Whole Number Pythagorean Triplets

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Abstract

Pythagorean triples are sets of three whole numbers that satisfy the equation. This paper introduces two new formulas related to Pythagorean triples. The first formula determines the largest possible value for a given, ensuring that all elements remain whole numbers. The second formula explores the special case where, identifying recurrent patterns within Pythagorean triples.

Introduction

A Pythagorean triple consists of three positive integers that satisfy the Pythagorean theorem. Typically, researchers study the generation of these triples or their special properties. Here, we focus on two aspects:

The greatest range of "a" and "b" values

Instances where the $b - a = \pm 1$

Greatest range of a and b values

Largest "b" value for a given "a" and $\{a, b, c\}$ are whole number values

$$b = \frac{3a^2 - 6 - (a^2 + 2)(-1)^a}{8}$$

Where $a = shortest\ side\ in\ a\ right\ angled\ triangle$

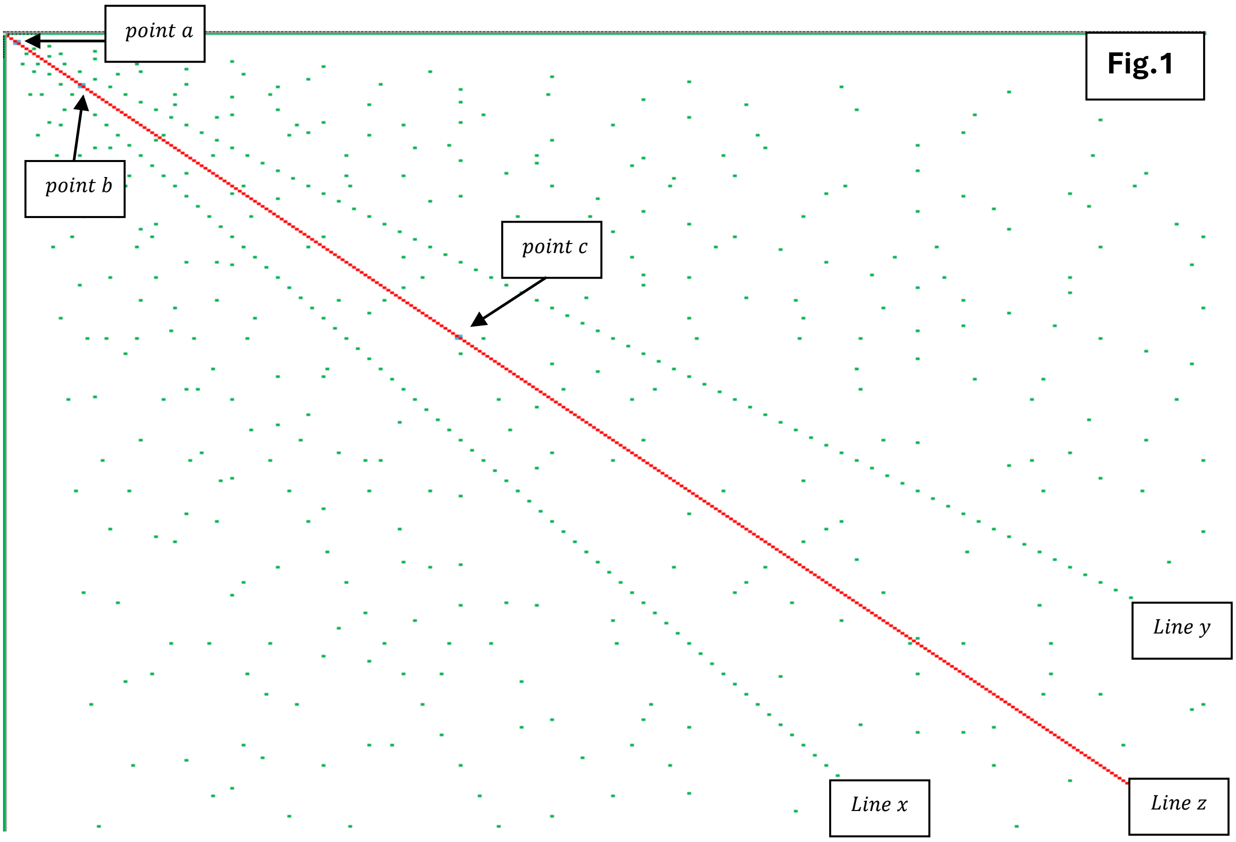
∴ Largest "c" value for a given "a" side is predetermined and $\{a, b, c\}$ are whole number values is as follows;

$$c = \sqrt{a^2 + \left(\frac{3a^2 - 6 - (a^2 + 2)(-1)^a}{8}\right)^2}$$

A value	B value	C value
3	4	5
4	3	5
5	12	13
6	8	10
7	24	25
8	15	17
9	40	41
10	24	26
11	60	61
12	35	37
13	84	85

By applying these formulas, we can determine the corresponding "b" value for any given "a" value. The sequence of "b" values exhibits an alternating pattern of increases and decreases, following a progression based on square numbers. Specifically, the term-to-term pattern follows the sequence: −1, +9, −4, +16, −9, +25, -16, and so forth.

