The three generations problem in particles physics

(A note on and a modest contribution to)

© Thierry PERIAT, 20 February 2025 – no affiliation

ThiLouArm@vodafonemail.de

The three generations problem is an open enigma concerning some types of leptons and of quarks in the standard model. The search for explanations dates back at least to the early 1980s. This document is a modest attempt to bring this research out of the nimbus of numerology. In a first step I introduce a special family of (3-3) matrices. Some of them are the representations of unit four-dimensional spheres via a Euler-Rodrigues parametrization. In a second step, I confront them with a condition formally preserving the definition of the Poynting vector in a changing geometrical context.

I. The standard model of particle physics

The particles can be classified into diverse categories. Inside the fermions, leptons (charged and neutral) and quarks (down type and up type) have identical electric and strong interactions but manifest themselves under three representations differing by their masses and their flavour quantum numbers. This fact has no explanation and this open question is what is called the unsolved three generations problem in particle physics.

II. The quest

Theoreticians hate unsolved problems and like to look for explanations, for combinations, for links between topics. This is why numerous tries have been made and are done to understand the existence of these three generations. In this document, I shall also propose my vision in focusing my research on the charged leptons.

III. The charged leptons

The attention of the scientific community has been drawn at the beginning of the eighties (20th century) to an empirical prediction made by a Japanese professor. At this time, the masses of the (classical) electron (m_e – first generation) and of the muon (m_μ - second generation) were already known. The prediction said that the mass of the tau ($m\tau$ - third generation) can be calculated with a specific formula which is nowadays labelled with the name of this professor [01; (I.1)]:

$$K = \frac{m_e + m_\mu + m_\tau}{\left(\sqrt[2]{m_e} + \sqrt[2]{m_\mu} + \sqrt[2]{m_\tau}\right)^2} = \frac{2}{3}$$

The first measurements of the mass of the tau were not convincing but their repetition and refinement ended up proving the prediction right; for now, with m_e = 0,511 MeV/c², m_μ = 105,66 MeV/c² [02] and m_τ = 1777,09 ± 0,14 MeV/c² [03], it can easily be verified that:

$$K = \frac{0{,}511 + 105{,}66 + 1777{,}09}{(0{,}7148 + 10{,}2791 + 42{,}1555)^2} = \frac{1883{,}261}{2824{,}8587} = 0{,}66667 \sim 2/3$$

Hazard, chance, obscure numerology or rational underground explanation? This is the question.

IV. The quarks

The next logical interrogation is to ask if the same formula applies to the quarks.

Considering the following measurements concerning the up-type quarks, quark up : 2,2 MeV/c², quark charm : 1 280 MeV/c² and quark top : 173 100 MeV/c², I state that K = 0,8486.

Considering the following measurements concerning the down-type quarks, quark down 4,67 MeV/c², quark strange : 93,4 MeV/c² and quark bottom : 4180 MeV/c², I state that K = 0,7314.

The ratio K(quark-up) and the ratio K(quark-down) are not the same as the one characterizing the charged leptons. At a first glance, this fact drastically reduces the generality of the formula.

V. Looking for an explanation concerning the charged leptons

1. Definition: The ratio K

Let generalize the formula connecting the masses of the three charged leptons in writing:

$$K(b^1, b^2, b^3) = \frac{(b^1)^2 + (b^2)^2 + (b^3)^2}{(b^1 + b^2 + b^3)^2}$$

The original formula concerning the charged leptons is recovered in writing:

$$b^1 = \sqrt[2]{m_e},\ b^2 = \sqrt[2]{m_\mu},\ b^3 = \sqrt[2]{m_\tau}$$

Let consider a space vector V = E(3, K). If the triple (b^1, b^2, b^3) represents the components of some vector b in V, then this generalized K-ratio can be synthetized as:

$$K(b) = \frac{||b||^2}{(b^\oplus)^2}$$

... where ||b|| denotes the classical Euclidean norm of b whilst $b^\oplus$ denotes the sum of its components.

2. Definition: a special set of matrices- the perian matrices

Let consider a space vector V = E(3, K) again and a pair (p, q) in V^2. A so-called perian matrix is, per convention, a representation of that pair in M(3, K) such that:

$$(a, b) \to [M(a, b)] = \alpha.\ Id_3 + \beta.\ T_2(\otimes)(b, b) + \chi.\ {}_{[J]}\Phi(b)$$

- The triple (α, β, χ) represents the components of p in the dual of V.
- Id_3 is the identity matrix in M(3, K), usually:

$$Id_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- The matrix $T_2(\otimes)(b, b)$ is a so-called Pythagorean table involving the components of q:

$$T_2(\otimes)(b, b) = \begin{bmatrix} b^1.b^1 & b^2.b^1 & b^3.b^1 \\ b^1.b^2 & b^2.b^2 & b^3.b^2 \\ b^1.b^3 & b^2.b^3 & b^3.b^3 \end{bmatrix} \in M(3, K)$$

- The matrix ${}_{[J]}\Phi(q)$ is the representation of an axial rotation:

$${}_{[J]}\Phi(q) = \begin{bmatrix} 0 & -b^3 & b^2 \\ b^3 & 0 & -b^1 \\ -b^2 & b^1 & 0 \end{bmatrix} \in M(3, K)$$

Remark: the sum of the entries of a perian matrix

The sum of the entries of a perian matrix is an application linking M(3, K) and K such that:

Equ.(0)

$$[M(a, b)] \to [M(a, b)]^{\oplus} = 3.\alpha + \beta.\ (b^{\oplus})^2$$

... where (recall) $b^{\oplus}$ is the sum of the components of q. It does not depend on the component χ.

