The Factor of 4/3 Paradox Between Electromagnetic Momentum and Electromagnetic Energy is Fully Resolved by Correct Choice of the Structure of the Electromagnetic Field Stress Energy Tensor

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Abstract

The ratio of electromagnetic momentum to electromagnetic energy is thought to be the velocity, but when calculated for a uniform velocity point charge is off by a factor of 4/3. The apparent problem is fully resolvable if we note that the electromagnetic stress energy tensor has a non-vanishing T^{ii} for a particle at rest, and if we then use the appropriate relationship for the stress energy components for an object which has non-vanishing T^{ii} in its rest frame.

1. Introduction

For a fluid without internal pressure the stress energy tensor is

$$T^{\mu\nu} = \rho \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds}. \tag{1}$$

Thus

$$T^{10} = \rho \frac{dx^{1}}{ds} \frac{dx^{0}}{ds} \tag{2}$$

and

$$T^{00} = \rho \frac{dx^{0}}{ds} \frac{dx^{0}}{ds} \tag{3}$$

In the frame where the particle is at rest, (3) becomes

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$$T^{00}_{[rest]} = \rho \tag{4}$$

Inserting (4) into (2), we get

$$T^{10} = \frac{dx^i}{ds} T^{00}_{[rest]} \frac{dx^0}{ds} \tag{5}$$

$$\int T^{10} d^3x = \frac{dx^i}{ds} \int T^{00}_{[rest]} \frac{dx^0}{ds} d^3x \tag{6}$$

The Lorentz contraction causes the volume element in the rest frame to differ from the volume element in the non-rest frame by a factor of γ:

$$d^3x = \frac{1}{\gamma} d^3x_{[rest]} \tag{7}$$

$$\gamma d^3x = d^3x_{[rest]} \tag{8}$$

Noting that $\frac{dx^0}{ds}$ is equal to γ, inserting (8) into (6) gives

$$\int T^{10} d^3x = \frac{dx^i}{ds} \int T^{00}_{[rest]} d^3x_{[rest]} \tag{9}$$

$$p^i = \frac{dx^i}{ds} p^0_{[rest]} \tag{10}$$

This is a familiar result.

However when the p^1 and $p^0_{[rest]}$ of the electromagnetic field for a point charge moving with uniform velocity are actually calculated we do not get the (10) result of $p^1 = \frac{dx^1}{ds} p^0_{[rest]}$ [1], but instead get the result of $p^1 = \frac{4}{3} \frac{dx^1}{ds} p^0_{[rest]}$. This is a longstanding paradox in electromagnetism.

Attempts to resolve the factor of 4/3 discrepancy have included efforts to include the effects of the forces holding the charged particle together [1, 2, 3]. We will see that in reality the paradox is caused by the inappropriate use of (10), which itself is based on the inappropriate (1), an equation only appropriate when rest frame T^{11} vanishes. It turns out that for a charged particle at rest T^{ii} actually does not vanish. If instead of the inappropriate (10), we use the appropriate equation for the realtionship between the components of the stress energy for a situation where $T^{ii}_{electromagnetic}$ does not vanish in the rest frame, (22) below, the paradox cleanly and completely disappears.

2. The Relationship Between Components of a Stress Energy Tensor When T^{ii} Does Not Vanish in the Rest Frame

2.1.

The correct relationship for the components of the stress energy tensor in a situation where a fluid has pressure in its rest frame is well-known [4, 5] to be

$$T^{\mu\nu} = (\rho + P_{[rest]})\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - \eta^{\mu\nu}P_{[rest]} \tag{11}$$

where ρ is the energy density in the rest frame and $P_{[rest]}$ is the pressure in the rest frame.