Smallest range of a and b values



By implementing the Pythagorean theorem within a spreadsheet (as illustrated in Figure 1), where column A consists of integers from 1 to 2000 and row 1 follows the same range, three distinct patterns emerge. Lines X and Y correspond to all ratios of the 3:4:5 Pythagorean triple, while Line Z represents points where

$$\sqrt{a^2 + b^2} = \sqrt{2}$$

During this process, anomalies were observed when a−b=±1a - b = \pm 1a−b=±1. Notable examples include:

- Point A: the 3:4:5 triangle
- Point B: the 20:21:29 triangle
- Point C: the 119:120:169 triangle

The occurrence of these anomalies appears to follow Pell's equation, specifically for the x values:

$$x^2 - Dy^2 = 1$$

with initial integer solutions:

$$(3,2), (17,12), (99,70)$$

However, slight adaptations were required to focus on the x values. The frequency of these occurrences aligns with cases where $b - 1 = a$ and $\{a, b, c\}$ form a set of integer values.

Recursive definitions

Let x_n be a sequence defined by the recurrent relation;

$$x_n = 6x_{n-1} - x_{n-2} \qquad\qquad for\ n \geq 2$$

$where\ n_0 = 1$ and $n_1 = 3$

Let a_n be another sequence relating to one non-hypotenuse side of a triangle, defined recursively as;

$$a_n = a_{n-1} + x_n \qquad\qquad for\ n \geq 1$$

Where a_0 = 0

Let b_n be another sequence relating to the other non-hypotenuse side of a triangle, defined recursively as;

$$b_n = a_n + 1$$

Let c_n be the final sequence relating to the hypotenuse side of a triangle, defined recursively as;

$$c_n = \sqrt{{a_n}^2 + {b_n}^2}$$

n	x(n)	a	b	c
0	1	0	1	1
1	3	3	4	5
2	17	20	21	29
3	99	119	120	169
4	577	696	697	985
5	3363	4059	4060	5741
6	19601	23660	23661	33461
7	114243	137903	137904	195025
8	665857	803760	803761	1136689
9	3880899	4684659	4684660	6625109
10	22619537	27304196	27304197	38613965
11	131836323	159140519	159140520	225058681
12	768398401	927538920	927538921	1311738121
13	4478554083	5406093003	5406093004	7645370045
14	26102926097	31509019100	31509019101	44560482149

Asymptotic Behaviour of the Recursive Sequences

As the recursive sequences a_n, b_n and c_n grow larger, the relationship between the sides of the Pythagorean triples approaches a limiting behaviour. Specifically, the term $\sqrt{a_n^2 + b_n^2}$ becomes asymptotically equivalent to $\frac{\sqrt{2}(a_n+b_n}{2}$.

To illustrate this, let us consider the asymptotic behaviour of a_n and b_n for large n. As n increases, the difference between a_n and b_n becomes negligible, i.e.,$a_n^2 + b_n^2$. Therefore, for large n, we approximate:

$$a_n^2 + b_n^2 \approx 2a_n^2 + 2a_n + 1$$

Taking the square root of both sides:

$$\sqrt{a_n^2 + b_n^2} \approx \sqrt{2a_n^2} = a_n\sqrt{2}$$

At the same time, the expression to $\frac{\sqrt{2}(a_n+b_n)}{2}$ behaves as:

$$\frac{\sqrt{2}(a_n + b_n)}{2} \approx \sqrt{2a_n^2} = a_n\sqrt{2}$$

Thus, as $n \to \infty$, both expressions approach the same value, $a_n\sqrt{2}$, confirming that:

$$\lim_{n\to\infty} \sqrt{a_n^2 + b_n^2} = \frac{\sqrt{2}(a_n + b_n)}{2}$$

This result reveals an intriguing asymptotic relationship between the sides of the Pythagorean triples, highlighting that, for large values of n, the geometric growth of the triples follows a predictable pattern influenced by the factor $\sqrt{2}$.

Conclusion

This research introduces new formulas and recursive definitions for generating whole-number Pythagorean triples. By deriving relationships for the largest possible value of b given a, as well as exploring special cases where$b - a = \pm 1$, this paper highlights both the general patterns and anomalies observed in the sequences of Pythagorean triples. The study of these patterns through Pell's equation uncovers a deeper connection between the integers involved, offering new insights into their underlying structure.

Furthermore, the asymptotic behaviour of the recursive sequences a_n, b_n and c_n has been analysed, revealing that, as the sequences grow large, the terms $\sqrt{a_n^2 + b_n^2}$ and $\frac{\sqrt{n}(a_n+b_n)}{2}$ converge. This provides a fascinating relationship for large values of$n$, where the geometric growth of the triples tends to follow a predictable pattern, influenced by the constant $\sqrt{2}$.

Through this study, the connections between Pythagorean triples and Pell's equation, along with the asymptotic growth of the sequences, enhance our understanding of these classic number theory objects. Further exploration into the properties and behaviours of Pythagorean triples, including potential applications in other areas of mathematics, remains an exciting avenue for future research.

References

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