3. Remark: perian matrix and Euler parametrization

Any four-dimensional sphere with radius 1 can be represented with a (3-3) matrix, thanks to the Euler-Rodrigues parametrization.

$$(u^0)^2 + (u^1)^2 + (u^2)^2 + (u^3)^2 = 1$$

$$\downarrow$$

$$M({}^{(4)}u)$$

$$=$$

$$\begin{bmatrix} (u^0)^2 + (u^1)^2 - (u^2)^2 - (u^3)^2 & 2.(u^2.u^1 - u^0.u^3) & 2.(u^3.u^1 + u^0.u^2) \\ 2.(u^1.u^2 + u^0.u^3) & (u^0)^2 - (u^1)^2 + (u^2)^2 - (u^3)^2 & 2.(u^3.u^2 - u^0.u^1) \\ 2.(u^1.u^3 - u^0.u^2) & 2.(u^2.u^3 + u^0.u^1) & (u^0)^2 - (u^1)^2 - (u^2)^2 + (u^3)^2 \end{bmatrix}$$

$$=$$

$$\{2.\ (u^0)^2 - 1\}.\ Id_3 + 2.\ T_2(\otimes)(b, b) + 2.\ u^0.\ {}_{[J]}\Phi(b)$$

$$=$$

$$[M({}^{(3)}\mathbf{a}, {}^{(3)}\mathbf{b})]$$

In writing:

Equ.(1)

$${}^{(3)}\mathbf{a}: (2.\ (u^0)^2 - 1, 2, 2.\ u^0) = ((u^0)^2 - ||{}^{(3)}\mathbf{b}||^2, 2, 2.\ u^0) = (<{}^{(4)}\mathbf{u}, {}^{(4)}\mathbf{u}>_{[\eta(+---)]}, 2, 2.\ u^0)$$

$${}^{(3)}\mathbf{b}: (u^1, u^2, u^3)$$

This matrix is obviously a special type of perian matrices.

Remark

The sum of the components of any element in this subset of perian matrices is:

$$M^{\oplus}({}^{(4)}\mathbf{u}) = [M({}^{(3)}\mathbf{a}, {}^{(3)}\mathbf{b})]^{\oplus} = 3.\ <{}^{(4)}\mathbf{u}, {}^{(4)}\mathbf{u}>_{[\eta(+---)]} + 2.\ ({}^{(3)}\mathbf{b}^{\oplus})^2$$

When the K-ratio of ${}^{(3)}\mathbf{b}$ is the one of the charged leptons, then:

$$K({}^{(3)}\mathbf{b}) = \frac{||b||^2}{(b^{\oplus})^2} = \frac{2}{3}$$

$$\Downarrow$$

$$M^{\oplus}({}^{(4)}\mathbf{u}) = [M({}^{(3)}\mathbf{a}, {}^{(3)}\mathbf{b})]^{\oplus} = 3.\ ((u^0)^2 - ||{}^{(3)}\mathbf{b}||^2) + 2.\ ({}^{(3)}\mathbf{b}^{\oplus})^2 = 3.\ (u^0)^2$$

The sum of the entries of any perian matrix $[M({}^{(3)}\mathbf{a}, {}^{(3)}\mathbf{b})]$

- which is the representation of a Euler-Rodrigues parametrization for some four-dimensional sphere with unit radius and ...
- of which the K-ratio of the second argument is the K-ratio of the charged leptons

... is equal to three times the square of the component u^0.

4. Remark: looking for a physical link with a 4D unit sphere

This first result gives rise to a question: “Is there an argument justifying the existence of a link between the K-ratio of the charged leptons and a physical unit four-dimensional sphere?”

Let examine the following example. In particle physics, the leptons respect the energy-impulse relation [08; p.4, (5)]:

$$E^2 = m^2.\ c^4 + c^4.\ p^2$$

It can be more precisely rewritten as:

$$\frac{E^2}{c^2} = m^2.\ c^2 + ||^{(3)}\mathbf{p}^2||$$

Hence when the 4D momentum $^{(4)}\mathbf{p}$ has the components:

$$(p^0 = m.\ c,\ p^1 = m.\ v^1,\ p^2 = m.\ v^2,\ p^3 = m.\ v^3)$$

... this relation can also be rewritten in a context related to a metric with the signature (+ + + +) as:

$$<{}^{(4)}\mathbf{p},\ {}^{(4)}\mathbf{p}>_{[+\,+\,+\,+]} = \frac{E^2}{c^2}$$

Or as:

$$E \neq 0 \Rightarrow \frac{c^2}{E^2}.\ <{}^{(4)}\mathbf{p},\ {}^{(4)}\mathbf{p}>_{[\eta(+\,-\,-\,-)]} = 1$$

interpreted as the equation of a four-dimensional unit sphere in a metric with signature (+ + + +) and there exists a vector $^{(4)}\mathbf{u}$:

$$u^0 = \frac{m.c^2}{E},\ u^1 = \frac{m.c}{E}.\ v^1,\ u^2 = \frac{m.c}{E}.\ v^2,\ u^3 = \frac{m.c}{E}.\ v^3 \Leftrightarrow {}^{(4)}\mathbf{u} = \frac{m.c}{E}.\ {}^{(4)}\mathbf{v}$$

... if I write ($v^0 = 1$).

