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A space-time equation for Electromagnetic Interaction.

Contents

1 Abstract

In the present work is explored the profound symmetry between the theories of gravity and electromagnetism, with particular attention to the similarity between the fundamental forces of nature: the weight force and the electric Lorentz force, Newton's law and Coulomb's law, as well as Maxwell's equations and the equations of gravitational electromagnetism. There are papers that explain how to obtain the gravitational electromagnetism equations by applying the weak field approximation to the Einstein field equations. Precisely in honour of this symmetry, this work aims to obtain Einstein field equations valid for

electromagnetism, i.e. field equations from which to derive, under weak field approximation, Maxwell's equations. This research suggests a parallel vision between electromagnetism and gravity, suggesting that perhaps there could be a curved electromagnetic field in 4 dimensions alongside the curved gravitational field (Einstein's space-time).

2 Introduction

The theories of gravity and electromagnetism represent two fundamental pillars of modern physics, but are often treated separately due to their apparent differences. However, there is a profound symmetry between these two forces that deserves more careful reflection. The gravitational force, described by Einstein's general relativity, and the electromagnetic force, described by Maxwell's equations, show notable similarities, both in their physical behavior and in the equations that govern them. In particular, the weight force and the electric Lorentz force are analogous in the mathematical formulation. Just as Newton's law for universal gravitation has a similar form to Coulomb's law for the electric force. Maxwell's equations and a particular formulation of Einstein's equations for the gravitational field also possess a symmetry that could prove useful for a more unified understanding of fundamental forces.
Despite the similarity between the two theories, gravity is usually treated as a geometric theory of space-time, while electromagnetism is described in terms of fields and particles. However, in recent decades, several theoretical approaches have suggested the possibility of "gravitoelectromagnetism", a model that aims to unify the two forces in a theoretical framework that respects the symmetry between gravitational and electromagnetic fields. Research in the field of gravitoelectromagnetism has led to the formulation of equations which, in certain approximations, could be reduced to Maxwell's equations for the electromagnetic case. These studies have mainly focused on the weak field approximation, where the gravitational effect is small enough to be treated as a perturbation.
In honor of this symmetry, this work aims to obtain Einstein field equations valid for electromagnetism. More precisely, the goal is to derive the field equations in a context that allows, under the weak field approximation, to obtain Maxwell's equations as a limit, maintaining coherence between the two theories. Our approach focuses on the symmetry between gravitational and electromagnetic fields, suggesting that there is a profound connection between the two phenomena. In particular, we propose the idea that there exists a curved electromagnetic field in four dimensions that coexists with the curved gravitational field described by general relativity. The charged material point with mass represents the junction point between the various fields.
This research not only sheds light on the analogies between gravity and electromagnetism, but also paves the way for possible future extensions of existing theories, offering a more complete picture of the fundamental forces of nature. The symmetry between these theories, if confirmed, could lead to new developments in understanding interactions at the cosmic and microscopic levels, as well as a more unified view of the physical laws that govern the universe.

3 Theoretical Model

The approach presented here is based on the symmetry between the equations describing gravity and the equations describing electromagnetism. Although formulated in different theoretical contexts, the two theories present a similar mathematical structure which suggests a possible unification. Both systems describe the interaction between bodies and forces first and between fields and matter then. However, while general relativity treats gravity as a curvature of space-time itself, electromagnetism is expressed through a vector field immersed in Euclidean space. In this sense, it seems that electromagnetism is described within a flat gravitational context.

3.1 Symmetry between Newton's Law and Coulomb's Law

Newton's law of universal gravitation:

$$\vec{F}_g = -G\frac{m_1 m_2}{r^2}\hat{r} \tag{1}$$

and Coulomb's law for the electrostatic force:

$$\vec{F}_e = -k\frac{q_1 q_2}{r^2}\hat{r} \tag{2}$$

they are similar in their formulation. In both cases, the force depends on the distance between two bodies and has an inverse relationship with the square of the distance. Furthermore, both depend on the intrinsic properties of objects: mass due to gravitation and charge due to electromagnetism, and manifest themselves as interactions at a distance.

