frac{1}{b^2}[(dE')^2 \ - \ (dP')^2] \ = \ \frac{1}{b^2}[(dE)^2 - (dP)^2] \ \cosh^2 \xi \ + \ [(dX)^2 - (dt)^2] \ \sinh^2 \xi \ -$$

$$\frac{1}{b} \ (dtdP - dXdE) \ \sinh(2\xi) \ \neq \ \frac{1}{b^2} \ [\ (dE)^2 \ - \ (dP)^2 \ ] \qquad (2.2)$$

which show that $(d\tau)^2 \equiv (dt)^2 - (dX)^2$, and $(d\mu)^2 \equiv (dE)^2 - (dP)^2$, the spacetime and energy-momentum infinitesimal displacement intervals, respectively, are *not* invariant under force/acceleration boosts. As expected, adding eqs-(2.1,2.2) furnish the $U(1,1)$ quadratic invariant in phase space

$$(dt')^2 - (dX')^2 + \frac{1}{b^2} \ [(dE')^2 - (dP')^2] \ = \ (dt)^2 - (dX)^2 + \frac{1}{b^2} \ [(dE)^2 - (dP)^2] \qquad (2.3a)$$

resulting from the identity $\cosh^2(\xi) - \sinh^2(\xi) = 1$.

Eqs-(2.1,2.2,2.3a) themselves are Lorentz invariant under velocity boost transformations[2] such that

$$(dt)^2 \ - \ (dX)^2 \ = \ (d\tilde{t})^2 \ - \ (d\tilde{X})^2; \ \ (dE)^2 \ - \ (dP)^2 \ = \ (d\tilde{E})^2 \ - \ (d\tilde{P})^2$$

$$(dt')^2 \ - \ (dX')^2 \ = \ (d\tilde{t}')^2 \ - \ (d\tilde{X}')^2; \ \ (dE')^2 \ - \ (dP')^2 \ = \ (d\tilde{E}')^2 \ - \ (d\tilde{P}')^2$$

$$dt \ dP \ - \ dX \ dE \ = \ d\tilde{t} \ d\tilde{P} \ - \ d\tilde{X} d\tilde{E} \qquad (2.3b)$$

A timelike interval in spacetime $(d\tau)^2 = (dt)^2 - (dX)^2 > 0$ is associated to a subluminal particle moving at speeds less than light. It is known that a non-inertial observer (in an accelerated frame of reference) assigns a pseudo-force acting on the particle. The centrifugal force is an example of a pseudo-force pointing in the opposite direction to the centripetal force. Hence a free particle from the point of view of a non-inertial observer will experience a pseudo-force. One could then envision that when the force/acceleration boost rapidity parameter tends to infinity $\xi \rightarrow \infty$ the particle's velocity relative to the accelerated frame of reference may reach the speed of light, and even surpass it.

[2] $d\tilde{t} = dt \cosh(\xi_v) - dX \sinh(\xi_v); d\tilde{E} = dE \cosh(\xi_v) - dP \sinh(\xi_v); d\tilde{X} = dX \cosh(\xi_v) - dt \sinh(\xi_v); d\tilde{P} = dP \cosh(\xi_v) - dE \sinh(\xi_v), \ldots$ and the same transformations for the primed variables $t', X', P', E' \rightarrow \tilde{t}', \tilde{X}', \tilde{P}', \tilde{E}'$

Namely, there could be a transition from a subluminal $(d\tau)^2 > 0$ to a superluminal regime $(d\tau')^2 < 0$. When $\xi \to \infty$ one has that $\cosh^2(\xi) \simeq \sinh^2(\xi)$, and $\sinh(2\xi) = 2\sinh(\xi)\cosh(\xi) \simeq 2\cosh^2(\xi)$, and eq-(2.1) becomes

$$(d\tau')^2 \ \simeq \ \left((d\tau)^2 \ - \ \frac{1}{b^2}(d\mu)^2 \ + \ \frac{2}{b} \ (dtdP - dXdE) \right) \cosh^2\xi \qquad (2.4)$$

At first glance, if one wishes to exclude the possibility that there is a crossover from the subluminal $(d\tau)^2 > 0$ to superluminal regime $(d\tau')^2 < 0$, and to a null regime $(d\tau')^2 = 0$, then one must have that $b >> 1$ in Planck units such that the leading term in eq-(2.4) becomes

$$(d\tau')^2 \simeq \left((d\tau)^2 \ + \ \mathcal{O}(\frac{1}{b}) \right) \cosh^2\xi \ > \ 0, \ \ with \ \ (d\tau)^2 > 0, \ \ (d\mu)^2 < 0 \quad (2.5a)$$

However, a more rigorous study reveals that one should factor out the $(d\tau)^2$ in eq-(2.4) leading to

$$(d\tau')^2 \ \simeq \ (d\tau)^2 \left(1 \ + \ \frac{F^2}{b^2} \ + \ \frac{2}{b} \ (\frac{dt}{d\tau}\frac{dP}{d\tau} - \frac{dX}{d\tau}\frac{dE}{d\tau}) \right) \cosh^2\xi \qquad (2.5b)$$

