Proof of the Collatz conjecture

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ABSTRACT

The Collatz conjecture, also known as the Syracuse conjecture or the $3x + 1$ problem, is a mathematical conjecture according to which the Collatz sequence always reaches the value 1, and then repeats the cycle $(1, 4, 2)$ indefinitely, regardless of the first term of the sequence as long as it is a strictly positive integer. It originated in the 1930s and its authors are mainly Lothar Collatz and Helmut Hasse. The latter shared it in the United States during a visit to Syracuse University, and the Collatz sequence then became known as the Syracuse sequence. To date, this conjecture has not been proven either true or false.

The purpose of this study is to prove, as clearly and precisely as possible, that this conjecture is true. The proof is based on classical mathematics which should not pose any major difficulties.

Contents

1 Introduction

First, we are going to define the Collatz sequence. Let the sequence be $\{u_n\}_{n\in\mathbb{N}}$ such that $u_0 = p$, where $p\in\mathbb{N}^*$, and such that:

$$\text{for } n\in\mathbb{N}^*, u_n = \begin{cases} 3u_{n-1}+1 & \text{if } u_{n-1} \text{ is odd,} \\ \frac{u_{n-1}}{2} & \text{if } u_{n-1} \text{ is even.} \end{cases}$$

Then, according to the conjecture, there exists $l\in\mathbb{N}^*$ such that $u_l = 1, u_{l+1} = 4, u_{l+2} = 2, u_{l+3} = 1, u_{l+4} = 4, u_{l+5} = 2$, *etc.* In other words, from rank l the sequence enters a cycle that repeats the numbers $1, 4, 2$ *ad infinitum.* We can express this sequence in another way, indeed, if p is odd then we have:

$$u_1 = 3p + 1 \text{ is even,}$$

$$u_2 = \frac{3p+1}{2^1}, \text{ if } u_2 \text{ is even then } u_3 = \frac{3p+1}{2^2},$$

$$\ldots,$$

$$\text{until } u_{1+\alpha_0} = \frac{3p+1}{2^{\alpha_0}} \text{ is odd.}$$

Let :

$$v_0 = u_{1+\alpha_0} = \frac{3p+1}{2^{\alpha_0}}$$

Where α_0 is the exponent corresponding to the number of times u_1 must be divided by 2 to obtain an odd number. Repeating the same process, we have:

$$u_{1+\alpha_0+1} = 3u_{1+\alpha_0} + 1 \text{ is even,}$$

$$u_{1+\alpha_0+1+1} = \frac{3u_{1+\alpha_0}+1}{2^1}, \text{if } u_{1+\alpha_0+1+1} \text{ is even then } u_{1+\alpha_0+1+2} = \frac{3u_{1+\alpha_0}+1}{2^2},$$

$$\ldots,$$

$$\text{until } u_{1+\alpha_0+1+\alpha_1} = \frac{3u_{1+\alpha_0}+1}{2^{\alpha_1}} \text{ is odd.}$$

Let:

$$v_1 = u_{1+\alpha_0+1+\alpha_1} = \frac{3u_{1+\alpha_0}+1}{2^{\alpha_1}}$$

Where α_1 is the exponent corresponding to the number of times $u_{1+\alpha_0+1}$ must be divided by 2 to obtain an odd number. By reformulating v_1, we have:

$$v_1 = \frac{\left(3\left(\frac{3p+1}{2^{\alpha_0}}\right)+1\right)}{2^{\alpha_1}} = \frac{3\,(3p+1)+2^{\alpha_0}}{2^{\alpha_0+\alpha_1}}$$

And by an easily verifiable recurrence (see Appendix 8.1), we obtain that for all $l \in \mathbb{N}^*$:

$$v_l = \frac{3^l\,(3p+1) + \sum_{i=0}^{l-1}(3^{l-1-i}(2^{\sum_{j=0}^{i}\alpha_j}))}{2^{\sum_{i=0}^{l}\alpha_i}} = \frac{3^l\,(3p+1)}{2^{\sum_{i=0}^{l}\alpha_i}} + \sum_{i=0}^{l-1}\frac{3^{l-1-i}}{2^{\sum_{j=i+1}^{l}\alpha_j}}$$

The resulting sequence $\{v_l\}_{l\in\mathbb{N}}$ has all its values in $2\mathbb{N}+1$, and it is therefore equal to the sequence $\{u_n\}_{n\in\mathbb{N}}$ without the even-valued terms of the latter. Thus, the cycle of length 3 and values $(1,4,2)$ of the sequence $\{u_n\}$ corresponds to the cycle of length 1 and value (1) of the sequence $\{v_l\}$. If p is even, there exists $\alpha\in\mathbb{N}^*$ such that $p = 2^\alpha q$, where $q\in 2\mathbb{N}+1$, and it suffices to replace p with q in the expression of the term v_0, which does not change the demonstration.