3.2 Symmetry between the Weight Force and the Electric Lorentz Force

The weight force and the electric Lorentz force show obvious similarities. The weight force in a gravitational field is expressed as:

$$\vec{F}_g = m\vec{g} \tag{3}$$

where m is the mass of the object and g is the acceleration due to gravity. The electric Lorentz force, however, is given by:

$$\vec{F}_e = q\vec{E} \tag{4}$$

where q is the electric charge of the object and E is the electric field. In both cases, the force acts on objects with specific properties (mass or charge) and depends on a field (gravitational or electric) that influences the matter.

3.3 Symmetry between Gravitoelectromagnetism (GEM) and Maxwell Equations

The gravitoelectromagnetism (GEM) equations have a similar structure to Maxwell's equations for electromagnetism. The gravitational acceleration described by

Newton's law can be seen as parallel to the electric field and the gravitomagnetic field, generated by a moving mass, can be considered the analogue of the magnetic field. In this framework, the gravitoelectric field (gravitational acceleration) and the gravitomagnetic field can be treated as a single entity that acts on particles of mass, while the electromagnetic field acts on charged particles. In other words, GEMs represent a "gravitational version" of Maxwell's equations, with similar properties of propagation and interaction between fields. This suggests that gravitoelectromagnetism could represent a more general manifestation of a gravitational behavior that is also found in electromagnetism. Furthermore, the presence of Einstein's field equations suggests that there could be a more general description of electromagnetism than the classic Maxwell equations.

3.4 Obtain Maxwell's equations from Einstein's equations in the weak field approximation

On the basis of the analogies illustrated above, the hypothesis is posed that Maxwell's equations can be obtained from a modified Einstein equation, similarly to how the GEM equations are obtained in the weak field limit.
The idea is to treat the electromagnetic field as a weak field perturbation of a suitably modified Einstein field equation. This would represent a limiting case in which the intensity of the electric charges or currents is sufficiently weak not to significantly perturb the "electromagnetic space-time". In other words, under the weak field approximation, the modifications to Einstein's equations could reduce to Maxwell's equations for the electromagnetic field.
This approach leads us to consider a "curved electromagnetic field" that coexists with the curved gravitational field described by general relativity, suggesting a unified vision of the fundamental forces that owes gravity and electromagnetism as two distinct but interrelated fields.

3.5 Derivation of the GEM Equations from Einstein's Equations

Gravitoelectromagnetism (GEM) is a theory that describes the analogy between the gravitational and electromagnetic fields. Its derivation from Einstein's field equations is based on the weak field approximation, in which the mass and energy density is low enough not to significantly deform space-time. This chapter will explore how, starting from Einstein's field equations, we can obtain the equations of gravitoelectromagnetism, a model which, under certain conditions, is similar to Maxwell's equations for the electromagnetic field.

3.6 Einstein's Equations for the Gravitational Field

Einstein's equations in vacuum are expressed as:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{5}$$

where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the metric tensor and R is the scalar curvature. In the presence of matter, this system is completed with the term of the energy-momentum tensor $T_{\mu\nu}$, which describes the distribution of matter and energy:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{6}$$

For our purpose, we assume that the gravitational field is weak and that the matter is distributed in such a way that the solution can be treated as a perturbation of flat space-time. In this situation, one can use a perturbed metric around the Minkowski metric, expressing the metric as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{7}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ is a small perturbation such that $|h_{\mu\nu}| \ll 1$.