Coming back to the main topic of this discussion, a logical link between the charged leptons, their K-ratio (2/3) and this specific unit 4D sphere implies:

$$<{}^{(4)}\mathbf{u},\ {}^{(4)}\mathbf{u}>_{[+\,-\,-\,-]} = (\frac{m.c^2}{E})^2 - (\frac{c}{E})^2.||\ {}^{(3)}\mathbf{p}|| = 2.\ (\frac{m.c^2}{E})^2 - 1$$

$${}^{(3)}\mathbf{a}\text{: } (<{}^{(4)}\mathbf{u},\ {}^{(4)}\mathbf{u}>_{[+\,-\,-\,-]},\ 2,\ 2.\ u^0) = (2.\ (\frac{m.c^2}{E}) - 1,\ 2,\ 2.\ \frac{m.c^2}{E})$$

$${}^{(3)}\mathbf{b} = \frac{m.c}{E}.\ {}^{(3)}\mathbf{v}$$

$$[M({}^{(3)}\mathbf{a},\ {}^{(3)}\mathbf{b})] = \{2.\ (\frac{m.c^2}{E})^2 - 1\}.\ Id_3 + 2.\ (\frac{m.c}{E})^2.\ T_2(\otimes)({}^{(3)}\mathbf{v},\ {}^{(3)}\mathbf{v}) + 2.\ \frac{m^2.c^3}{E^2}.\ {}_{[J]}\Phi({}^{(3)}\mathbf{v})$$

$${}^{(3)}\mathbf{b}^{\oplus} = \frac{m.c}{E}.\ {}^{(3)}\mathbf{v}^{\oplus},\ ||{}^{(3)}\mathbf{b}|| = \frac{m.c}{E}.\ ||{}^{(3)}\mathbf{v}||,\ K({}^{(3)}\mathbf{b}) = K({}^{(3)}\mathbf{v}) = \frac{||v||^2}{(v^{\oplus})^2}$$

In that case, the K-ratio at hand coincides with the one of the spatial speed of the particle at hand, here: an electron, a muon or a tau. But this way of thinking could also be applied to any particle for which the energy-momentum relation is true.

5. Discussion

Let analyse the plausibility of an intervening of the energy-momentum relation further.

A. The eventual effective existence of a direct link between the K-ratio related to the energy-momentum relation and the original one would then automatically impose a supplementary condition looking like this one:

$$\forall\ a = 1,\ 2,\ 3\text{: } v^a \sim \sqrt{m_a}$$

It immediately introduces a problem related to the logic because word for word this condition would mean that a given charged lepton (precisely its mass) must be associated with a given components of the spatial speed of … what? a collective triple of particles? Formulated in that way, this condition makes seemingly no sense.

Each particle (precisely its mass) is usually associated with the three components of a spatial speed, not with only one component of a collective speed. This basic fact induces that the discussion must introduce three matrices of that type.

But it also means that a link between the original K-ratio related to the masses and each of the three ratios $K({}^{(3)}\mathbf{v}_e)$, $K({}^{(3)}\mathbf{v}_\mu)$ and $K({}^{(3)}\mathbf{v}_\tau)$ related to the existence of an energy-momentum relation can only be indirect.

B. Another problem is that there is no reason to believe that the three generations of charged leptons exist and interact at a same instant in each frame. Such eventuality can better be envisaged for the quarks from which we know that they are confined in protons or in neutrons.

Having these thoughts in mind, I shall now look for a rational argument explaining the simultaneous existence of three perian matrices.

VI. The evolutions of the Poynting vector in a changing geometry

The Morley and Michelson experiments [04] are at the origin of a deep revolution in our understanding of space-time. Not only the electromagnetic waves propagate at a given speed (it is not infinite) – see J. C. Maxwell's work [05]- but this speed is invariant for observers situated at the origin of inertial frames. This experimental fact gives rise to the Lorentz-Poincare transformations which indirectly appear in the first version of the Einstein's theory of relativity [06].