Definition 1.1. *Let $f = 2^\alpha p$, where $\alpha\in\mathbb{N}$ and $p\in 2\mathbb{N}+1$, the sequence $\{v_l\}_{l\in\mathbb{N}}$, the so-called reformulated Collatz sequence, is defined for all $l\in\mathbb{N}$ as follows:*

$$v_l = \begin{cases} \frac{3p+1}{2^{\alpha_0}} & \text{if } l = 0, \\ \frac{3^l(3p+1)}{2^{\sum_{i=0}^{l}\alpha_i}} + \sum_{i=0}^{l-1}\frac{3^{l-1-i}}{2^{\sum_{j=i+1}^{l}\alpha_j}} & \text{if } l > 0. \end{cases}$$

Where α_0 is the exponent such that v_0 is odd and the α_i, for $i \in \{1,\ldots,l\}$, are the exponents such that $v_i = \frac{3v_{i-1}+1}{2^{\alpha_i}}$ is odd.

Let $A_l = \frac{3^l(3p+1)}{2^{\sum_{i=0}^{l}\alpha_i}}$ and $B_l = \sum_{i=0}^{l-1}\frac{3^i}{2^{\sum_{j=l-i}^{l}\alpha_j}}$, then for $l > 0$, $v_l = A_l + B_l$.

2 Sums of the exponents

The aim here is to study the sums of the exponents $\sum_{i=r}^{l}\alpha_i$ of the terms A_l and B_l, where $r \in [0,l]$, from a probabilistic point of view, in order to establish conjectures that will be demonstrated later. Let $\gamma\in\mathbb{R}_+^*$ such that $2^\gamma q = 3q+1$, where q is odd, then $\gamma = \frac{\ln\left(3+\frac{1}{q}\right)}{\ln(2)}$ and converges decreasingly to $\beta = \frac{\ln(3)}{\ln(2)} \cong 1,58496$ when q tends to infinity. Similarly, for each term in the sequence $\{v_l\}_{l\in\mathbb{N}^*}$, we have:

$$2^{\gamma_l} v_{l-1} = 3v_{l-1} + 1 \implies \gamma_l = \frac{\ln\left(3+\frac{1}{v_{l-1}}\right)}{\ln(2)}$$

This means that multiplying the term v_{l-1} by 3 and then adding 1 is equivalent to multiplying it by 2^{γ_l}. Therefore, to the extent that for any $i\in[0,l]$ the exponent α_i is a non-zero natural number and that γ_i is always closer to 2 than to 1, the probability that $\alpha_i \geq 2$, such that the term :

$$v_i = \begin{cases} \frac{3p+1}{2^{\alpha_0}} & \text{if } i = 0, \\ \frac{3v_{i-1}+1}{2^{\alpha_i}} & \text{if } i > 0. \end{cases}$$

is odd, is significantly greater than the probability that $\alpha_i = 1$ (see Appendix 8.2). Consequently, we can formulate the following conjecture:

Conjecture 2.1. *There exists a rank L^p of the sequence $\{v_l\}$ such that for any $l \geq L^p$:*

$$\beta(l+1) < \sum_{i=0}^{l} \alpha_i \iff \frac{1}{\sum_{i=0}^{l} \alpha_i} < \frac{1}{\beta(l+1)}$$

Where $\beta = \frac{\ln(3)}{\ln(2)}$ and L^p means that L depends on p (see Definition 1.1).