3.7 The Linear Approach and the Perturbation of the Gravitational Field

In the context of gravitoelectromagnetism, it can be assumed that the terms of the energy-momentum tensor are all negligible except for the term T_{00}, which represents the energy density, and the term T_{0i}, which represents the mass flow:

$$T_{00} \simeq \rho c^2 \tag{8}$$

$$T_{0i} \simeq j^i v \tag{9}$$

Having reached this point we can introduce the gravitational quadripotential in the following way:

$$\phi = -\frac{c^2}{4} h_{00} \tag{10}$$

$$A_i = \frac{c^2}{2} h_{0i} \tag{11}$$

And, by doing so, the gravitoelectric field and the gravitomagnetic field can be obtained through the classic definition based on the quadripotential.
In the weak field regime, the curvature of spacetime is small, and therefore the perturbation $h_{\mu\nu}$ is also small. Developing Einstein's equations in terms of this perturbation, we obtain a linear system which can be written as:

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} \tag{12}$$

With $\Box = \partial^\mu \partial_\mu$ d'Alembertian operator and $\bar{h}_{\mu\nu} = h_{\mu\nu} - 1/2\eta_{\mu\nu} h$ is the centered perturbation.
Under this approximation, Einstein's equations for the gravitational field reduce to a form similar to Maxwell's equations for the electromagnetic field. These can be written as:

$$\begin{array}{c|c} \nabla \cdot E = 4\pi G \rho & \nabla \cdot B = 0 \\ \hline \nabla \times E = -\frac{\partial B}{\partial t} & \nabla \times B = \frac{4\pi G}{c^2} j + \frac{1}{c^2}\frac{\partial E}{\partial t} \end{array} \tag{13}$$

3.8 Derivation of Einstein's Equations for Electromagnetism

The profound symmetry with Maxwell's laws valid for electromagnetism can be appreciated in the table below [**GEM1**][**GEM2**].

GEM	$Maxwell's$	GEM	$Maxwell's$
$\nabla \cdot E = 4\pi G\rho$	$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$	$\nabla \cdot B = 0$	$\nabla \cdot B = 0$
$\nabla \times E = -\frac{\partial B}{\partial t}$	$\nabla \times E = -\frac{\partial B}{\partial t}$	$\nabla \times B = \frac{4\pi G}{c^2} j + \frac{1}{c^2}\frac{\partial E}{\partial t}$	$\nabla \times B = \mu_0 j + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$

(14)

From this table it can be deduced that there is a parallelism between the constants in the two sets of equations:

$$4\pi G \rightarrow \frac{1}{\varepsilon_0} \tag{15}$$

This correspondence would be respected by substituting $4\pi k$ in place of $4\pi G$, knowing that $4\pi k = \frac{4\pi}{4\pi\varepsilon_0} = \frac{1}{\varepsilon_0}$.

$$\frac{4\pi G}{c^2} \rightarrow \mu_0 \tag{16}$$

This corrispondence would be respected by substituting $\frac{4\pi k}{c^2}$ in place of $\frac{4\pi G}{c^2}$.

$$\frac{1}{c^2} \rightarrow \mu_0 \varepsilon_0 \tag{17}$$

Which leads to deducing the correct version of Einstein's field equation in the form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{4\pi k}{c^4} T_{\mu\nu} \tag{18}$$

4 Equations of Gravitoelectromagnetism in a Curved Electromagnetic Space-Time

In general relativity, Maxwell's electromagnetism is formulated in curved space-time, with the equations of the theory expressible in terms of the covariant derivative:

$$F^{\alpha\beta}_{\ \ ;\beta} = 4\pi J^\alpha \tag{19}$$

where $F^{\alpha\beta}$ is the electromagnetic tensor, J^α is the four-vector current density and the semicolon indicates the covariant derivative with respect to the gravitational metric $g_{\mu\nu}$. This formalism allows us to describe the behavior of the electromagnetic field in the presence of a gravitational curvature.
Following a principle of symmetry, we can derive an analogous equation that describes the behavior of the gravitational field, formulated in the language of gravitoelectromagnetism (GEM), in a curved space-time due to the effect of the electromagnetic field. Let us then consider the GEM tensor $G^{\alpha\beta}$, built in analogy with $F^{\alpha\beta}$, and we impose that it satisfies an equation of the type

$$G^{\alpha\beta}_{\ \ ;\beta} = 4\pi \mathcal{J}^\alpha \tag{20}$$

where $\mathcal{J}^\alpha$ is the gravitational fourcurrent, i.e. the source of the GEM field, which can be related to the mass and momentum distribution.