Eq-(2.5b) results after invoking the relations : when $(d\tau)^2 > 0 \Rightarrow (d\mu)^2 < 0$; and when $(d\tau)^2 < 0 \Rightarrow (d\mu)^2 > 0$ such that $1 - \frac{1}{b^2}\frac{(d\mu)^2}{(d\tau)^2} = 1 + \frac{F^2}{b^2}$. The first two terms inside the parenthesis in eq-(2.5b) are positive. This leaves the analysis of the last term inside the parenthesis. Let us evaluate this last term in the case of hyperbolic (Rindler) trajectories associated with a particle moving with a uniform proper acceleration g and proper force $F = mg$. The equations of motion in $c = 1$ units lead to

$$t \ = \ \frac{1}{g}\sinh(g\tau); \ \ X \ = \ \frac{1}{g}\cosh(g\tau); \ \ P \ = \ \gamma m \frac{dX}{dt} \ = \ m \ \cosh(g\tau) \ \tanh(g\tau) \ =$$

$$m \ \sinh(g\tau); \qquad (2.6a)$$

$$E \ = \ m\gamma \ = \ m \ \cosh(g\tau); \ \ \frac{dt}{d\tau} \ = \ \cosh(g\tau); \ \ \frac{dX}{d\tau} \ = \ \sinh(g\tau);$$

$$\frac{dP}{d\tau} \ = \ mg \ \cosh(g\tau); \ \ \frac{dE}{d\tau} \ = \ mg \ \sinh(g\tau) \qquad (2.6b)$$

γ above is the Lorentz dilation factor $(1 - v^2)^{-1/2} = \cosh(g\tau)$. Hence, the last term inside the parenthesis in eq-(2.5b) turns out to be *positive* for all values of τ,

$$\frac{2}{b} \ (\frac{dt}{d\tau}\frac{dP}{d\tau} - \frac{dX}{d\tau}\frac{dE}{d\tau}) \ = \ \frac{2mg}{b} \ [\cosh^2(g\tau) \ - \ \sinh^2(g\tau)] \ = \ \frac{2mg}{b} \ > \ 0 \quad (2.7)$$

Therefore, all the terms inside the parenthesis in eq-(2.5b) are positive

$$1 \ + \ (\frac{mg}{b})^2 \ + \ \frac{2mg}{b} \ = \ (1 \ + \ \frac{mg}{b})^2 \ > \ 0, \quad F = mg \tag{2.8}$$

so that if $(d\tau)^2 > 0 \Rightarrow (d\tau')^2 > 0$; and if $(d\tau)^2 < 0 \Rightarrow (d\tau')^2 < 0$, consequently there is no crossover in the spacetime intervals.

The pending question is what happens when $\xi \rightarrow -\infty$? In that case there is a crucial sign change due $\sinh(\xi) < 0$ when $\xi < 0$, and the terms inside the parenthesis of eq-(2.5b) become

$$1 \ + \ \frac{F^2}{b^2} \ - \ \frac{2mg}{b} \ = \ (1 \ - \ \frac{mg}{b})^2 \ \geq \ 0, \quad F = mg \tag{2.9a}$$

Thus, even with the presence of the minus sign in (2.9a) there will *not* be a crossover in the spacetime intervals. Similar results are found for *all* the values of the rapidity parameter ξ. After reinstating the appropriate $\cosh(\xi), \sinh(\xi)$ terms in eq-(2.1), it leads to

$$(d\tau')^2 \ = \ (d\tau)^2 \ \left(\ \cosh^2(\xi) \ + \ (\frac{mg}{b})^2 \sinh^2(\xi) \ + \ \frac{2mg}{b} \sinh(\xi)\cosh(\xi) \ \right) \ \Rightarrow$$

$$(d\tau')^2 \ = \ (d\tau)^2 \ (\ \cosh(\xi) \ + \ \frac{mg}{b} \sinh(\xi) \ )^2, \quad (d\tau)^2 \neq 0 \tag{2.10}$$

To sum up, in this example we found that there are *no* crossovers in the spacetime intervals when one performs a force boost transformation for any value of the ξ rapidity parameter. Consequently, in eq-(2.10) one has that if $(d\tau)^2 > 0 \Rightarrow (d\tau')^2 > 0$. And viceversa, if $(d\tau)^2 < 0 \Rightarrow (d\tau')^2 < 0$ because the parenthesis-squared in eq-(2.10) is always positive-definite, except in the very special case when $mg = b$ and $\xi = -\infty$ to be discussed below, leading to a null interval $(d\tau')^2 = 0$.

