The Balance Paradox and Fermat's Theorem

Vladislav Koshchakov

Annotation

The possibility of using the principle of balancing scales to represent Fermat's theorem for the cases n=2 and n=3 is considered. In doing so, the fundamental difference between these two representations is revealed.

It is shown that for the case n=2 a new approach to obtaining known formulas for Pythagorean triples is possible, and for the case n=3 its unsolvability in integers is proved by elementary means using Fermat's infinite descent method.

Moreover, the case $n=3$ is easily extended to any n>2, which makes it possible to confirm that Pierre Fermat had a complete proof of his famous statement.

This physical and geometrical representation of the mathematical formulas of Fermat's theorem makes it possible to approach it in a completely new and unexpected way, providing the opportunity to see what was previously hidden in purely mathematical approaches.

The Notes show some similarities between these arguments and Pierre Fermat's mentions of his theorem in some letters.

Introduction

As early as the fifth millennium BC, scales were used in Mesopotamia that used the principle of a lever to achieve balance.

The first explanation of the lever rules was given in the 3rd century BC by Archimedes, who linked the concepts of force, load and leverage. Using the principle of the balance beam, Archimedes was the first to determine the volume of a sphere.

In the modern world, the principle of the lever is used everywhere - internal combustion engines, pliers, scissors...

However, beam scales also have a certain feature, which I called a paradox, if you use not a one-dimensional lever, but, for example, a two-dimensional one.

This paradox provides the possibility of unifying all cases of Fermat's theorem for n>2 by finding that they have two different shift coefficients to the right and left, in contrast to the case n=2, in which the shift coefficients are the same.

In the simplest example, this looks like this: three rods are attached to a moving axis - on the left in the figure, and three similar triangles - on the right. The system is in equilibrium in both cases.

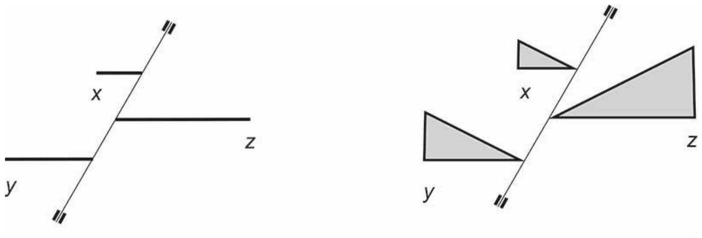

Fig. 1 On the left is a one-dimensional lever, on the right is a two-dimensional lever

It will be shown below that the rods on the scales, under equilibrium conditions, represent the equation

$$x^2 + y^2 = z^2 \qquad (1)$$

with his decisions:

$$x = 2ab$$
$$y = b^2 - a^2 \qquad (2)$$
$$z = b^2 + a^2$$

and triangles are an equation

$$x^3 + y^3 = z^3$$

Weighing of rods

Let us imagine a massless, dimensionless in diameter movable axis, to which three rods with the same linear mass are attached perpendicularly and in one plane. Let us place this structure in a uniform gravitational field.

Indeed, since the moments of forces on the right and left are equal and the mass of the rod is proportional to its length, and the center of gravity is the middle of the rod, then we look at Fig. 2:

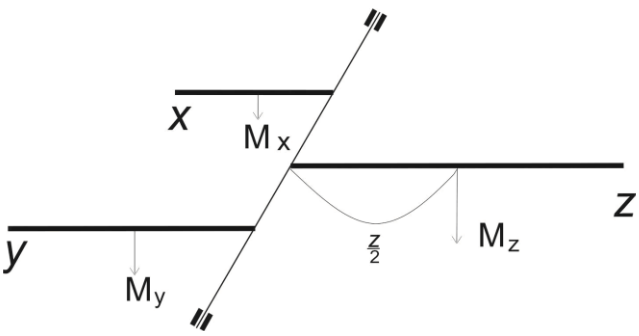

Fig.2

and we arrive at our expression (1).