The leptons are the source of electromagnetic fields; concretely: they can be understood as a set of pairs of spatial vectors $({}^{(3)}E, {}^{(3)}B)$. A Poynting vector ${}^{(3)}S$ can be associated with each state of these fields. Its usual definition is a classical cross product [07-G; p.90, (31,2)]:

$$ {}^{(3)}S = \frac{c}{4.\pi}.\ {}^{(3)}E \wedge {}^{(3)}B $$

It gives information on the energy carried by them [07-G; p.111, (47,5)]:

$$ {}^{(3)}S = \frac{\rho.\ c}{4.\pi}.\ {}^{(3)}n,\ \|{}^{(3)}n\| = < {}^{(3)}n, {}^{(3)}n >_{Id3} = 1 $$

Starting from here, within a 3D classical Euclidean geometry, I can write a scalar product:

$$ < {}^{(3)}S, {}^{(3)}S >_{Id3} = \left(\frac{c}{4.\pi}\right)^2.\ < {}^{(3)}E \wedge {}^{(3)}B, {}^{(3)}E \wedge {}^{(3)}B >_{Id3} = \left(\frac{c}{4.\pi}\right)^2.\ \rho^2 $$

The geometry, even the one of the empty regions of the universe, is not necessary invariant. The cross product can be understood as a tensor product which has been deformed by an antisymmetric (3-3-3) cube. Therefore, I propose to consider that any variation of the geometry deforms the Levi-Civita cube into some (3-3-3) cube D (hypothesis 1).

$$\text{cube } \varepsilon \rightarrow \text{cube D}$$

And I suppose (hypothesis 2) that this modification formally preserves the definition of the Poynting vector:

$$< {}^{(3)}S, {}^{(3)}S >_{Id3} = (\frac{c}{4.\pi})^2.\ \rho^2 \rightarrow < {}^{(3)}S', {}^{(3)}S' >_{[G]} = (\frac{c}{4.\pi})^2.\ \rho'^2$$

Since the speed of light remains a universal invariant for observers at the origins of inertial frames, the hypotheses 1 and 2 together allow:

Equ.(2)

$$\frac{1}{\rho^2}.\ < {}^{(3)}E \wedge {}^{(3)}B, {}^{(3)}E \wedge {}^{(3)}B >_{Id3} = (\frac{c}{4.\pi})^2 = \frac{1}{\rho'^2}.\ < \otimes_D({}^{(3)}E', {}^{(3)}B'), \otimes_D({}^{(3)}E', {}^{(3)}B') >_{[G]}$$

In the language of components, this relation is equivalent to:

$$\frac{1}{\rho^2}.\ \delta_{\alpha\beta}.\ (\varepsilon_{\alpha\lambda\mu}.\ E^{\lambda}.\ B^{\mu}).\ (\varepsilon_{\beta\nu\omega}.\ E^{\nu}.\ B^{\omega}) = \frac{1}{\rho'^2}.\ g_{\alpha\beta}.\ (d_{\alpha\lambda\mu}.\ E'^{\lambda}.\ B'^{\mu}).\ (d_{\beta\nu\omega}.\ E'^{\nu}.\ B'^{\omega})$$

Some manipulations transform the left-hand term in:

$$\text{l.h.t.} = \frac{1}{\rho^2}.\ (\delta_{\lambda\nu}.\ \delta_{\mu\omega} - \delta_{\mu\nu}.\ \delta_{\lambda\omega}).\ E^{\lambda}.\ B^{\mu}.\ E^{\nu}.\ B^{\omega}$$

Concerning the right-hand term, because the exact formalism of the transformations linking the pair (${}^{(3)}E', {}^{(3)}B'$) to the pair (${}^{(3)}E, {}^{(3)}B$) is unknown, I shall only presuppose (hypothesis 3) the existence of two matrices without logical connection with the Lorentz-Poincare transformations and such that:

Equ.(Hypothesis 3)

$$|{}^{(3)}E'> = [\theta].\ |{}^{(3)}E> \text{ and } |{}^{(3)}B'> = [\Xi].\ |{}^{(3)}B>$$

This hypothesis allows the concrete calculation of the r.h.t. and delivers the relations:

Equ.(3)

$$\frac{1}{\rho^2}.\ (\delta_{\lambda\nu}.\ \delta_{\mu\omega} - \delta_{\mu\nu}.\ \delta_{\lambda\omega}) = \frac{1}{\rho'^2}.\ g_{\alpha\beta}.\ A_{\alpha\lambda\mu}.\ A_{\beta\nu\omega}$$

$$A_{\alpha\lambda\mu} =$$

$$A_{\beta\nu\omega} =$$

The l.h.t. can always be understood as a peculiar representation of the object:

Equ.(4)

$$T_{\lambda\mu\nu\omega} = g_{\lambda\nu}.\ g_{\mu\omega} - g_{\mu\nu}.\ g_{\lambda\omega}$$

Precisely, the one which is obtained when $^{(3)}[G]$ can be identified with the Euclidean geometry:

Equ.(5)

$$\frac{1}{\rho^2}.\ \mathrm{Lim}_{[G]\rightarrow Id3}\ T_{\lambda\nu\mu\omega} = \frac{1}{\rho'^2}.\ g_{\alpha\beta}.\ A_{\alpha\lambda\mu}.\ A_{\beta\nu\omega}$$

As a matter of mathematical facts, the object "T" owns the same properties than the Rieman's curvature tensor:

Equ.(6)

$$T_{\lambda\mu\nu\omega} = - T_{\mu\lambda\nu\omega}$$

$$T_{\lambda\mu\nu\omega} = - T_{\lambda\mu\omega\mu}$$

$$T_{\lambda\mu\nu\omega} = T_{\nu\omega\lambda\mu}$$

$$T_{\lambda\mu\nu\omega} + T_{\lambda\omega\mu\nu} + T_{\lambda\nu\omega\mu} = 0$$