Following the same steps for the energy-momentum displacements in eq-(2.2) one arrives at

$$(d\mu')^2 \ = \ b^2 \ (d\tau)^2 \ \left(\ - \ (\frac{mg}{b})^2 \cosh^2(\xi) \ - \ \sinh^2(\xi) \ - \ \frac{2mg}{b} \sinh(\xi)\cosh(\xi) \ \right) \ \Rightarrow$$

$$(d\mu')^2 \ = \ -b^2 \ (d\tau)^2 \ (\ \frac{mg}{b} \ \cosh(\xi) \ + \ \sinh(\xi) \ )^2, \quad (d\tau)^2 \neq 0 \tag{2.11}$$

And, as expected, the sign of $(d\mu')^2$ is the *opposite* of $(d\tau')^2$. From eqs-(2.10,2.11) one can infer that in the accelerated frame of reference the proper force acting on the particle is now seen as $F' \neq mg$, and given by

$$\frac{1}{b^2} \ \frac{(d\mu')^2}{(d\tau')^2} \ = \ - \ \frac{F'^2}{b^2} \ = \ - \ \frac{(\ \frac{mg}{b} \ \cosh(\xi) \ + \ \sinh(\xi) \ )^2}{(\ \cosh(\xi) \ + \ \frac{mg}{b} \sinh(\xi) \ )^2} \tag{2.12}$$

The physical interpretation of eq-(2.12) is the following. One has a massive particle moving with respect to a reference frame S with a uniform proper acceleration and force given by g and $F = mg$, respectively. A new observer S' comes into the picture moving with respect to S with a uniform proper force/acceleration associated to a force boost rapidity parameter ξ given by $\tanh(\xi) = \frac{ma}{b}$. Therefore, the expression in eq-(2.12) depicts the "addition" of *proper* forces/accelerations describing the net proper force/acceleration of the massive particle with respect to the second observer S'.

The reader might raise an immediate *objection* to the expression in eq-(2.10) in the extremal case when the force boost rapidity parameters is $\xi = \infty$. In this extreme case the parenthesis in eq-(2.10) *blows* up leading to $(d\tau')^2 = (d\tau)^2 \times \infty$ and furnishing a *divergent* value for $(d\tau')^2 = \infty$ (since $(d\tau)^2 \neq 0$) which was supposed to be an infinitesimal squared-displacement. To settle this pathological paradox one must not forget to include also the key divergent contribution of $(d\mu')^2 = (dE')^2 - (dP')^2 = -\infty$, with an *opposite* sign in eq-(2.11), to the actual invariant phase space infinitesimal interval $(d\omega)^2$ giving

$$(d\omega)^2 = (d\tau)^2 + \frac{(d\mu)^2}{b^2} = (d\tau')^2 + \frac{(d\mu')^2}{b^2} = \infty - \infty = (d\tau')^2 \left(1 + \frac{1}{b^2}\frac{(d\mu')^2}{(d\tau')^2}\right) =$$

$$(d\tau')^2 \left(1 - \frac{F'^2}{b^2}\right) = (d\tau')^2 (1-1) = \infty \times 0 \neq \infty \tag{2.13}$$

The last terms of (2.13) result after taking the $\xi = \infty$ limit in the numerator and denominator of eq-(2.12) giving unity for their ratio[3] so that $-\frac{F'^2}{b^2} = -1$. Therefore, despite the *divergent* values of $(d\tau')^2, (d\mu')^2$, their net contribution to the phase space infinitesimal interval in eq-(2.13) $(d\omega)^2 = \infty \times 0$ is *not* divergent nor zero due to their *negative* relative signs.

An analogy of a product of the form $\infty \times 0$ occurs in special relativity when there is an infinite Lorentz time-dilation factor due to an infinite velocity boost rapidity parameter corresponding to the speed of light and giving
$dt = \gamma(v = c) d\tau = \infty \times 0 \neq \infty$ (nor zero) since the infinitesimal proper time interval is $d\tau = 0$ for a light-like path. Another pathological behavior occurs when there are spacetime singularities. A Schwarzschild black hole has a spatial singularity $(d\tau)^2 = -\infty$ at $r = 0$ and with $(dt)^2 \neq 0$.

Let us continue with the physical picture behind the expression in eq-(2.12). The motion of an accelerated particle with a proper uniform acceleration g, and proper uniform force $F = mg$, with respect to a reference frame S can be also be reproduced by starting with a massive particle at rest, and free of any forces (a free particle), and then performing a force boost transformation whose force boost rapidity parameter is given by $\tanh(\xi) = \frac{F'}{b} = \frac{mg}{b}$. In this scenario, eqs-(2.1,2.2) lead now to the following relations

$$(dt')^2 - (dX')^2 = [(dt)^2 - (dX)^2] \cosh^2 \xi = (d\tau)^2 \cosh^2(\xi) \tag{2.14}$$