The irrational nature of $\beta(l+1)$ (see Appendix 8.3) and the fact that $\sum_{i=0}^{l} \alpha_i$ is a sum of strictly positive integers ensure that $\beta(l+1)$ cannot be equal to $\sum_{i=0}^{l} \alpha_i$. This justifies, if the conjecture is true, that:

$$\beta(l+1) < \sum_{i=0}^{l} \alpha_i$$

Just as for $i \in [0, l]$, for $i \in [r, l]$, the probability that $\alpha_i \geq 2$, such that the term v_i mentioned above is odd, is also significantly greater than the probability that $\alpha_i = 1$, we can therefore formulate another conjecture concerning the partial sums of the exponents of the term B_l:

Conjecture 2.2. *There exists a rank $L^p > 0$ of the sequence $\{v_l\}$ such that for any $l \geq L^p$:*

$$\sum_{i=0}^{l-1} \frac{1}{\sum_{j=l-i}^{l} \alpha_j} < \sum_{i=0}^{l-1} \frac{1}{\beta(i+1)}$$

Where $\beta = \frac{\ln(3)}{\ln(2)}$ and L^p means that L depends on p (see Definition 1.1).

The proof of these conjectures will be given in Section 4.

3 Behavior of the sequence

Considering Conjectures 2.1 and 2.2 as true, we are going to demonstrate that regardless of the value of p, the sequence $\{v_l\}$ is always upper bounded.

Theorem 3.1. *For all $p \in 2\mathbb{N}+1$, such that $v_0 = \frac{3p+1}{2^{\alpha_0}}$, there exists a rank ω^p of the sequence $\{v_l\}$ such that for any $l \in \mathbb{N}$, $v_l \leq v_{\omega^p}$.*

Proof. According to Definition 1.1, for all $l > 0$:

$$v_l = \frac{3^l(3p+1)}{2^{\sum_{i=0}^{l} \alpha_i}} + \sum_{i=0}^{l-1} \frac{3^{l-1-i}}{2^{\sum_{j=i+1}^{l} \alpha_j}} = \frac{3^l(3p+1)}{2^{\sum_{i=0}^{l} \alpha_i}} + \sum_{i=0}^{l-1} \frac{3^i}{2^{\sum_{j=l-i}^{l} \alpha_j}}.$$

Note that we refer in this section to the terms $A_l = \frac{3^l(3p+1)}{2^{\sum_{i=0}^{l} \alpha_i}}$ and $B_l = \sum_{i=0}^{l-1} \frac{3^i}{2^{\sum_{j=l-i}^{l} \alpha_j}}$ defined in section 1, and let us remind that p, which appears as the exponent of a variable, means that this variable depends on p (see Definition 1.1).

Conjecture 2.1 says that there exists $L_0^p \in \mathbb{N}$ such that for any $l \geq L_0^p$:

$$\frac{1}{\sum_{i=0}^{l} \alpha_i} < \frac{1}{\beta(l+1)} \iff \frac{1}{2^{\sum_{i=0}^{l} \alpha_i}} < \frac{1}{2^{\beta(l+1)}} \iff A_l < \frac{3^l(3p+1)}{2^{\beta(l+1)}}$$

And according to Conjecture 2.2, there exists $L_1^p \in \mathbb{N}$ such that for any $l \geq L_1^p$:

$$\sum_{i=0}^{l-1} \frac{1}{\sum_{j=l-i}^{l} \alpha_j} < \sum_{i=0}^{l-1} \frac{1}{\beta(i+1)} \iff \sum_{i=0}^{l-1} \frac{1}{2^{\sum_{j=l-i}^{l} \alpha_j}} < \sum_{i=0}^{l-1} \frac{1}{2^{\beta(i+1)}} \iff B_l < \sum_{i=0}^{l-1} \frac{3^i}{2^{\beta(i+1)}}$$

As a result, for all $l \geq (L^p = \max\{L_0^p, L_1^p\})$:

$$A_l + B_l < \frac{3^l(3p+1)}{2^{\beta(l+1)}} + \sum_{i=0}^{l-1} \frac{3^i}{2^{\beta(i+1)}}$$

On the other hand, given that there is an infinity of real numbers between $\beta(l+1)$ and $\sum_{i=0}^{l} \alpha_i$, as well as between $\beta(i+1)$ and $\sum_{j=l-i}^{l} \alpha_j$, for $i \in \{0, \ldots, l-1\}$, and because of the facts related to the sequence $\{v_l\}$, there exists $\beta^p > \beta$ such that for all $l \geq L^p$:

$$v^{\beta^p} = \frac{3^l(3p+1)}{2^{\beta^p(l+1)}} + \sum_{i=0}^{l-1} \frac{3^i}{2^{\beta^p(i+1)}} < \frac{3^l(3p+1)}{2^{\beta(l+1)}} + \sum_{i=0}^{l-1} \frac{3^i}{2^{\beta(i+1)}} = v^{\beta}$$