This implies that

$$T^{i0} = (\rho + P_{[rest]})\frac{dx^i}{ds}\frac{dx^0}{ds} \tag{12}$$

and that

$$T^{00} = (\rho + P_{[rest]})\frac{dx^0}{ds}\frac{dx^0}{ds} - P_{[rest]} \tag{13}$$

In the rest frame, (13) becomes

$$T^{00}_{[rest]} = \rho \tag{14}$$

Inserting (14) into (12) we get

$$T^{i0} = (T^{00}_{[rest]} + P_{[rest]})\frac{dx^i}{ds}\frac{dx^0}{ds} \tag{15}$$

$$\int T^{10}d^3x = \int (T^{00}_{[rest]} + P_{[rest]})\frac{dx^1}{ds}\frac{dx^0}{ds}d^3x \tag{16}$$

Again noting that $\frac{dx^0}{ds}$ is γ, inserting (8) into (16) gives

$$\int T^{10}d^3x = \frac{dx^1}{ds}\int \left(T^{00}_{[rest]} + P_{[rest]}\right) d^3x_{[rest]} \tag{17}$$

$$p^1 = \frac{dx^1}{ds}\left(p^0_{[rest]} + \left(\int P_{[rest]}d^3x_{[rest]}\right)\right) \tag{18}$$

Comparing (18), the equation for a system where there can be pressure in the rest frame, to (10), the special case where rest frame pressurelessness is assumed, we see from the mathematical structures that (10) is indeed the special case of (18) for a system with no pressure. (10) is only valid if the pressure in the rest frame vanishes. Use of (10) for a situation where there is pressure in the rest frame is incorrect, and can lead to incorrect results.

In the language of stress energy tensors, the "rest frame pressure" is $T^{11}_{[rest]} = T^{22}_{[rest]} = T^{33}_{[rest]}$. (We will see that if these three quantities are not equal, although "$P_{[rest]}$" is no longer definable, a more general relationship, (22), of which (18) is a special case, can be used.)

We will show by direct calculation that the T^{ii} quantities are indeed non-vanishing for a point charge at rest. Thus (10), the assumption used in creating the "paradox", is wrong to be used in this situation. (18) must be used instead. It will be shown that when (18) is used there is no paradox. The paradox occurred due to use of an invalid equation for the situation examined.

2.2.

It might seem that we must get $p^1 = \frac{dx^1}{ds} p^0_{[rest]}$ by making a Lorentz transformation of p^0 for the particle at rest to a moving frame. However, we will presently show that if T^{ii} does not vanish in the rest frame we do not actually get that result. Instead, we get a result that is a generalization of (18).

The rule for transforming a 2-tensor is $A'^{\mu\nu} = \frac{\partial x^\mu}{\partial x^\alpha}\frac{\partial x^\nu}{\partial x^\beta} A^{\alpha\beta}$. Letting the $T^{\mu\nu}$ tensor be the $A^{\mu\nu}$, and having the coordinate transformation be the Lorentz transformation, we get

$$\begin{aligned} T'^{10} = & \left(\beta\gamma\right)\left(\gamma\right) T^{00}_{[rest]} \quad + \\ & \left(\gamma\right)\left(\beta\gamma\right) T^{11}_{[rest]} \end{aligned} \tag{19}$$

$$\begin{aligned} \int T'^{10} d^3x' = & \int \left(\beta\gamma\right)\left(\gamma\right) T^{00}_{[rest]} d^3x' \quad + \\ & \int \left(\gamma\right)\left(\beta\gamma\right) T^{11}_{[rest]} d^3x' \end{aligned} \tag{20}$$

In this case the non-primed frame volume element is the rest frame volume element, and the primed volume element is the non-rest frame volume element. Thus using (8), we get

$$\int T'^{10} d^3x' = \int (\beta\gamma)(T^{00}_{[rest]}) d^3x_{[rest]} \quad + \\ \int (\beta\gamma) T^{11}_{[rest]} d^3x_{[rest]} \tag{21}$$

$$p^1 = \frac{dx^1}{ds}(p^0_{[rest]} + \int T^{11}_{[rest]} d^3x_{[rest]}) \tag{22}$$

We see that (18) was a special case of (22) where $T^{11}_{[rest]} = T^{22}_{[rest]} = T^{33}_{[rest]}$, in which case a quantity "$P_{[rest]}$" can be defined as $P_{[rest]} \equiv T^{11}_{[rest]} = T^{22}_{[rest]} = T^{33}_{[rest]}$. If it is not the case that $T^{11}_{[rest]} = T^{22}_{[rest]} = T^{33}_{[rest]}$ then there is no definable $P_{[rest]} \equiv T^{11}_{[rest]} = T^{22}_{[rest]} = T^{33}_{[rest]}$, in which case (18) cannot be used, but (22) is still valid, and (22) must be used.