[3] In the $\xi = \infty$ limit the $\cosh(\xi)$ and $\sinh(\xi)$ functions coincide

and

$$\frac{1}{b^2}(d\mu')^2 \ = \ \frac{1}{b^2}[(dE')^2 - (dP')^2] \ = \ [(dX)^2-(dt)^2] \ \sinh^2 \xi \ = \ - \ (d\tau)^2 \ \sinh^2(\xi) \tag{2.15}$$

because in the rest frame (or an inertial frame) in flat space one has $dE = dP = 0$. Eqs-(2.14, 2.15) yield

$$\frac{1}{b^2}\frac{(d\mu')^2}{(d\tau')^2} \ = \ - \ \frac{F'^2}{b^2} \ = \ - \ \tanh^2(\xi) \ \Rightarrow \ \tanh(\xi) \ = \ \frac{F'}{b} = \frac{mg}{b} \tag{2.16}$$

Hence, as expected, a free particle in one frame of reference (with $F = 0$) is subjected to a non-vanishing uniform proper force $F' = mg$ in an accelerated frame of reference. One has to be careful with the issue of signs. Because when $\xi < 0 \Rightarrow \frac{F'}{b} < 0$, one has to choose the appropriate negative sign for the value of $F' = -mg$, after *reversing* the sign of g in eq-(2.16). And when $\xi > 0$ one has instead $F' = mg$. The non-relativistic analog of this negative sign subtlety is associated with the notion of a pseudo-force in non-inertial frames of references. The centrifugal force is a pseudo-force whose sign is *opposite* to the physical centripetal force.

This is also consistent with taking the negative sign under the square root in the definition of the proper acceleration $-g = -\sqrt{|(d^2t/d\tau^2)^2 - (d^2X/d\tau^2)^2|}$. Reversing the sign of $F = mg$ has also an analogy in special relativity. The relation between the velocity boost parameter ξ_v and the velocity is $\tanh(\xi_v) = \frac{v}{c}$. When the observer moves in the positive, negative x-direction, one has $v > 0, v < 0$ and $\xi_v >, \xi_v < 0$, respectively.

Having discussed how performing a force boost transformation of a massive particle at rest mimics the motion of an accelerated particle with a proper uniform acceleration g, and proper uniform force $F = mg$, we shall explore two very interesting scenarios behind eq-(2.12) after setting : (i) $\tanh(\xi) = \frac{mg}{b}$, and (ii) $\tanh(\xi) = -\frac{mg}{b}$. Plugging $\tanh(\xi) = \frac{mg}{b}$ into eq-(2.12) yields after some straightforward algebra to

$$\frac{F'^2}{b^2} \ = \ \tanh^2(2\xi) \ \Rightarrow \ \frac{F'}{b} \ = \ \tanh(2\xi), \ \ \xi > 0 \tag{2.17}$$

which is consistent with the addition of two positive force boost rapidity parameters $\xi + \xi = 2\xi > 0$. Whereas, after plugging $\tanh(\xi) = -\frac{mg}{b}$ (corresponding to $\xi < 0$) into eq-(2.12) yields

$$F' = 0, \ (d\mu')^2 \ = \ (dE')^2 - (dP')^2 \ = \ 0, \ (d\tau')^2 \ = \ \frac{(d\tau)^2}{\cosh^2(\xi)} \ = \ (d\tau)^2 \ (1 - \frac{(mg)^2}{b^2}) \tag{2.18}$$

which is consistent with the subtraction of the force boost rapidity parameters $\xi - \xi = 0$, and which is tantamount of *undoing* the initial force boost transformation of a massive particle at rest, and returning the particle to its initial rest state from a state of uniform proper acceleration. Once the particle returns to rest (its inertial state) it ends up with $(dE') = (dP') = 0 \Rightarrow (dE')^2 - (dP')^2 = 0$, as expected.

Let us proceed with the following very interesting findings after examining eq-(2.12) more closely :

(i) when $F = mg$ attains its *maximal* value $F = mg = b$, one can infer from eq-(2.12) that $F' = b$ for *all* positive values of ξ ($F' = -b$ for all negative values of ξ). This is consistent with the *invariant* maximal proper force postulate which states that the value of $F = mg = b = F'$ must remain invariant.

(ii) when $F = mg < b$, in the extreme limits $\xi \to \pm\infty$, the value of F' in eq-(2.12) becomes $F' \to \pm b$, and one finds that the maximal proper force behaves like an *attractor* when $mg < b$. $F' \to -b$ for $\xi \to -\infty$, and $F' \to b$ for $\xi \to \infty$.

(iii) when $\xi = -\infty$, and $F = mg = b$, in this very special case the parenthesis in eqs-(2.10,2.11)) become null : $\cosh(\xi) + \frac{mg}{b}\sinh(\xi) \to 0$; and $\frac{mg}{b}\cosh(\xi) + \sinh(\xi) \to 0$, such that both $(d\tau')^2 \to 0$; $(d\mu')^2 \to 0$. Therefore, in this very special case a time-like interval $(d\tau)^2 > 0$ ($(d\mu)^2 < 0$) in one frame of reference will appear null-like $(d\tau')^2 = (d\mu')^2 = 0$ in an accelerated frame of reference. Note once again, that there is still *no* crossover from $(d\tau)^2 > 0$ to $(d\tau')^2 < 0$ in this very special extreme case. We shall study next the converse situation, where a null-like interval in one frame may appear massive and subluminal in another accelerated frame of reference.