And such that:

$$v_l \leq v^{\beta^p}$$

Then for all $l \geq L^p$ we have (see Appendix 8.4 on the existence of β^p):

$$v_l \leq v_l^{\beta^p} < v_l^{\beta}$$

Where the sequences $\{v_l^{\beta}\}_{l \in \mathbb{N}^*}$ and $\{v_l^{\beta^p}\}_{l \in \mathbb{N}^*}$ are defined as follows:

$$v_l^{\beta} = \frac{3^l(3p+1)}{2^{\beta(l+1)}} + \sum_{i=0}^{l-1} \frac{3^i}{2^{\beta(i+1)}}$$

$$v_l^{\beta^p} = \frac{3^l(3p+1)}{2^{\beta^p(l+1)}} + \sum_{i=0}^{l-1} \frac{3^i}{2^{\beta^p(i+1)}}$$

Let $A_l^{\beta^p} = \frac{3^l(3p+1)}{2^{\beta^p(l+1)}}$ and $B_l^{\beta^p} = \sum_{i=0}^{l-1} \frac{3^i}{2^{\beta^p(i+1)}}$, and let's study the evolution of these two terms when $l \to +\infty$.

The term $A_l^{\beta^p}$

This term is a geometric sequence with a common ratio of $\frac{3}{2^{\beta^p}}$:

$$A_l^{\beta^p} = \frac{3^l(3p+1)}{2^{\beta^p(l+1)}} = \frac{3p+1}{2^{\beta^p}} \left(\frac{3}{2^{\beta^p}}\right)^l$$

Considering that $\frac{3}{2^{\beta^p}} < 1$, since $\beta^p > \frac{\ln(3)}{\ln(2)}$, $A_l^{\beta^p}$ converges to 0 when $l \to +\infty$.

The term $B_l^{\beta^p}$
The term $B_l^{\beta^p} = \sum_{i=0}^{l-1} \frac{3^i}{2^{\beta^p(i+1)}}$ is a geometric series:

$$\sum_{i=0}^{l-1} \frac{3^i}{2^{\beta^p(i+1)}} = \frac{1}{2^{\beta^p}} \sum_{i=0}^{l-1} \left(\frac{3}{2^{\beta^p}}\right)^i = \frac{1}{2^{\beta^p}} \cdot \frac{\left(\frac{3}{2^{\beta^p}}\right)^l - 1}{\frac{3}{2^{\beta^p}} - 1}$$

As before, considering that $\frac{3}{2^{\beta^p}} < 1$ and $\beta^p > \frac{\ln(3)}{\ln(2)}$, when $l \to +\infty$:

$$B_l^{\beta^p} = \frac{1}{2^{\beta^p} - 3} < +\infty$$

Hence:

$$\lim_{l\to+\infty} \left(v_l^{\beta^p}\right) = \lim_{l\to+\infty} \left(A_l^{\beta^p}\right) + \lim_{l\to+\infty} \left(B_l^{\beta^p}\right) < +\infty$$

Finally, since the sequence $\{v_l\}$ takes values in $2\mathbb{N}+1$, there exists $M^p \in 2\mathbb{N}+1$ such that:

$$\lim_{l\to+\infty} (v_l) \leq \lim_{l\to+\infty} \left(v_l^{\beta^p}\right) \leq M^p$$

Insofar as the values in the sequence $\{v_l\}$ are natural numbers, from a certain rank, all its terms will be less than or equal to M^p. This implies that there exists $\omega^p \in \mathbb{N}$, such that for any $l \in \mathbb{N}$, $v_l \leq v_{\omega^p}$, which is equivalent to saying that the values taken by the sequence are in a finite subset of $\mathbb{N}$. Consequently, assuming that Conjectures 2.1 and 2.1 are true, the sequence $\{v_l\}$ is always upper bounded. □

4 Proof of Conjectures 2.1 and 2.2

To establish that Conjectures 2.1 and 2.2 are true, we will study the hypothetical cases where one of the conjectures is false and where both conjectures are false. As in Section 3, $A_l = \frac{3^l(3p+1)}{2^{\sum_{i=0}^{l}\alpha_i}}$ and $B_l = \sum_{i=0}^{l-1} \frac{3^i}{2^{\sum_{j=l-i}^{l}\alpha_j}}$.