The latter allows to go further:

$$\frac{1}{\rho^2}.\ (\mathrm{Lim}_{[G]\rightarrow Id3}\ T_{\lambda\nu\mu\omega} + \mathrm{Lim}_{[G]\rightarrow Id3}\ T_{\lambda\omega\mu\nu} + \mathrm{Lim}_{[G]\rightarrow Id3}\ T_{\lambda\nu\omega\mu})$$

$$=$$

$$\frac{1}{\rho'^2}.\ g_{\alpha\beta}.\ (A_{\alpha\lambda\mu}.\ A_{\beta\nu\omega} + A_{\alpha\lambda\omega}.\ A_{\beta\omega\nu} + A_{\alpha\lambda\nu}.\ A_{\beta\omega\mu})$$

$$=$$

$$0$$

Hence, the definition of the Poynting vector remains formally unchanged when, whatever the geometry is, the entries of these cubes respect the relation:

Equ.(7)

$$A_{\alpha\lambda\mu}.\ A_{\beta\nu\omega} + A_{\alpha\lambda\omega}.\ A_{\beta\omega\nu} + A_{\alpha\lambda\nu}.\ A_{\beta\omega\mu} = 0$$

After some cumbersome manipulations, this relation can be reformulated as:

Equ.(8)

$$\forall\ n = 1, 2, 3: \{[_nA]^{\oplus}\}^2 + \{[_nA]^2 + [_nA]^t.\ [_nA]\}^{\oplus} = 0$$

8

VII. Looking for the formalism of the admissible matrices

1. Definition: admissible cubes and admissible matrices

The next logical step of this quest is to look for the generic formalism of the matrices composing the admissible cubes A, i.e.: the cubes preserving the definition of the Poynting vector in a changing background (synonym: geometry, provided the geometry is indirectly represented by the cubes).

2. Behaviour of the perian matrices

Let start in considering any perian matrix. A lot of annoying calculations delivers the condition defining the subset containing the perian matrices which can be included into the admissible cubes; here are the details of the calculations:

$$[A] = \alpha.\, Id_3 + \beta.\, T + \chi.\, \Phi$$

$$[A]^t = \alpha.\, Id_3 + \beta.\, T - \chi.\, \Phi$$

$$[A]^t.\, [A]$$

$$=$$

$$(\alpha.\, Id_3 + \beta.\, T - \chi.\, \Phi).\, (\alpha.\, Id_3 + \beta.\, T + \chi.\, \Phi)$$

$$=$$

$$(\alpha^2 + \chi^2.\, ||b||^2).\, Id_3 + (\alpha.\, \beta + \beta.\, \alpha + \beta^2.\, ||b||^2 - \chi^2).\, T + (\alpha.\, \chi - \chi.\, \alpha).\, \Phi$$

$$[A]^2$$

$$=$$

$$(\alpha.\, Id_3 + \beta.\, T + \chi.\, \Phi).\, (\alpha.\, Id_3 + \beta.\, T + \chi.\, \Phi)$$

$$=$$

$$(\alpha^2 - \chi^2.\, ||b||^2).\, Id_3 + (\alpha.\, \beta + \beta.\, \alpha + \beta^2.\, ||b||^2 + \chi^2).\, T + (\alpha.\, \chi + \chi.\, \alpha).\, \Phi$$

$$[A]^2 + [A]^t.\, [A]$$

$$=$$

$$2.\, \alpha^2.\, Id_3 + 2.\, (\alpha.\, \beta + \beta.\, \alpha + \beta^2.\, ||b||^2).\, T + 2.\, \alpha.\, \chi.\, \Phi$$

Therefore:

$$[A]^{\oplus} = 3.\, \alpha. + \beta.\, (b^{\oplus})^2$$

$$\{[A]^{\oplus}\}^2 = 9.\, \alpha^2 + 6.\, \alpha.\, \beta.\, (b^{\oplus})^2 + \beta^2.\, (b^{\oplus})^4$$

$$\{[A]^2 + [A]^t.\, [A]\}^{\oplus} = 6.\, \alpha^2 + 2.\, (\alpha.\, \beta + \beta.\, \alpha + \beta^2.\, ||b||^2).\, (b^{\oplus})^2$$

And, when the discussion occurs on a set equipped with a commutative multiplication, the condition is:

Equ.(9)

$$\beta^2. (b^{\oplus})^4 + (10. \alpha. \beta + 2. \beta^2. ||b||^2). (b^{\oplus})^2 + 15. \alpha^2 = 0$$

It does not depend on χ.