We also note that in the derivation leading to (22) there is no assumption of a fluid, and thus it is valid for any stress energy scenario, such as the stress energy of an electromagnetic field.[1]

3. The Point Charge Paradox is Resolved When the Appropriate Relationship Between Stress Energy Components for a Situation with a Non-Vanishing T^{ii} in the Rest Frame is Used

As previously explained, the paradox is that when p^1 and ${p^0}_{[rest]}$ are calculated [1] for a point charge moving with uniform velocity it turns out that $p^1 = \frac{4}{3}\frac{dx^1}{ds}{p^0}_{[rest]}$. This was supposedly contradicted by the belief, based on (19) that the relationship must be $p^1 = \frac{dx^1}{ds}{p^0}_{[rest]}$. However, we have shown that the appropriate relationship between the components of the stress energy tensor when the T^{ii} do not vanish in the rest frame is the relationship from (22), for which (10) is a special case which is only true when the T^{ii} vanish in the rest frame.

Using the correct relationship between stress energy components for a situation where T^{ii} does not vanish in the rest frame we substitute the calculated result $p^1 = \frac{4}{3}{p^0}_{[rest]}$ into the left-hand side of the valid (22) relationship that $p^i = \frac{dx^i}{ds}\left({p^0}_{[rest]} + \int T^{11}_{[rest]} d^3x\right)$. Doing this, we get

[1]Actually, while (11), the equation that led to (18) is typically characterized as the relationship for a fluid, its derivation does not at all depend on the characterized entity being a fluid.

$$\frac{4}{3}\frac{dx^1}{ds}p^0{}_{[rest]} = \frac{dx^1}{ds}p^0{}_{[rest]} \quad + \\ \frac{dx^1}{ds}\int T^{11}_{[rest]}d^3x_{[rest]} \tag{23}$$

From (23) we see that if and only if $\int T^{11}_{[rest]}d^3x_{[rest]}$ is equal to $\frac{1}{3}p^0_{[rest]}$, then use of (22), the equation that is the appropriate equation if $\int T^{11}d^3x$ is non-vanising in the rest frame, would resolve the paradox.

To see if $\int T^{11}_{[rest]}d^3x_{[rest]}$ is indeed equal to $\frac{1}{3}p^0_{[rest]}$, we will need to calculate $p^0_{[rest]}$ and to calculate $\int T^{11}_{[rest]}d^3x_{[rest]}$.

To calculate $p^0_{[rest]}$ we use the $T^{\mu\nu}_{[electromagnetic]}$ formula $T^{\mu\nu}_{[electromagnetic]} = \frac{1}{4}\pi\left(F^{\mu\sigma}F^{\nu}_{\sigma} - \frac{1}{4}\eta^{\mu\nu}F^{\lambda\omega}F_{\lambda\omega}\right)$ to calculate $T^{00}_{[rest]}$, and then integrate. Since the point charge is at rest in the rest frame, $B_{[rest]} = 0$. Inserting into the $T^{00} = \frac{1}{4}\pi\left(F^{0\sigma}F^{0}_{\sigma} - \frac{1}{4}\eta^{00}F^{\lambda\omega}F_{\lambda\omega}\right)$ formula, we then get

$$T^{00}_{[rest]} = \frac{1}{4}\pi\left(\left((E_{[rest]})^2\right) - \frac{1}{4}\left(2(E_{[rest]})^2\right)\right) \tag{24}$$

$$T^{00}_{[rest]} = \frac{1}{4}\pi(\frac{1}{2}(E_{[rest]})^2) \tag{25}$$

$$\int T^{00}_{[rest]}d^3x_{[rest]} = \frac{1}{4}\pi\int(\frac{1}{2}(E_{[rest]})^2)d^3x_{[rest]}. \tag{26}$$

$$p^0_{[rest]} = \frac{1}{4}\pi\int(\frac{1}{2}(E_{[rest]})^2)d^3x_{[rest]} \tag{27}$$

We note that this result actually is valid for any charge distribution at rest.