First case: Conjecture 2.1 is true and Conjecture 2.2 is false
If Conjecture 2.1 is true, then there exists $L_0 \in \mathbb{N}$ such that for any $l \geq L_0$:

$$\frac{1}{\sum_{i=0}^{l}\alpha_i} < \frac{1}{\beta(l+1)} \iff A_l < \frac{3^l(3p+1)}{2^{\beta(l+1)}}.$$

Given $\frac{3}{2^\beta} = 1$ and A_l is always positive, we have:

$$0 \leq \lim_{l \to +\infty} (A_l) \leq \lim_{l \to +\infty} \left(\frac{3^l(3p+1)}{2^{\beta(l+1)}} \right) = \frac{3p+1}{3}$$

If Conjecture 2.2 is false, then for any $L_1 \in \mathbb{N}$ there exists $l \geq L_1$ such that:

$$\sum_{i=0}^{l-1} \frac{1}{\sum_{j=l-i}^{l} \alpha_j} \geq \sum_{i=0}^{l-1} \frac{1}{\beta(i+1)} \iff B_l \geq \sum_{i=0}^{l-1} \frac{3^i}{2^{\beta(i+1)}}.$$

Considering that $\frac{3}{2^\beta} = 1$ and that we can take L_1 as large as desired, when $l \to +\infty$, the term:

$$\sum_{i=0}^{l-1} \frac{3^i}{2^{\beta(i+1)}} = \frac{1}{2^\beta} \sum_{i=0}^{l-1} \left(\frac{3}{2^\beta} \right)^i = \frac{1}{2^\beta} l$$

tends toward infinity. Hence:

$$\lim_{l \to +\infty} (v_l) = \lim_{l \to +\infty} (A_l) + \lim_{l \to +\infty} (B_l) = +\infty$$

If this case were proven the sequence $\{v_l\}$ would diverge.

Second case: Conjecture 2.1 is false and Conjecture 2.2 is true

If Conjecture 2.1 is false, then for any $L_0 \in \mathbb{N}$, there exists $l \geq L_0$ such that:

$$\frac{1}{\sum_{i=0}^{l} \alpha_i} \geq \frac{1}{\beta(l+1)} \iff A_l \geq \frac{3^l(3p+1)}{2^{\beta(l+1)}}$$

And we have:

$$\lim_{l \to +\infty} (A_l) \geq \lim_{l \to +\infty} \left(\frac{3^l(3p+1)}{2^{\beta(l+1)}} \right) = \frac{3p+1}{3}$$

If Conjecture 2.2 is true, there exists $L_1 \in \mathbb{N}$ such that for any $l \geq L_1$:

$$\sum_{i=0}^{l-1} \frac{1}{\sum_{j=l-i}^{l} \alpha_j} < \sum_{i=0}^{l-1} \frac{1}{\beta(i+1)} \iff B_l < \sum_{i=0}^{l-1} \frac{3^i}{2^{\beta(i+1)}}$$

And we have:

$$0 < \lim_{l \to +\infty} (B_l) \leq \lim_{l \to +\infty} \left(\sum_{i=0}^{l-1} \frac{3^i}{2^{\beta(i+1)}} \right) = \lim_{l \to +\infty} \left(\frac{1}{2^\beta l} \right) = +\infty$$

Hence:

$$\lim_{l \to +\infty} (v_l) = \lim_{l \to +\infty} (A_l) + \lim_{l \to +\infty} (B_l) > \frac{3p+1}{3}$$

If this case were proven, there would exist a rank L of the sequence $\{v_l\}$ such that for any $l \geq L$, $v_l > \frac{3p+1}{3}$. Since $p \geq 1$ and the sequence $\{v_l\}$ takes values in $2\mathbb{N}+1$, from rank L all the terms in the sequence would be greater than or equal to 3.