In the most general case there is no crossover in the spacetime intervals under acceleration boosts, in the asymptotic limit $\xi \to \infty$, if the following condition is satisfied for all values of proper time τ during the motion of a particle

$$1 + \frac{F^2(\tau)}{b^2} + \frac{2}{b}\,\gamma(\tau)\,\frac{dP(\tau)}{d\tau}\,(\,1 - v^2(\tau)\,) \geq 0, \quad f(\tau) \equiv \frac{dP(\tau)}{d\tau} \tag{2.19}$$

Eq-(2.19) restricts the dynamics of the particle, namely one is looking for trajectories with $(dP(\tau)/d\tau) \geq 0$. A free particle will not experience a crossover. Naturally, setting $b \to \infty$ yields $(d\omega)^2 \simeq (d\tau)^2$ and the invariance $U(1,3)$ group effectively "contracts" to the $SO(1,3)$ group and BRRT "reduces" to special relativity and no crossover will occur.

2.2 Massless Particles and the null condition $(d\omega) = 0$ in Phase Space

We shall examine next what happens to massless particles under force boosts transformations. If one wants to preserve the null like conditions $(d\tau)^2 = 0, (d\mu)^2 = 0$ in eqs-(2.1,2.2) one must have $dtdP - dXdE = 0$ which is only satisfied in *two* cases out of *four* branches resulting from the relations $dt = \pm dX; dP = \pm dE$, and which in turn, are a consequence of the null like conditions $(dt)^2 - (dX)^2 = 0; (dE)^2 - (dP)^2 = 0$. One finds that there are two cases where $dtdP - dXdE = 0$, namely when $dt = dX, dP = dE$, and $dt = -dX, dP = -dE$.

And two cases where $dtdP - dXdE \neq 0$, namely when $dt = dX, dP = -dE$, and $dt = -dX, dP = dE$. The former two branches do lead to $(d\tau')^2 = 0$ in eq-(2.1), while the latter two branches do *not* lead to $(d\tau')^2 = 0$ in eq-(2.1).

Consequently, if one wishes one could discard those two branches which do not retain the null conditions. However one finds that this is not necessary because the null condition in phase space $(d\omega)^2 = (d\tau)^2 + \frac{1}{b^2}(d\mu)^2 = 0$ is still be valid in the primed reference frame : $(d\tau')^2 + \frac{1}{b^2}(d\mu')^2 = 0$ despite that each individual piece $(d\tau')^2, \frac{1}{b^2}(d\mu')^2$ may cease to be null. If one is positive, the other is negative, and vice versa, they cancel each other. This can be verified by simple inspection of eqs-(2.1,2.2) when $(d\tau)^2 = (d\mu)^2 = 0$ giving

$$(d\tau')^2 \ = \ \frac{1}{b} \ (dtdP - dXdE) \ sinh(2\xi) \neq 0 \qquad (2.20a)$$

$$\frac{1}{b^2}(d\mu')^2 \ = \ - \ \frac{1}{b} \ (dtdP - dXdE) \ sinh(2\xi) \neq 0 \qquad (2.20b)$$

as expected the expression in eq-(2.20b) has the opposite sign as that in eq-(2.20a) so $(d\omega)^2 = (d\tau')^2 + \frac{1}{b^2}(d\mu')^2 = 0$ remains null as it should because it is invariant under force boosts and Lorentz transformations.

In addition to these four cases above, there is the trivial solution $dP = 0, dE = 0$ which retains always the null like conditions. Its physical interpretation in terms of a massless photon is that the photon's frequency does *not* change as it propagates : there is no blue-shift nor red-shift in flat backgrounds.

The situation changes in curved backgrounds $dP \neq 0, dE \neq 0$. An expanding de Sitter universe leads to a photon redshift; an in-falling photon into a black hole is blue-shifted; while an outgoing photon is red-shifted, then in these cases when the spacetime background is *not* flat the situation changes because when $dtdP - dXdE \neq 0$ one finds that $(d\tau')^2 = -\frac{1}{b^2}(d\mu')^2 \neq 0$ and leading to important physical implications : a massless photon in one frame of reference is *no* longer massless in an accelerated frame.

Despite that so far we have been working in $flat$ phase space backgrounds, for the sake of the argument, let us examine the *hypothetical* case when $dP \neq 0, dE \neq 0$ (which occurs naturally in a curved backgrounds) and yielding $dtdP - dXdE = 2dtdP = -2dXdE \neq 0$. We then find in eqs-(2.2,2.20b), for any value of ξ, that

$$(d\mu')^2 \ = \ (dE')^2 \ - (dP')^2 \ \neq 0 \ \Rightarrow \ dE' \neq \pm dP' \ \Rightarrow \ E' \neq \pm P' \ \Rightarrow$$

$$(E')^2 - (P')^2 \ = \ (m')^2 \neq 0 \qquad (2.21)$$

therefore, one arrives at $m' \neq 0$ (the photon no longer appears massless in the accelerated frame) despite that $m = 0$. This is not surprising because m^2 is Lorentz invariant but is *not* $U(1,1)$-invariant.