It is possible to reduce the number of rods while maintaining the equilibrium position: let's move a rod, for example, y, parallel to itself along the moving axis so that it coincides with rod z. We now have only two rods: x and (z+y).
The new rod has a moment in the form of the product of its mass and the force arm, which is still equal to the moment of the segment x:

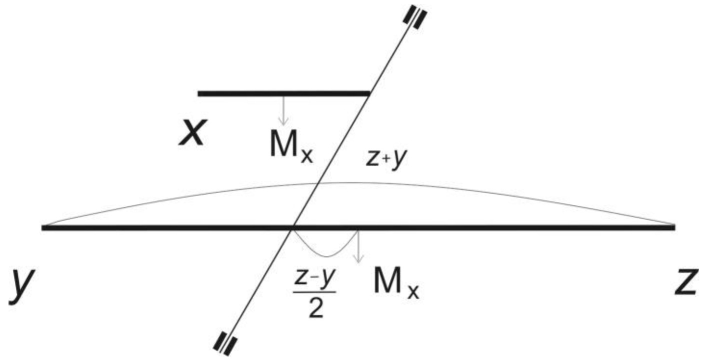

Fig.3

$$M_x = M_{z+y} \rightarrow x\frac{x}{2} = (z+y)(\frac{z-y}{2}) \qquad (3)$$

One more operation can be performed with the displaced segment *y*: instead of adding it to segment z, we can subtract segment *y* from it, and the system will still remain in equilibrium.

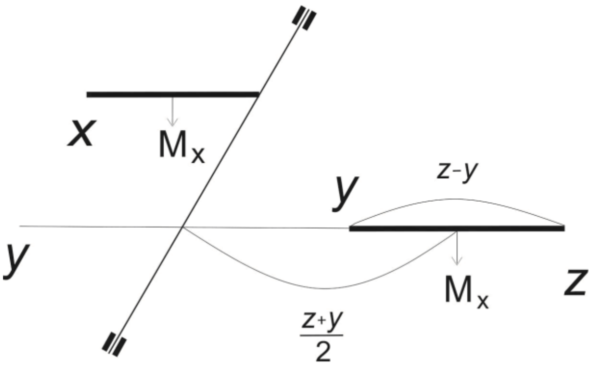

Fig.4

$$M_x = M_{z-y} \rightarrow x\frac{x}{2} = (z-y)(\frac{z+y}{2}) \qquad (4)$$

Further, the following display of the x, y and z segments will be more useful to us:

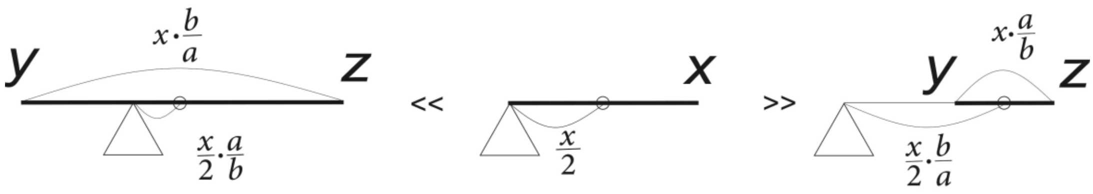

Fig.5

By moving the segment x to the right or left behind the center of gravity by any value while preserving the moment, we obtain all the values of the Pythagorean triples. In both cases, the connection between z and y is easily found relative to x through values a and b – we arrive to formulas (2) in a visual way: from the right end of the segment x we move through its center of gravity to the right to the center of gravity of the displaced segment x, then, through its changed length, to the left – to y, or to the right – to z. All these movements are rational.

Essentially, we are depicting a simple algebraic formula on the drawings:

$$x^2 = (\frac{a}{b}x)(\frac{b}{a}x) \qquad (5)$$

This graphical representation of this formula is more informative, because it allows us to clearly represent each of the factors in the form of a specific graphical object: the segment itself in the form of a linear mass and its arm – the location (through its center of gravity) relative to another object – the moving axis.

Note that it is possible to take not only y (see Fig. 3) but also x for alignment with the segment z. It is clear that the values of the right-left shift will differ from the values of a and b, since $x \neq y$, the values of this Pythagorean triplet themselves must remain unchanged. Let us designate this new shift as c and d .

Let us show the relationship between c and d and the existing a and b:

$$\frac{c}{d} = \frac{b-a}{b+a} \qquad (6) \qquad \text{And, accordingly: } \frac{a}{b} = \frac{d-c}{d+c} \qquad (7)$$

By substituting the values from formula (7) into formulas (2) instead of a and b, we notice how elegantly nature gets out of this awkward situation:

$$\begin{aligned} x &= 2ab = 2(d-c)(d+c) = 2(d^2 - c^2) \\ y &= b^2 - a^2 = (d+c)^2 - (d-c)^2 = 4dc \\ z &= b^2 + a^2 = (d+c)^2 + (d-c)^2 = 2(d^2 + c^2) \end{aligned} \qquad (8)$$

Note that the numbers c and d are odd, and the values of x and y in formulas (8) have changed places.
For example, for the simplest Pythagorean triple, one must choose a=1 and b=2 in formulas (2). However, the same triple can be obtained if one chooses a=1 and b=3 (in our notation, c=1 and d=3). Algebraically, this replacement and its result are not entirely noticeable, so formulas (6) and (7) have not been discovered until now. Using the physical-geometric approach brings clarity to this simple interdependence.