To determine the explicit form of this equation, we proceed by analogy with electromagnetism in general relativity. In Maxwell's case, the field tensor is defined in terms of the four-potential A^μ:

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \tag{21}$$

Similarly, we can define the tensor GEM $G^{\mu\nu}$ as a function of a gravitational quadripotential Φ^μ:

$$G^{\alpha\beta} = \partial^\alpha \Phi^\beta - \partial^\beta \Phi^\alpha \tag{22}$$

Where the potential four-vector Φ^μ it is constructed based on the components of the metric perturbation $h_{\mu\nu}$ in the weak field regime:

$$\Phi_0 = -\frac{c^2}{4} h_{00}, \quad A_i = \frac{c^2}{2} h_{0i} \tag{23}$$

Starting from this definition, we can write the equations of the GEM field in a curved space-time as

$$G^{\alpha\beta}_{\ \ ;\beta} = 4\pi \mathcal{J}^\alpha \tag{24}$$

where now the covariant derivative is calculated with respect to the affine connection determined by the curved electromagnetic field.
This equation implies that the presence of an electromagnetic space-time can modify the propagation of the GEM field, just as gravitational curvature modifies the behavior of electromagnetism. The interpretation is that the geometry of space-time can be influenced not only by the energy-momentum tensor of matter, but also by the electromagnetic field itself, leading to a natural extension of the equations of general relativity.

5 Unit of Measurement and Analogy between Gravity and Electromagnetism

We feel the need to say a few more words regarding the aspect of the units of measurement in the various laws studied so far. In fact, it can be noted that there is a constant with units $\frac{kg}{C}$ that returns repeatedly in the various equations.
Going into detail, a comparison between the gravitational force and the Coulomb force highlights a similar formal structure:

- Gravitational force (according to Newton):

$$F_g = G\frac{m_1 m_2}{r^2} \tag{25}$$

  Coulomb force (electricity):

$$F_e = k\frac{q_1 q_2}{r^2} \tag{26}$$

  Both forces consist of a constant multiplied by the product of two specific quantities (mass for gravity and charges for electromagnetism), together

with a field that depends only on the source.
The gravitational field is defined as:

$$\begin{array}{c} g = \frac{F_g}{m} = G\frac{M}{r^2} \\ [g] = \frac{\text{N}}{\text{kg}} = \frac{\text{m}}{\text{s}^2} \end{array} \tag{27}$$

The electric field is defined as:

$$\begin{array}{c} E = \frac{F_e}{q} = k\frac{Q}{r^2} \\ [E] = \frac{\text{N}}{\text{C}} = \frac{\text{kg}}{\text{C}} \cdot \frac{\text{m}}{\text{s}^2} \end{array} \tag{28}$$

We note that and have the same unit of measurement except for a conversion factor between mass and charge:

$$\frac{\text{kg}}{\text{C}} \tag{29}$$

This suggests that, if a physical constant with units of measurement were introduced, one field could be directly transformed into another, emphasizing the analogy between gravity and electromagnetism.