What are now the novel physical implications of in-falling/outgoing photons in a black hole gravitational background ? Before answering this question one may recall that a photon is its own antiparticle and that virtual photons can carry positive and negative energies (frequencies) [13]. If virtual photons exchanged between particles have a positive energy, they contribute to the electromagnetic force as a repulsive force. This means that the two charged particles are repelled from each other and the electromagnetic force pushes them apart. On the other hand, if the virtual photons have a negative energy, they contribute to the electromagnetic force as an attractive force. This means that the two charged particles are attracted to each other and the electromagnetic force pulls them towards each other [13].

Bearing this in mind one finds that the sign of $(d\tau')^2$ in eqs-(2.1,2.20a) is determined by the sign of $dtdP - dXdE = 2dtdP = -2dXdE \neq 0$. There are 4 scenarios for $\xi > 0$ in eqs-(2.1,2.20a) :

(i) An in-falling photon $dX < 0$ (X plays a similar role as the radius r) of *negative* energy (frequency) $E < 0$ leads to $-dXdE > 0$ in eq-(2.20a) because as it gains energy one has $dE > 0$, so that $-dXdE > 0 \Rightarrow (d\tau')^2 > 0$, and the negative energy photon from the point of view of an accelerated frame (with respect to a static spherically symmetric Schwarzschild black hole, for example) will appear massive and subluminal.

(ii) An outgoing photon $dX > 0$ of *positive* energy (frequency) $E > 0$ leads to $-dXdE > 0$. As the energy decreases (red-shift) one has $dE < 0$, and from eq-(2.20a) it gives $-dXdE > 0 \Rightarrow (d\tau')^2 > 0$, so that the red-shifted photon from the point of view of an accelerated frame (with respect to a static spherically symmetric Schwarzschild black hole, for example) will appear massive and subluminal.

(iii) The converse scenario of (i), when the in-falling photon ($dX < 0$) has positive energy. As the energy increases one has $dE > 0 \Rightarrow -dXdE > 0$, leading to a massive subluminal regime $(d\tau')^2 > 0$.

(iv) The converse scenario of (ii) when the out-going photon ($dX > 0$) has negative energy, it leads again to a massive subluminal regime. As the outgoing negative-energy photon is losing energy one has $dE < 0$ so that $-dXdE > 0 \Rightarrow (d\tau')^2 > 0$.

Concluding, in the *hypothetical* case when $dP \neq 0, dE \neq 0$ (which occurs naturally in a curved backgrounds) all these four scenarios arrive at the same result : massless photons in one frame of reference will appear massive in an accelerated frame of reference when $dtdP - dXdE = 2dtdP = -2dXdE \neq 0$. This is not the first unexpected finding that might occur in non-inertial frames of reference. For example, one has the Fulling-Davies-Unruh effect when an accelerated observer in Minkowski space no longer experiences a vacuum but a thermal bath of photons at an absolute temperature proportional to the proper acceleration $T = \frac{a}{2\pi}$.

What we found in this subsection is the *converse* picture to what was found in the previous subsection **2.1** where we have shown that in the very special case (iii) when $mg = b$ and $\xi = -\infty$, a time-like interval $(d\tau)^2 > 0$ ($(d\mu)^2 < 0$) in one frame of reference will appear null-like $(d\tau')^2 = (d\mu')^2 = 0$ in an accelerated frame of reference.

After examining eqs-(2.20a,2.20b) further, one arrives at $\frac{1}{b^2}\frac{(d\mu')^2}{(d\tau')^2} = -\frac{F'^2}{b^2} = -1$ irrespective of the values of ξ. In the accelerated frame of reference the photons not only appear massive but are subjected to the maximal proper force $F' = b$. Whereas in the original inertial frame of reference because $(d\tau)^2 = \frac{1}{b^2}(d\mu)^2 = 0$, their ratio $\frac{0}{0}$ seems to be undetermined. Nevertheless there is a way to determine this $\frac{0}{0}$ ratio by looking at the hyperbolic trajectories of a massive particle moving with a uniform proper acceleration in a $D = 1 + 1$-dim spacetime. One can see that the hyperbolas (Rindler trajectories) asymptote (degenerate) to the null photon lines when $g \to \infty$. Because the photon is massless one can recover the null photon trajectories from the double scaling limit of mg such that as $m \to 0, g \to \infty \Rightarrow mg \to b$. In other words, a massless photon effectively experiences an infinite proper acceleration as seen with respect to an *inertial* frame of reference, let us say an observer S at rest.