Extension of the method to Fermat's equation for *n*=3

The weight shift method ***WSM***, which we used earlier, we can try to apply to the next indefinite equation in order. We will call the value of the shift $\frac{b}{a}$ (or $\frac{a}{b}$) *the shift coefficient* ***SC***, the sum of its numerator and denominator – *the sum of the shift* ***SS***.

In this case, we will weigh the mass of a certain plane multiplied by its arm relative to the moving axis, by analogy with Fig. 1.

$$x^3 = \left(\frac{a}{b}x^2\right)\left(\frac{b}{a}x\right) \qquad (9)$$

It will be convenient to represent the area as an isosceles right triangle (half a square), attached to the axis by one of its acute angles, similar to Fig. 5.
Let the mass be evenly “smeared” over the area of all figures in the following drawing:

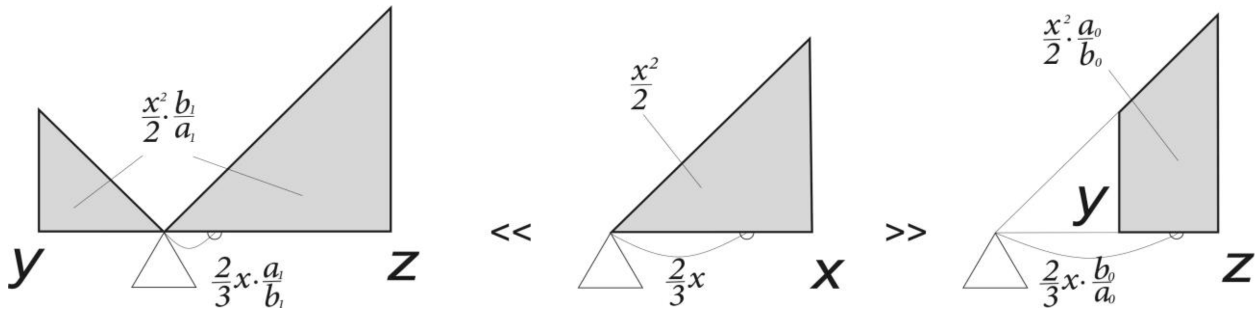

Fig.6

It is not difficult to see that this figure depicts the equation of Fermat's theorem for n=3

$$x^3 = z^3 - y^3 \qquad (10)$$

This equation is homogeneous, the mass density per unit area of each figure is the same, the size of the arm of each figure is directly proportional to x, y and z respectively, therefore their mutual reduction to the values of formula (10) occurs.

We shift the triangle $\frac{x^2}{2}$ to the right and left by analogy with the rods behind the center of gravity so as to obtain the same values of z and y for both shifts.

When shifting to the right, the shoulder increases by $\frac{b_0}{a_0}$ a factor of 1, and the area (its mass) decreases by the same amount to preserve the moment of force:

$$x^2 \frac{a_0}{b_0} = z^2 - y^2 \qquad (11)$$

If desired, here too, the values of y and z can be expressed relative to x through the values of a_0 and b_0, similar to the square case. After all, for the resulting figure, consisting of a rectangle and an isosceles right triangle, both the center of gravity and the mass in the form of an area can easily be found (see Fig. 6).

We will do it even more simply: since (10) is a product in accordance with formula (9), and (11) is one of the factors, then the second factor, the shoulder, is easily found:

$$x \frac{b_0}{a_0} = \frac{z^3 - y^3}{z^2 - y^2} \qquad (12)$$

So, we have divided equation (10) into two rational factors in accordance with formula (9). And the product of two rational factors is always rational, and this is possible only if there is a solution to equation (10).

More useful for us is the converse statement: only a rational number can be divided into two rational factors by using rational shift coefficients according to formula (9).

Therefore, there must be at least one solution of equation (10) in integers to be able to divide it into two rational factors – the shoulder and the “mass” of the area. If this solution (10) does not exist, then there will be nothing to divide.