- The same analogy is also found in the case of Maxwell's equations and GEM equations. For simplicity, the terms on the right of the equal will be compared in their relativistic tensor formulation. In the case of Maxwell's equations we obtain the term

$$\frac{\rho}{\epsilon_0} \rightarrow \frac{\text{C}}{\text{m}^3} \cdot \frac{\text{Nm}^2}{\text{C}^2} = \frac{\text{kg}}{\text{s}^2\text{C}} \tag{30}$$

While in the case of GEM equations we obtain:

$$4\pi G J^\mu \rightarrow \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2} \frac{\text{kg}}{\text{m}^3} = \text{s}^{-2} \tag{31}$$

Also in this case we note how the $\frac{\text{kg}}{\text{C}}$ factor differentiates the two equations.
Similarly, the equations for electromagnetic space-time that are proposed here will have the dimensions $\frac{\text{kg}}{\text{m}^2\text{C}}$, while the Einstein tensor has the dimensions m^{-2}.

6 Cases of charged masses

As previously discussed, the electromagnetic interaction can be formalized with a four-dimensional spacetime that curves in response to electric charge density and current.
This concept leads us to say that there are two overlapping four-dimensional space-times, one gravitational and one electromagnetic, which interact with each other. In fact, in the case in which the sources are both objects with mass and electric charge, the gravitational and electromagnetic field equations must be considered simultaneously, giving rise to the following system:

$$\begin{cases} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \\ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{4\pi k}{c^4}T_{\mu\nu} \end{cases} \tag{32}$$

Which can become

$$2 \cdot G_{\mu\nu} = \frac{8\pi G}{c^4} T_{g\mu\nu} + \frac{4\pi k}{c^4} T_{e\mu\nu} \tag{33}$$

The factor 2 in front of the Einstein tensor scales the intensity of the curvature of space-time, meaning that, in the presence of charged mass, the intensity of the curvature doubles.
It is essential to underline that the sources of the two fields are represented by the same matter. This imposes an additional constraint on the two equations, tying the geometry of one spacetime to the geometry of the other. The sources of the two fields cannot be separated, but must coincide at the same point in space-time, otherwise the physical behavior would be inconsistent.
In particular, in the case of objects that possess both mass and charge, it is crucial that the two descriptions, gravitational and electromagnetic, are consistent with each other. This means that the sources of the two fields – the mass for the gravitational field and the charge for the electromagnetic field – must be at the same point in both space and time. This constraint is essential to maintain system consistency and avoid discrepancies in field models. In other words, there are no "separate sources" for the two phenomena: the mass and the charge must be located at the same point in space-time, otherwise the equations are not compatible.
Taking into consideration concrete examples, such as the case of a charged black hole, one can observe how the presence of electric charges in gravitating objects directly influences the field equations. In this scenario, the black hole not only curves space-time through general relativity, but also generates an electric field that interacts with the gravitational field, modifying the surrounding dynamics. This requires a synthesis between the two theories, so that the geometry of space-time can simultaneously reflect the effect of mass (in the case of the gravitational field) and charge (in the case of the electromagnetic field). The combination of the two equations therefore provides a coherent description of the interaction between gravity and electromagnetism, where both sources influence the same space-time in an intertwined manner.

7 Possible experimental checks

To explore the interaction between the gravitational and electromagnetic fields, there are some particularly promising physical contexts. Charged black holes, such as those described by the Reissner-Nordström solution, offer an ideal framework for studying how the curvature of space-time and the electric field generated by the charge interact with each other. Similar situations also occur in charged neutron stars, where the density of matter and the presence of a residual charge could influence the radiation emitted and the gravitational dynamics. Another interesting context is that of relativistic plasma in the presence of strong magnetic fields, such as in the regions surrounding black holes or in magnetic pulsars, where the interaction between the electromagnetic field and the curvature of space-time could modify the propagation of the radiation. Finally, gravitational and electromagnetic oscillations in binary systems of compact objects, such as black holes or neutron stars, could reveal the combined effects of gravity and electromagnetism through the distortion of the emitted waves. Each of these environments offers potentially revealing clues to better understand the interactions between the two fundamental fields.

References

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[GEM2] B. Mashhoon, F. Gronwald, H.I.M. Lichtenegger (1999), *Gravitomagnetism and the Clock Effect*, Arxiv.