To calculate $\int T^{11}_{[rest]}d^3x_{[rest]}$ we again use the $T^{\mu\nu}_{electromagnetic} = \frac{1}{4}\pi\left(F^{\mu\sigma}F^{\nu}_{\sigma} - \frac{1}{4}\eta^{\mu\nu}F^{\lambda\omega}F_{\lambda\omega}\right)$ formula (and again $B_{[rest]} = 0$, being that B vanishes in the rest frame).

$$T^{11}_{[rest]} = \frac{1}{4}\pi\left(\left(-(E^x_{[rest]})^2\right) + \frac{1}{4}\left(2(E_{[rest]})^2\right)\right) \tag{28}$$

$$\int T^{11}_{[rest]}d^3x_{[rest]} = \\ \frac{1}{4}\pi\int\left(\left(-(E^x_{[rest]})^2\right) + \frac{1}{4}\left(2(E_{[rest]})^2\right)\right)d^3x_{[rest]} \tag{29}$$

Because of the symmetry of the electric field for a point charge at rest, $\int (E^x_{[rest]})^2 d^3x_{[rest]} = \int (E^y_{[rest]})^2 d^3x_{[rest]} = \int (E^z_{[rest]})^2 d^3x_{[rest]}$; and thus $\int (E_{[rest]})^2 d^3x_{[rest]}$, which is $\int (E^x_{[rest]})^2 d^3x_{[rest]} + \int (E^y_{[rest]})^2 d^3x_{[rest]} + \int (E^z_{[rest]})^2 d^3x_{[rest]}$, is equal to $3 \int (E^x_{[rest]})^2 d^3x_{[rest]}$. Thus $\int (E^x_{[rest]})^2 d^3x_{[rest]} = \frac{1}{3} \int (E_{[rest]})^2 d^3x_{[rest]}$. Inserting this result into (29) we get

$$T^{11}_{[rest]} d^3x_{[rest]} = \frac{1}{4}\pi \left(- \left(\frac{1}{3} (E_{[rest]})^2 \right) + \frac{1}{4} \left(2 (E_{[rest]})^2 \right) \right) d^3x_{[rest]} \tag{30}$$

$$\int T^{11}_{[rest]} d^3x_{[rest]} = \frac{1}{4}\pi \int (\frac{1}{6} E^2) d^3x_{[rest]} \tag{31}$$

Comparing (31) to (27), we see that $\int T^{11}_{[rest]} d^3x_{[rest]}$ is indeed equal to $\frac{1}{3} p^0_{[rest]}$, exactly the result we had needed for the paradox to be resolved by use of the correct equation for situations where T^{11} is non-vanishing in the rest frame, (22). The paradox is cleanly and simply resolved by using the correct energy momentum relationship for situations with non-vanishing T^{ii} in the rest frame.

4. Appendix - Creating and Resolving Other Electromagnetic Energy and Momentum Paradoxes

If we use other shapes and motions for the electric charge distribution we can create other paradoxes if we naively assume the (10) $p^1 = \frac{dx^i}{ds} p^0_{[rest]}$ relationship that is not actually valid for situations where there is non-vanishing T^{ii} in the rest frame. (Interestingly the discrepancy factor is not always $\frac{4}{3}$–the $\frac{4}{3}$ is not some special characteristic of electromagnetism.)

In these paradoxes, just as was the case with the point charge paradox, the paradoxes are cleanly and simply resolved by using (22), the correct energy momentum relationship for situations with non-vanishing T^{ii} in the rest frame

4.1.

Consider a plane of charge in the yz plane moving in the x direction.

As noted previously, $p^0_{[rest]}$ is always that given by (27). Thus

$$p^0_{[rest]} = \frac{1}{4}\pi \int \frac{1}{2}(E_{[rest]})^2 d^3x_{[rest]} \tag{32}$$

To calculate T^{10} we note that in this case there is no magnetic field perpendicular to the electric field, and thus ExB is zero. Thus the Poynting vector, the T^{10} from the $T^{10} = \frac{1}{4}\pi \left(F^{1\sigma}F^0_\sigma - \frac{1}{4}\eta^{10}F^{\lambda\omega}F_{\lambda\omega}\right)$ formula, is zero. So

$$p^1 = 0 \tag{33}$$

Comparing (33) with (32) we see that

$$p^1 = \frac{dx^1}{ds}(0)p^0_{[rest]} \tag{34}$$

From the perspective of (the inappropriate) (10) we see a discrepency from the (erroneously) expected $p^1 = \frac{dx^1}{ds}p^0_{[rest]}$ result. The discrepancy this time is a factor of zero, analogous to the discrepancy factor of $\frac{4}{3}$ for the point charge.