Going back to eqs-(2.20) we found that the ratio of the expressions in (2.20b) and (2.20a) yields $\frac{1}{b^2}\frac{(d\mu')^2}{(d\tau')^2} = -\frac{F'^2}{b^2} = -1 \Rightarrow F' = b$, *irrespective* of the values of the force boost rapidity parameter ξ. In other words, ξ decouples after taking the ratios. Therefore one finds that the proper force $F = mg = 0 \times \infty = b$ experienced by the massless photon in the inertial frame of reference, and the proper force $F' = b$ experienced in any accelerated frame of reference (for any values of ξ) is the same $F = F' = b$, as it should, reflecting the invariance of the maximal proper force b. Since we have shown how a massless particle can appear massive in an accelerated frame, one ends up having $F' = m'g' = b = F$. Namely, under *any* force boost transformation (for all values of ξ) it takes $m = 0, g = \infty \to m' \neq 0; g' \neq \infty$ such that $F' = m'g' = b = F$ remains *invariant*.

Furthermore, by varying the values of ξ from 0 to $+\infty$, one can assign the running values of $m' = m'(\xi), g' = g'(\xi)$ along the points of a hyperbola depicted by the algebraic relation $m'(\xi)g'(\xi) = b$, and constrained to obey the reciprocal conditions $m'(\xi = 0) = 0; g'(\xi = 0) = \infty$, and $m'(\xi = \infty) = \infty; g'(\xi = \infty) = 0$ in the extreme limit. It has been known for a long time that very large masses moving with very low accelerations fall in the realm of modified Newtonian dynamics (MOND) [10].

One should add that the two scenarios (i), (ii) are very appealing because they involve negative energy photons radially in-falling into the black hole, and positive energy out-going photons leaving the black hole, respectively. Despite the similarity one must not confuse these findings with the Hawking evaporation process of black holes due to *quantum* effects. An heuristic explanation of the Hawking evaporation can be envisioned as virtual pairs of particles, say an electron-positron pair, popping out of the vacuum in the vicinity of the horizon. The positron carrying *negative* energy falls inside the horizon (decreasing the mass of the black hole), and the positive energy electron escapes outwards.

In view of the novel findings of this work that massless particles can appear massive in accelerated frames, and vice versa, it should have many important consequences in cosmology (dark energy, dark matter problem) and QFT. For a recent study of the astrophysical implications of a photon mass see [12].

Let us summarize the study of the spacelike, timelike and null intervals in phase space taking into account the above findings under force/acceleration boosts transformations. It would be interesting to find, if possible, if there is a particular *subgroup* of $U(1,1)$ involving both velocity and force/acceleration boosts preserving $(d\tau)^2 > 0$, for example. An analogous situation occurs with the Lorentz group which is not compact, nor connected. The subgroup of all Lorentz transformations in four dimensions preserving both orientation and direction of time is called the proper, orthochronous Lorentz group or restricted Lorentz group, and is denoted by $SO^+(1,3)$.

Given the $U(1,1)$ invariant interval in phase space $(d\omega)^2 = (d\tau)^2 + \frac{1}{b^2}(d\mu)^2$, when $(d\tau)^2 \neq 0$, it allows the *factorization* $(d\omega)^2 = (d\tau)^2[1 + \frac{1}{b^2}\frac{(d\mu)^2}{(d\tau)^2}]$. Once again, it is very important to emphasize once again that there are *no* factors of $(1 + F^2/b^2)$ appearing in the factorization process because in the superluminal case one has $m^2 a^2 < 0$, despite that $a^2 > 0$, because $m^2 < 0$ due to the imaginary mass of tachyons. Hence we always have the factor $(1 - F^2/b^2)$.

The above factorization leads to the following 2 cases to explore :

Case **1** : The timelike interval $(d\omega)^2 > 0$ (in phase space) leads to the following two sub-cases

$$\mathbf{1a} \ : \ (d\tau)^2 > 0, \ \ 1 - \frac{F^2}{b^2} > 0 \qquad (2.22a)$$

and

$$\mathbf{1b} \ : \ (d\tau)^2 < 0, \ \ 1 - \frac{F^2}{b^2} < 0 \qquad (2.22b)$$

The case **1b** must be disregarded because it implies that F is *larger* than b violating the maximal force postulate, in addition to having a superluminal particle (tachyon). Therefore, this leaves the case **1a** where the timelike interval $(d\omega)^2 > 0$ has also a correspondence with the special relativistic timelike interval $(d\tau)^2 > 0$ (subluminal velocities) and with the maximal force condition $F^2 < b^2$. As we have shown in the previous section, one can then assure that force/acceleration boosts transformations will *not* lead to a crossover from case **1a** to the unphysical case **1b** in the case of hyperbolic trajectories; for free particles and when $f(\tau) = (dP(\tau)/d\tau) \geq 0$.

Case **2** : The spacelike interval $(d\omega)^2 < 0$ (in phase space) leads to the following two sub-cases

$$\mathbf{2a} \ : \ (d\tau)^2 < 0, \ \ 1 - \frac{F^2}{b^2} > 0 \qquad (2.23a)$$

and

$$\mathbf{2b} \ : \ (d\tau)^2 > 0, \ \ 1 - \frac{F^2}{b^2} < 0 \qquad (2.23b)$$

The case **2a** involves the (spacetime) spacelike interval $(d\tau)^2 < 0$ corresponding to superluminal velocities, and to $F^2 < b^2$ obeying the maximal force postulate. Whereas, one finds in case **2b** that despite that $(d\tau)^2 > 0$ involving subluminal speeds, F^2 is *larger* than b^2 leading to a *violation* of the maximal force postulate.