3. The perian matrices representing a Euler-Rodrigues parametrization

When, for example, this perian matrix is the representation of some Euler-Rodrigues parametrization of a four-dimensional unit sphere, then due to Equ.(1):

$$a: (\alpha, \beta, \chi) = (2. (u^0)^2 - 1, 2, 2. u^0)$$

And the condition of admissibility writes more precisely:

$$4. (b^{\oplus})^4 + \{20. (2. (u^0)^2 - 1) + 8. ||b||^2\}. (b^{\oplus})^2 + 15. ((2. (u^0)^2 - 1)^2 = 0$$

But in that case, because of the equation of the unit 4D-sphere (recall):

$$(u^0)^2 + ||b||^2 = 1$$

This equation transforms the condition in:

$$4. (b^{\oplus})^4 + \{20. (1 - 2. ||b||^2) + 8. ||b||^2\}. (b^{\oplus})^2 + 15. (1 - 2. ||b||^2)^2 = 0$$

It can be rewritten as:

Equ.(10)

$$4. (b^{\oplus})^4 + \{20 - 32. ||b||^2\}. (b^{\oplus})^2 + \{60. ||b||^4 - 60. ||b||^2 + 15\} = 0$$

a) If furthermore the K(b) exists and is equal to 2/3:

$$3. ||b||^2 = 2. (b^{\oplus})^2, b^{\oplus} \neq 0$$

Then the condition is reformulated as:

Equ.(11)

$$9. ||b||^4 + \{10 - 16. ||b||^2\}. 3. ||b||^2 + \{60. ||b||^4 - 60. ||b||^2 + 15\} = 0$$

$$3. ||b||^4 + \{10 - 16. ||b||^2\}. ||b||^2 + \{20. ||b||^4 - 20. ||b||^2 + 5\} = 0$$

$$(3 - 16 + 20). ||b||^4 + (10 - 20). ||b||^2 + 5 = 0$$

$$7. ||b||^4 - 10. ||b||^2 + 5 = 0$$

b) If the sum of the components of b vanishes ($b^{\oplus} = 0$), then *the K-ratio of b does not exist* and Equ.(10) imposes:

$$||b||^2 = ½$$

The system:

Equ.(12)

$$||b||^2 = ½ \text{ and } b^{\oplus} = 0$$

... has solutions; they respect the relations:

Equ.(13)

$$b^1. b^2 + b^2. b^1 + b^2. b^3 + b^3. b^2 + b^3. b^1 + b^1. b^3 = - ½$$

$$b^1 + b^2 + b^3 = 0$$

4. Proposition

There are perian matrices such that (i) K(b) = 2/3, $b^{\oplus} \neq 0$ and (ii) the condition preserving the definition of the Poynting vector in a changing background is true *which are not* the representation of some Euler-Rodrigues parametrization related to a four-dimensional unit sphere.

Proof: A separate calculation proves that the polynomial:

Equ.(14)

$$4. (b^{\oplus})^4 - 16. (b^{\oplus})^2 + 15 = 0$$

... has two solutions:

Equ.(15,1 and 2)

$$(b^{\oplus})^2 = 3/2 \text{ and } (b^{\oplus})^2 = 5/2$$

Hence with (15,1) and the supplementary condition:

Equ.(16)

$$||b|| = 1$$

All perian matrices preserving the formal definition of the Poynting vector have a K(b) ratio equal to 2/3 when, because of Equ.(9):

$$\beta^2. (b^{\oplus})^4 + (10. \alpha. \beta + 2. \beta^2. ||b||^2). (b^{\oplus})^2 + 15. \alpha^2 = 0$$

$$\beta^2. (3/2)^4 + (10. \alpha. \beta + 2. \beta^2). (3/2)^2 + 15. \alpha^2 = 0$$

$$\{(81/16) + (18/4)\}. \beta^2 + (90/4). \alpha. \beta + 15. \alpha^2 = 0$$

Equ.(17)

$$153. \beta^2 + 360. \alpha. \beta + 240. \alpha^2 = 0, \forall \chi$$

They are never identifiable with a representation of some Euler-Rodrigues parametrization because when - recall Equ.(1):

$$^{(3)}\mathbf{a}: (2.\,(u^0)^2 - 1,\, 2,\, 2.\,u^0) = ((u^0)^2 - ||^{(3)}\mathbf{b}||^2,\, 2,\, 2.\,u^0) = (<^{(4)}\mathbf{u},\, ^{(4)}\mathbf{u}>_{[\eta(+\,-\,-\,-)]},\, 2,\, 2.\,u^0)$$

Then, due to Equ.(16) and to the equation of the unit 4D-sphere (recall):

$$(u^0)^2 + ||b||^2 = 1$$

It follows that:

$$(u^0)^2 = 0,\ \alpha = 0,\ \beta = 2$$

And that:

$$153.\,\beta^2 + 360.\,\alpha.\,\beta + 240.\,\alpha^2 = 612 \neq 0,\ \forall\,\chi$$

5. Lemma

The set of perian matrices is greater than the set of matrices representing Euler-Rodrigues parametrization.