Let us assume that such a solution exists. Then, using formula (11), we obtain the corresponding rational shift coefficient $\frac{a_0}{b_0}$. And vice versa – by shifting the triangle $\frac{x^2}{2}$ to the right by the center of gravity $\frac{b_0}{a_0}$ and reducing its area by $\frac{a_0}{b_0}$ times, in order to preserve the moment of force, we must obtain the assumed solution in whole numbers (10).
Indeed, let's see how points *X* and, for example, *Z* are connected in the case of the presence of an integer (or rational, which is unimportant for a homogeneous equation) solution of equation (10). Let us designate the points of the centers of gravity of the original triangle, which we need for the analysis – *Cx* and the same triangle, but shifted to the right – *Cyz*.
For clarity, we transfer the points of interest to the ray emanating from the zero point. We connect the points *X* and *Z* through the segments: *XCx*, *CxCyz*, *CyzZ*. The first segment is independent of the shift, the second increases by $\frac{b_0 - a_0}{a_0}$ times with the shift, and the last (let's designate it t) decreases by some rational number of times according to the rational law from $\frac{a_0}{b_0}$, because the points *Cyz* and *Z* are rational, and the latter is rational according to our assumption.

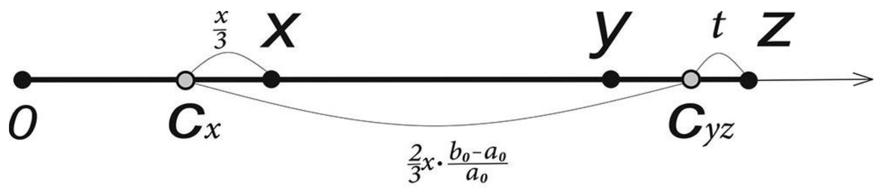

Fig.7

Consequently, the points *X* and *Z* have a rational connection between themselves, i.e. the relation $\frac{z}{x}$ is rational, and this relation of rationality includes exclusively the shift coefficient $\frac{b_0}{a_0}$ in the form of operations of addition, subtraction, multiplication and division of the integers that make it up. Only these operations are possible for rational functions.
For point *Y* the reasoning is similar.

Thus, assuming the existence of at least one solution (10), we must admit the existence of some formulas for the inverse transformation from the value of the ***SC*** to the integer solution (10) by analogy with a quadratic equation, where $\frac{a_0}{b_0}$ there is an argument, and $\frac{z}{x}$ and $\frac{y}{x}$ the essence of their functions.

Let us write out these hypothetical formulas, based on the assumption of the existence of a solution (10):

$$\frac{z}{x} = \varphi_z\left(\frac{a_0}{b_0}\right)$$
$$\frac{y}{x} = \varphi_y\left(\frac{a_0}{b_0}\right) \qquad (13)$$

Then any value $\frac{a_0}{b_0}$ will give its own triple of integers for $n=3$, similar to the square case.

Or, if these formulas do not exist, i.e. our proposition about the existence of at least one integer solution (10) is not true, then not a single triple in integers can be obtained with any rational shift. The irrational shift coefficient for obtaining rational x and y in formula (10) is excluded by a simple glance at formula (11).

Now let's look at the left shift in Fig. 6:

$$x^2 \frac{b_1}{a_1} = z^2 + y^2 \qquad (14)$$

Why do we indicate different coefficients for the shift values to the right and to the left here? After all, in the case of segments, this was not observed and the shift value was the same.

This is the fundamental difference between the second degree and all other degrees.

Indeed, multiplying the formulas for changing the length of a segment in the square case with different shifts (see Fig. 5):

to the right: $x\frac{a}{b} = z - y$

left: $x\frac{b}{a} = z + y$

we get our original equation: $x^2 = z^2 - y^2$.

And what do we get from multiplying formulas (11) and (14) in the case of identical shift coefficients? We get this:

$$x^4 = z^4 - y^4$$

which is not at all the purpose of our current discussions.

Therefore, the right-left shift coefficients to obtain equal y and z for any degree of a homogeneous equation of the form (10), except for one equal to two, they can't be the same!

So, any possible solution of a homogeneous equation for $n=3$ in integers has two different ***SC*** . This is its peculiarity, there is no way to avoid it, we should accept it and see what can be extracted from it.

Let's consider all the options for approaching it in light of the above:

A. The simplest thing is that such a solution does not exist, and the problem of the simultaneous existence of two pairs of shift coefficients disappears from the agenda.

B. We take only the shift to the right as a basis and consider exclusively the coefficients a_0 and b_0 **.** We do not take the shift to the left into account in our reasoning, and consider it insignificant.
The approach is refuted by the presence of equation (14).

C . We take only the shift to the left as a basis and consider exclusively the coefficients a_1 and b_1 . We do not take the shift to the right into account, we consider it insignificant.
The approach is refuted by the presence of equation (11).