Case **3a** : The null case $(d\omega)^2 = (d\tau)^2 + \frac{1}{b^2}(d\mu)^2 = 0$ with $(d\tau)^2 = 0$, and $(d\mu)^2 = 0$ corresponds to the null lines of a massless particle.

Case **3b** : The null case $(d\omega)^2 = 0$ with $(d\tau)^2 > 0 \Rightarrow (d\omega)^2 = (d\tau)^2(1 - \frac{F^2}{b^2}) = 0 \Rightarrow F = b$ involves a subluminal particle experiencing the maximal proper force $F = F_{max} = b$.

Case **3c** : The null case $(d\omega)^2 = 0$ with $(d\tau)^2 < 0 \Rightarrow (d\omega)^2 = (d\tau)^2(1 - \frac{F^2}{b^2}) = 0 \Rightarrow F^2 = b^2$, involves a superluminal particle (tachyon) experiencing the maximal proper force. To sum up, out of all of these cases only three cases **1a**, **3a** , **3b** are physically viable under force (acceleration) boosts and also trivially so under Lorentz transformations.

So far we have studied the *flat* Born geometry. A curved geometry of the phase space (cotangent space) requires the tools of Hamilton-Cartan geometry (Lagrange-Finsler geometry in the case of tangent space). The Born interval in an 8-dim curved phase space (cotangent space) is given by

$$(d\omega)^2 \;=\; g_{\mu\nu}(x,p)\, dx^\mu\, dx^\nu + h_{ab}(x,p)\, (dp^a + A^a_\mu(x,p)\, dx^\mu)\, (dp^b + A^b_\nu(x,p)\, dx^\nu) \tag{2.24}$$

$g_{\mu\nu}(x,p)$ is the horizontal base spacetime metric; $\mu, \nu = 0, 1, 2, 3$. $h_{ab}(x,p)$ is the vertical space (fiber) metric; $a, b = 0, 1, 2, 3$. $A^a_\mu(x,p)$ is the *nonlinear* connection. The flat space limit occurs when $g_{\mu\nu} = \eta_{\mu\nu}; h_{ab} = \frac{1}{b^2}\eta_{ab}$; $A^a_\mu = 0$. See [6] and the many references therein for more details. A curved geometry of the phase space has many other applications like in resolving the problem of UV divergences in QFT. Recently, an intrinsic regularization via curved momentum space as a geometric solution to divergences in Quantum Field Theory has been proposed by [14].

To finalize, defining the components of the generalized momentum Π in the 4-dim cotangent space (phase space) associated with the 2-dim space time, after taking derivatives with respect to the phase space invariant variable ω, as

$$\Pi \;=\; \mathcal{M}\, (\frac{dt}{d\omega}, \frac{dX}{d\omega}, \frac{1}{b}\frac{dE}{d\omega}, \frac{1}{b}\frac{dP}{d\omega}) \;\Rightarrow\; \Pi^2 \;=\; \mathcal{M}^2 \tag{2.25}$$

leads in phase space to the analog of the mass-shell condition in Minkowski spacetime. $\mathcal{M}$ is the $U(1,1)$ invariant version of mass, and is also $SO(1,1)$ invariant; whereas m is $SO(1,1)$ invariant but is not $U(1,1)$ invariant. These results can be extended to higher-dimensional phase space of dimension $2D = 2(d+1)$ and associated with a spacetime of dimension $D = d+1$. The invariance group is $U(1,d)$ and its Lorentz subgroup is $SO(1,d)$.

The analog of the Klein-Gordon equation corresponding to the on-shell condition in phase space (2.25) is [2]

$$(\Pi^2 \;-\; \mathcal{M}^2)\Psi(t,X,E,P) \;=\; 0 \;\Rightarrow$$

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial X^2} + b^2\frac{\partial^2}{\partial E^2} - b^2\frac{\partial^2}{\partial P^2} + \mathcal{M}^2\right)\Psi(t,X,E,P) \;=\; 0 \tag{2.26}$$

When Ψ only depends on t, X one recovers a Klein-Gordon equation with m replaced by $\mathcal{M}$. A rigorous study of the world-line quantization of a reciprocally invariant system can be found in [11]. One may note that a $b = \infty$ limit in eq-(2.26) would require $\frac{\partial^2 \Psi}{\partial E^2} = \frac{\partial^2 \Psi}{\partial P^2} = 0$ in order to avoid divergences, and in turn, it would lead to a field $\Psi(t, X)$ depending on t, X only. This would be consistent with the special relativistic regime of BRRT when $b \to \infty$. The physical implications of eq-(2.26) will be the subject of future investigations.

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