6. Perian matrices not representing a Euler-Rodrigues parametrization

Since I am mainly interested in looking for specific matrices [M(a, b)] allowing (i) the definition of a K-ratio for b and (ii) preserving the definition of the Poynting vector in a changing background, the main condition must be calculated in starting with Equ.(9):

$$\forall\,(\alpha, \beta, \chi) \in K$$

$$\beta^2.\,(b^{\oplus})^4 + (10.\,\alpha.\,\beta + 2.\,\beta^2.\,||b||^2).\,(b^{\oplus})^2 + 15.\,\alpha^2 = 0$$

... and with the condition of existence for a K-ratio:

$$\forall\,||b||,\ b^{\oplus} \neq 0$$

Recalling now the information contained in Equ.(14) and (16), I consider the system:

Equ.(18,1 to 17,4)

$$\forall\,k$$

$$\beta^2 = 4.\,k$$

$$10.\,\alpha.\,\beta + 2.\,\beta^2.\,||b||^2 = -16.\,k$$

$$15.\,\alpha^2 = 15.k$$

... because it is equivalent to the polynomial:

$$\{4.\,(b^{\oplus})^4 - 16.\,(b^{\oplus})^2 + 15\}.\,k = 0$$

... which has the ad hoc solution (see § VII 4.), at least when k = 1 and ||b|| = 1. The Equ.(18,3) imposes:

$$\alpha = \pm\sqrt{k}$$

Let inject this condition into Equ.(18,3), due to Equ.(18,2), I obtain:

$$\pm 20.\ k + 8.\ k.\ ||b||^2 = -16.\ k$$

Or equivalently:

$$\forall\ k: \pm 20 + 8.\ ||b||^2 = -16$$

The consequence of which is either:

$$||_1b||^2 = -\frac{9}{2}$$

Or:

$$||_2b||^2 = +\frac{1}{2}$$

These results are compatible with a ratio K(b) equal to 2/3 when respectively:

Equ.(19)

$$\{ K(b) = 2/3,\ ||_1b||^2 = -\frac{9}{2} \} \Rightarrow ({}_1b^{\oplus})^2 = -\frac{27}{4}$$

$$\{ K(b) = 2/3,\ ||_2b||^2 = +\frac{1}{2} \} \Rightarrow ({}_2b^{\oplus})^2 = +\frac{3}{4}$$

These results seem to be in contradiction with the ones of § VII 4. although they were inspired by them! Are they? Why? In fact: no, there is no contradiction because the Equ.(14) has nothing to do with the problem at hand.

Perian matrices	Any matrix
$[M(a, b)] = \alpha.\ Id_3 + \beta.\ T_2(\otimes)(b, b) + \chi.\ {}_{[J]}\Phi(b)$	$[A]$
Preservation of the Poynting vector	
Equ.(9) $\beta^2.\ (b^{\oplus})^4 + (10.\ \alpha.\ \beta + 2.\ \beta^2.\ \|\|b\|\|^2).\ (b^{\oplus})^2 + 15.\ \alpha^2 = 0$	$\{[A]^{\oplus}\}^2 + \{[A]^2 + [A]^t.\ [A]\}^{\oplus} = 0$

Table 1

For the perian matrices only			
Euler parametrization $< \mathbf{u}, \mathbf{u} >_{Id4} = 1$ $(u^0)^2 + \|\|b\|\|^2 = 1$ (α, β, χ) $=$ $((u^0)^2 - \|\|^{(3)}\mathbf{b}\|\|^2, 2, 2.\ u^0)$ Equ.(10) $4.\ (b^{\oplus})^4 + \{20 - 32.\ \|\|b\|\|^2\}.\ (b^{\oplus})^2 + \{60.\ \|\|b\|\|^4 - 60.\ \|\|b\|\|^2 + 15\}$ $=$ 0		Not a Euler parametrization	
$b^{\oplus} \neq 0$ $\exists$ K(b) = 2/3 Equ.(11)	$b^{\oplus} = 0$ not $\exists$ K(b) $\|\|^{(3)}\mathbf{b}\|\|^2 = ½$	$b^{\oplus} \neq 0$ $\exists$ K(b) = 2/3 Equ.(19)	$b^{\oplus} = 0$ not $\exists$ K(b) $\alpha = 0$

Table 2

VIII. End of the discussion and perspective

The purpose of this document was to propose rational arguments connecting the three masses of a given type of particles. Unfortunately, only a small part of the travel on the road going to the goal has been done. I explain why.

If it is believed that the masses of the three generations of a given type of particles are related with the help of a K-ratio, then there exist elements taken in a specific family of (3-3) matrices (the so-called perian matrices) depending on the components of a pair of vectors which (i) preserve the formal definition of the Poynting vector despite of geometric deformations and (ii) are the ad hoc mathematical objects to build these K-ratios.

This document illustrates this affirmation in focusing attention on the charged leptons because their masses are effectively related via a K-ratio equal to 2/3.

A subset of these specific matrices can be identified with representations of Euler-Rodrigues parametrizations. But an identification with this kind of representations is not a necessary condition. It is only a sufficient one.

At this stage, the approach developed in the first part of this document allows the definition of perian matrices. Each of them can represent a Euler-Rodrigues parametrization related to the energy-momentum relation associated with one given type of particles. Focusing attention on charged leptons, this is resulting in three K-ratios, $K(^{(3)}\mathbf{v}_e)$, $K(^{(3)}\mathbf{v}_\mu)$ and $K(^{(3)}\mathbf{v}_\tau)$, at each given instant of some chronology.

But except if the discussion does no more concern the charged leptons, there is no reason to believe that these particles exist at the same time in the frame of an observer and that the three generations interact in that frame. If they do, prior calculations carry no information on this topic.