D. We take into account the presence of two different pairs of shift coefficients in each possible integer solution of the equation for $n=3$ and see what this can lead to.

To begin with, we note that the sums of the shift of the ***SS*** for the right (11) and left (14) graphical representation of formula (10) cannot be equal, because the inequality is always satisfied:

$$a_1 + b_1 > a_0 + b_0 \qquad (15)$$

Indeed, from formulas (11) and (14) we obtain:

$z_0^2 + y_0^2 + x_0^2 > z_0^2 - y_0^2 + x_0^2$, therefore $2y_0^2 > 0$, and this is always true.

A natural question arises: what happens if we apply the right shift coefficient to the left one? Both ***SC*** are the internal essence of the assumed integer solution of equation (10) and have the same rights, so we have the ability to apply either of them to any direction of shift, because they do not contain any element indicating the direction of the shift that generated them.
Naturally, the formulas for the left shift will be different, but they must exist under the assumption of the presence of a solution to equation (10). Reasoning by analogy with the right shift (see Fig. 7), we obtain our hypothetical formulas for the left shift:

$$\frac{z}{x} = \phi_z\left(\frac{a_1}{b_1}\right)$$
$$\frac{y}{x} = \phi_y\left(\frac{a_1}{b_1}\right) \qquad (16)$$

As a result of such a shift, we obtain a new solution to equation (10):

$$x_1^3 = z_1^3 - y_1^3 \qquad (17)$$

We have already removed possible common factors in it.

Obtaining a solution to equation (17) in integers is possible because we previously assumed the existence of one solution to equation (10) in integers and, accordingly, made formulas (13) and (16) real.

We find new shift coefficients for this solution using formulas (11) and (14).

For right shift: a_2 and b_2

For left shift: a_3 and b_3

It is easy to see that $a_2 + b_2 < a_0 + b_0$

A similar process can be repeated an infinite number of times, but the number of integers smaller than the original sum $a_0 + b_0$ is finite.

Therefore, our assumption about the existence of a solution to the equation in integers for *n*=3 wrong - such solutions do not exist.

Extension to any *n*

For a homogeneous Fermat equation of degree n, one can use the area under the curve of the function $u = v^{n-2}$

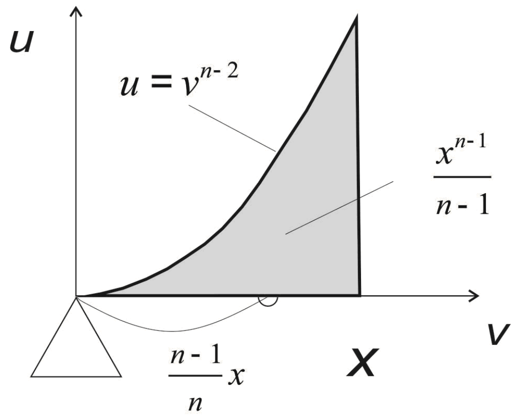

Fig.8

It is known that the area under the curve is equal to: $\frac{x^{n-1}}{n-1}$, and the center of gravity relative to the support is equal to: $\frac{n-1}{n}x$.

When shifting this area to the right by the value, $\frac{b_0}{a_0}$ we obtain the following values of the “area mass” (18) and the arm (19), by analogy with the case of Fermat’s equation for n=3.

$$x^{n-1}\frac{a_0}{b_0} = z^{n-1} - y^{n-1} \qquad (18)$$

$$x\frac{b_0}{a_0} = \frac{z^n - y^n}{z^{n-1} - y^{n-1}} \qquad (19)$$

And the fundamental formula that connects them into a single whole:

$$x^n = (\frac{a_0}{b_0}x^{n-1})(\frac{b_0}{a_0}x) \qquad (20)$$

Further reasoning is similar to the case n=3.

Notes

Let us point out the similarity of this approach with the individual references in Pierre Fermat's letters to his theorem that have come down to us:

1. In both cases, the method of infinite descent is used for a third-degree equation.

2. Having a proof of the theorem for n=3, it is not at all necessary to prove it for individual cases n>3, since the proof method (the equilibrium paradox) allows it to be used for any n. Fermat was able to find the areas under curves, find their centers of gravity, so extending the statement to any n should not have caused him any difficulties.

3. The physical-geometric approach to proving Fermat's theorem proposed here is indeed unusual for pure mathematical science and number theory in particular, and therefore sometimes encounters a certain misunderstanding. It is appropriate to recall the words of Pierre Fermat himself: "... a truly amazing proof...".

It can be assumed with some degree of probability that Fermat used similar or similar reasoning.