Let us see if we can resolve this new paradox by again using the appropriate (22) $p^1 = \frac{dx^1}{ds}p^0_{[rest]} + \frac{dx^1}{ds}\int T^{11}_{[rest]}d^3x_{[rest]}$ relationship, instead of the inappropriate (10) $p^i = \frac{dx^i}{ds}p^0_{[rest]}$ relationship that led to the supposed paradox. Substituting (34) into (22), we get

$$0 = \frac{dx^1}{ds}p^0_{[rest]} + \frac{dx^1}{ds}\int T^{11}_{[rest]}d^3x_{[rest]} \tag{35}$$

Thus our new paradox will be resolved if and only if $\int T^{11}_{[rest]}d^3x_{[rest]}$ is equal to negative $p^0_{[rest]}$.

So we now calculate $p^0_{[rest]}$, and calculate $\int T^{11}_{[rest]}d^3x$.

As noted earlier, (27) is valid for any situation where the charge distribution is at rest. So

$$p^0_{[rest]} = \frac{1}{4}\pi \int (\frac{1}{2}(E_{[rest]})^2) d^3x_{[rest]} \tag{36}$$

We again will calculate $\int T^{11}_{[rest]}d^3x_{[rest]}$ from the
$\frac{1}{4}\pi \left(F^{1\sigma}F^1_\sigma - \frac{1}{4}\eta^{11}F^{\lambda\omega}F_{\lambda\omega}\right)$ formula. This again gives (28) leading to (29)

$$\int T^{11}_{[rest]} d^3x_{[rest]} =$$
$$\frac{1}{4}\pi \int \left((-(E^x_{[rest]})^2) + \frac{1}{4}(2(E_{[rest]})^2) \right) d^3x_{[rest]} \tag{37}$$

This time the electric field is totally in the x direction, so $(E^x_{[rest]})^2 = (E_{[rest]})^2$. Thus (37) becomes

$$\int T^{11}_{[rest]} d^3x_{[rest]} =$$
$$\frac{1}{4}\pi \int \left((-(E_{[rest]})^2) + \frac{1}{4}\left(2(E_{[rest]})^2\right) \right) d^3x_{[rest]} \tag{38}$$

$$\int T^{11}_{[rest]} d^3x_{[rest]} = \frac{1}{4}\pi(-\frac{1}{2}(E_{[rest]})^2) d^3x_{[rest]} \tag{39}$$

Comparing (39) to (36), we see that $\int T^{11}_{[rest]} d^3x_{[rest]}$ is indeed negative $p^0_{[rest]}$, exactly the result we had needed for the paradox to be resolved by use of the correct equation for situations where T^{11} is non-vanishing in the rest frame, (22). The paradox is cleanly and simply resolved by using the correct energy momentum relationship for situations with non-vanishing T^{ii} in the rest frame.

4.2.

Now let us consider a plane of charge– this time in the xy plane – moving in the x direction.

As noted previously, $p^0_{[rest]}$ is always that given by (27) . Thus

$$p^0_{[rest]} = \frac{1}{4}\pi \int \frac{1}{2}(E_{[rest]})^2 d^3x_{[rest]} \tag{40}$$

To calculate T^{10} we note that B has a magnitude of $\frac{dx^1}{ds}$ times $E_{[rest]}$, and is in the y direction, always completely perpendicular to E. Thus ExB has a magnitude of $\frac{dx^1}{ds}$ times $E_{[rest]}$ times E, and is in the x dir.