The second part of this exploration suggests that the admissible matrices should preserve the formalism of the Poynting vector in a changing background; that's all. Until now, I did not verify if the matrices which have been discovered in the first part preserve this formalism. If they do, the Equ.(10) must be true (recall):

$$4. ({}^{(3)}\mathbf{b}^{\oplus})^4 + \{20 - 32. ||{}^{(3)}\mathbf{b}||^2\}. ({}^{(3)}\mathbf{b}^{\oplus})^2 + \{60. ||{}^{(3)}\mathbf{b}||^4 - 60. ||{}^{(3)}\mathbf{b}||^2 + 15\} = 0$$

With here (recall):

$${}^{(3)}\mathbf{b} = \frac{m.c}{E}. {}^{(3)}\mathbf{v}$$

If the particle at hand is moving at c speed in the frame where the discussion occurs, then:

$$||{}^{(3)}\mathbf{b}||^2 = \frac{m^2.c^4}{E^2}$$

Let remark that the energy-momentum relation for such a particle writes:

$$E^2 = m^2. c^4 + c^4. p^2 = m^2. c^4. (1 + c^2) \sim m^2. c^6$$

$$\Downarrow$$

$$||{}^{(3)}\mathbf{b}||^2. c^2 = \frac{m^2.c^6}{E^2} \sim 1$$

$$\Downarrow$$

$$||{}^{(3)}\mathbf{b}||^2 \sim 0$$

Note that the same approximative result is obtain when a particle moving at c speed is massless ($m \sim 0$). The condition preserving the Poynting vector formalism is then approximately:

$$4. ({}^{(3)}\mathbf{b}^{\oplus})^4 + 20. ({}^{(3)}\mathbf{b}^{\oplus})^2 + 15 = 0$$

It implies that either the components of the speed or, but more probably, the masses of the particles at hand must be pure imaginary complex numbers! Imaginary energies can be envisaged in quantum mechanics. They describe virtual particles and/or energetic transitions; see example in [09; Tome 1, complement H_{IV} ; pp. 468-473].

If the approach presented in this document is correct, then it only applies to virtual particles. But are the charged leptons or the quarks virtual particles? The leptons are quite certainly not, the quarks are perhaps. The question is open.

Furthermore, the question concerning the link between the diverse charged leptons (resp. quarks), more precisely between their masses and their speeds is yet not clear.

All these thoughts introduce deep doubts:

- Doubts on the choice which has been made to privilege the energy-momentum relation. Other objects perhaps better reflect the physical situation concerning the masses of the charged leptons (resp. of the quarks).
- More dramatically, doubts on the way of thinking which is promoted here.

This is the reason why a future exploration will study the links between the perian matrices and the main parts of the decompositions of some deformed cross product. The confrontation will prove that these matrices can be linked with the polynomials systematically associated with the decompositions.

The hope is that some of these polynomials will give useful information concerning the masses of the charged leptons (resp. of the quarks). At the end of the day, it will no matter if they don't because the failure of some essays save time for others.

IX. Bibliography

[01] The Koide Lepton Mass Formula and Geometry of Circle; arXiv:1201.2067v1 [physics.gen-ph] 5 Jan 2012.

[02] Measurement of Muon Properties in the Advances Students Laboratory, Heidelberg University, version 1.3, 16 July 2021, 56 pages.

[03] DESY, 31 mai 2023, Belle II collaboration (KEK, Japon).

[04] Michelson, A. and Morley, E: "On the Relative Motion of the Earth and the Luminiferous Ether''. Originally published in "The American Journal of Science", N° 203 November 1887 (Editors James D. and Edward S. Dana; associated editors: Prof. A. Gray, J. P. Cooke and J. Trowbridge, of Cambridge, Prof. H.A. Newton and A. E. Verrill of New Haven; Prof. G. F. Barker of Philadelphia. Third series, Vol. XXXIV.- (Whole number, CXXXIV.)

[05] Maxwell, J. C.: A Dynamical Theory of the Electromagnetic Field; Philosophical Transactions of the Royal Society of London, 1865, 155: 459 – 512.

[06] Einstein, A. : On the Electrodynamics of Moving Bodies, published in Annalen der Physik 17 (1905): 891-921.

[07-GB] Landau, L. D. and Lifschitz, E. M.: The Classical Theory of Fields, Course of Theoretical Physics, Third Revised English Edition, © 1971 Pergamon Press Ltd, 387 pages. [07-G: German Version] Lehrbuch der theoretischen Physik, Band II, Klassische Feldtheorie: L.D. Landau, E.M. Lifschitz; 12. überarbeitete Auflage, in dt. Sprache hrsg.

von Hans-Georg Schöpf [Übers. aus dem russ. von Georg Dautcourt], © Akademie Verlag GmbH, Berlin 1992, ISBN 3-05-501550-9.

[08] Dyson, Freeman, J.: Quantenfeld-theorie (Die Weltbekannte Einführung von einem der Väter der QED); Springer Spektrum, ISBN 978-3-642-37677-1, © Springer Verlag Berlin Heidelberg 2014, 288 pages.

[09] Cohen-Tanoudji, B. Diu et Fr. Lalöe : Mécanique quantique, tome I, édition revue, corrigée et augmentée de 1977, ISBN 2 7056 5733 9; © 1973 Herman Paris, 2 tomes au total, 1491 pages.