Third case: both conjectures are false
In the case where both conjectures would be false, considering that for any $L \in \mathbb{N}$ there would exist $l \geq L$ such that:

$$A_l \geq \frac{3^l(3p+1)}{2^{\beta(l+1)}} \text{ and } B_l \geq \sum_{i=0}^{l-1} \frac{3^i}{2^{\beta(i+1)}}$$

And given that:

$$\lim_{l \to +\infty} (v_l^{\beta}) = \lim_{l \to +\infty} \left(\frac{3^l(3p+1)}{2^{\beta(l+1)}} \right) + \lim_{l \to +\infty} \left(\sum_{i=0}^{l-1} \frac{3^i}{2^{\beta(i+1)}} \right) = +\infty$$

the sequence $\{v_l\}$ would diverge.

These three cases show that for any $p \in 2\mathbb{N}+1$, either the sequence $\{v_l\}$ diverges or there is a rank from which its terms are always greater than or equal to 3. This contradicts the fact that the sequence eventually enters the cycle of length 1 and value (1), whenever p is of the form $\frac{2^{2a}-1}{3}$, for $a \in \mathbb{N}^*$, or $p \in \{1, 3, 5, 7, 9, \dots, 8400511, \dots\}$. This leads us to conclude that both conjectures are necessarily true, and that the sequence $\{v_l\}$ is truly upper bounded for all $p \in 2\mathbb{N}+1$, as demonstrated in Section 3. Therefore, Theorem 3.1 is true. □

5 Cycles of the sequence

We are going to study whether the sequence $\{v_l\}$ can enter a cycle, under what conditions and what cycles are possible. We will start with cycles of lengths 1, 2 and 3, and then study the general case. The mathematical expression of the terms in the sequence, with the exponents α_i, is the same as the one presented in the introduction, and to simplify matters, we will start counting the exponents from 0. This is equivalent to initialise the term v_0 with the first value of a possible cycle.

The cyclicity condition common to the cycles of lengths 2, 3 and t lies in the fact that the cycle values (within a cycle) must be different from one another. Otherwise, considering the definition of the sequence $\{v_l\}$, the repetition of a value would form a cycle of a shorter length.

5.1 Cycle of length 1

It will be demonstrated that for any odd value of p, the one and only cycle of length 1 into which the sequence $\{v_l\}$ can enter is the cycle of value (1).

Theorem 5.1. *For all $p \in 2\mathbb{N}+1$, such that $v_0 = \frac{3p+1}{2^{\alpha_0}}$, the one and only cycle of length 1 in the sequence $\{v_l\}$ can be the cycle of value* (1).

Proof. The sequence $\{v_l\}$ has a cycle of length 1 if there exists $L \in \mathbb{N}$ such that for any $l \geq L, v_{l+1} = v_l$. Let q the value of the term of rank l in the sequence, then $v_{l+1} = v_l$ if:

$$v_l = q = \frac{3q+1}{2^{\alpha_0}} = v_{l+1} \implies q = \frac{1}{2^{\alpha_0} - 3}$$

This is only possible if $2^{\alpha_0} = 4 \implies \alpha_0 = 2$, and implies that $q = 1$. Reciprocally, we check that if $v_l = 1$ then for any $k \in \mathbb{N}^*, v_{l+k} = 1$. The only cycle of length 1 into which the sequence $\{v_l\}$ can enter from a certain rank is therefore the cycle of value (1), which corresponds to the cycle of values $(1, 4, 2)$ of the sequence $\{u_n\}$. □

5.2 Cycle of length 2 or 3

It will be demonstrated that for any odd value of p, the sequence $\{v_l\}$ has neither a cycle of length 2 nor a cycle of length 3.

Theorem 5.2. *For any $p \in 2\mathbb{N} + 1$, such that $v_0 = \frac{3p+1}{2^{\alpha_0}}$, the sequence $\{v_l\}$ has neither a cycle of length 2 nor a cycle of length 3.*

Proof there is no cycle of length 2. Similarly, the sequence $\{v_l\}$ has a cycle of length 2 if there exists $L \in \mathbb{N}$ such that for any $l \geq L, v_{l+2} = v_l$. Let q the value of the term of rank l in the sequence, then $v_{l+2} = v_l$ if:

$$q = \frac{3\left(3q+1\right) + 2^{\alpha_0}}{2^{\alpha_0+\alpha_1}} \implies q = \frac{3 + 2^{\alpha_0}}{2^{\alpha_0+\alpha_1} - 9}$$