Therefore the Poynting vector is $\frac{1}{4}\pi\frac{dx^1}{ds}E_{[rest]}$ times E, and is in the x direction.

$$T^{10} = \frac{1}{4}\pi(\frac{dx^1}{ds})(E_{[rest]})E \tag{41}$$

$$p^1 = \frac{1}{4}\pi\int(\frac{dx^1}{ds})(E_{[rest]})Ed^3x. \tag{42}$$

The Lorentz transformation equation for the electric field is $E' = \gamma\,(E + \beta \times B)$. Letting E' be the electric field in the non-rest-frame, and letting E and B be the electric and magnetic fields in the rest frame, then this Lorentz transformation formula gives $E' = \gamma(E_{[rest]} + \beta \times B_{[rest]})$. Since $B_{[rest]} = 0$, the formula thus yields $E = \gamma E_{[rest]}$. Substituting this into (42) we get

$$p^1 = \frac{1}{4}\pi\int(\frac{dx^1}{ds})(E_{[rest]})(E_{[rest]}\gamma)d^3x. \tag{43}$$

Using (8) again, we get

$$p^1 = \frac{1}{4}\pi\int(\frac{dx^1}{ds})(E_{[rest]})(E_{[rest]})d^3x_{[rest]} \tag{44}$$

Comparing (44) to (40), we see that

$$p^1 = \frac{dx^i}{ds}2p^0_{[rest]} \tag{45}$$

From the perspective of (the inappropriate) (10) we see a discrepency from the (erroneously) expected $p^1 = \frac{dx^1}{ds}p^0_{[rest]}$ result. The discrepancy this time is a factor of two, analogous to the discrepency factor of $\frac{4}{3}$ for the point charge (and the factor of zero for the plane of charge moving in the perpendicular direction).

Let us see if we can resolve this new paradox by again using the appropriate (22) $p^1 = \frac{dx^1}{ds}p^0_{[rest]} + \frac{dx^1}{ds}\int T^{11}_{[rest]}d^3x_{[rest]}$ relationship, instead of the inappropriate (10) $p^i = \frac{dx^i}{ds}p^0_{[rest]}$ relationship that led to the supposed paradox. Substituting (45) into (22), we get

$$2\frac{dx^1}{ds}p^0_{[rest]} = \frac{dx^1}{ds}p^0_{[rest]} + \frac{dx^1}{ds}\int T^{11}_{[rest]}d^3x_{[rest]} \tag{46}$$

Thus our new paradox will be resolved if and only if $\int T^{11}_{[rest]}d^3x_{[rest]}$ is equal to $p^0_{[rest]}$.

To perform the test, we will need to calculate the quantitative values for p^0 and for $\int T^{11}_{[rest]} d^3x_{[rest]}$.

We already have $p^0_{[rest]}$ as

$$p^0_{[rest]} = \frac{1}{4}\pi \int \frac{1}{2}(E_{[rest]})^2 d^3x_{[rest]} \tag{47}$$

To get $\int T^{ii}_{[rest]} d^3x_{[rest]}$ we again use the $\frac{1}{4}\pi \left(F^{1\sigma}F^1_\sigma - \frac{1}{4}\eta^{11}F^{\lambda\omega}F_{\lambda\omega}\right)$ formula. This again gives (28) leading to (29):

$$\int T^{11}_{[rest]} d^3x_{[rest]} = \frac{1}{4}\pi \int \left(-(E^x{}_{[rest]})^2 + \frac{1}{4}(2(E_{[rest]})^2)\right) d^3x_{[rest]} \tag{48}$$

This time there is no electric field in the x direction, so $(E^x)^2 = 0$. Thus (48) becomes

$$\int T^{11}_{[rest]} = \frac{1}{4}\pi \left(-0 + \frac{1}{4}(2(E_{[rest]})^2)\right) \tag{49}$$

$$T^{11}_{[rest]} d^3x_{[rest]} = \frac{1}{4}\pi \int \frac{1}{2}(E_{[rest]})^2 d^3x_{[rest]} \tag{50}$$

Comparing (50) to (47), we see that $\int T^{11}_{[rest]} d^3x_{[rest]}$ is indeed equal to $p^0_{[rest]}$, exactly the result we had needed for the paradox to be resolved by use of the correct equation for situations where T^{11} is non-vanishing in the rest frame, (22). The paradox is cleanly and simply resolved by using the correct energy momentum relationship for situations with non-vanishing T^{ii} in the rest frame.

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