This is only possible if $q \in 2\mathbb{N} + 1$ and $2^{\alpha_0+\alpha_1} > 9 \implies \alpha_0 + \alpha_1 \geq 4$. For $\alpha_0 + \alpha_1 = 4$, we get that $q = 1$ for $(\alpha_0, \alpha_1) = (2, 2)$ and is not integer for the other values of (α_0, α_1). And for $\alpha_0 + \alpha_1 > 4$, q is not integer because $2^{\alpha_0+\alpha_1} - 9 > 3 + 2^{\alpha_0}$, for any $\alpha_0 \geq 1$. Since for $q = 1$ we have established that the sequence becomes stationary from rank l, the sequence $\{v_l\}$ cannot enter a cycle of length 2. □

Proof there is no cycle of length 3. The sequence $\{v_l\}$ has a cycle of length 3 if there exists $L \in \mathbb{N}$ such that for any $l \geq L, v_{l+3} = v_l$. Let q the value of the term of rank l in the sequence, then $v_{l+3} = v_l$ if:

$$q = \frac{3^2\left(3q+1\right) + 3.2^{\alpha_0} + 2^{\alpha_0+\alpha_1}}{2^{\alpha_0+\alpha_1+\alpha_2}} \implies q = \frac{9 + 3.2^{\alpha_0} + 2^{\alpha_0+\alpha_1}}{2^{\alpha_0+\alpha_1+\alpha_2} - 27}$$

This is only possible if $q \in 2\mathbb{N} + 1$ and $2^{\alpha_0+\alpha_1+\alpha_2} > 27 \implies \alpha_0 + \alpha_1 + \alpha_2 \geq 5$. For $\alpha_0 + \alpha_1 + \alpha_2 = 5$, q is not integer. For $\alpha_0 + \alpha_1 + \alpha_2 = 6$, we obtain that $q = 1$ for $(\alpha_0, \alpha_1, \alpha_2) = (2, 2, 2)$, and is not integer for the other values of $(\alpha_0, \alpha_1, \alpha_2)$. Finally, for $\alpha_0 + \alpha_1 + \alpha_2 > 6$, q is not integer because $2\left(2^{\alpha_0+\alpha_1+\alpha_2} - 27\right) > \left(9 + 3.2^{\alpha_0} + 2^{\alpha_0+\alpha_1}\right)$. Therefore, we conclude that the sequence $\{v_l\}$ cannot have a cycle of length 3. □

5.3 Cycle of length t

After proving that the sequence $\{v_l\}$ cannot enter a cycle of length 2 or 3, we are going to demonstrate that the sequence $\{v_l\}$ cannot have a cycle of length greater than or equal to 4.

Theorem 5.3. *Let $t \geq 4$, for any $p \in 2\mathbb{N} + 1$, such that $v_0 = \frac{3p+1}{2^{\alpha_0}}$, the sequence $\{v_l\}$ has no cycle of length t.*

Proof. The sequence $\{v_l\}$ has a cycle of length $t \geq 4$ if there exists $L \in \mathbb{N}$, such that for any $l \geq L, v_{l+t} = v_l$. Let q the value of the term of rank l in the sequence, such that $q > 1$ to discard the cycle of length 1 and value (1), then $v_{l+t} = v_l$ if:

$$q = \frac{3^{t-1}(3q+1) + \sum_{i=0}^{t-2}\left(3^{t-2-i}\left(2^{\sum_{j=0}^{i}\alpha_j}\right)\right)}{2^{\sum_{i=0}^{t-1}\alpha_i}} \implies q = \frac{3^{t-1} + \sum_{i=0}^{t-2}\left(3^{t-2-i}\left(2^{\sum_{j=0}^{i}\alpha_j}\right)\right)}{2^{\sum_{i=0}^{t-1}\alpha_i} - 3^t}$$

The cyclicity condition specific to the cycle of length t can also be expressed, such that for any $k \in \mathbb{N}^*$,$v_{l+kt} = v_l$, which is equivalent to:

$$q = \frac{3^{kt-1} + \sum_{i=0}^{kt-2}\left(3^{kt-2-i}\left(2^{\sum_{j=0}^{i}\alpha_j}\right)\right)}{2^{\sum_{i=0}^{kt-1}\alpha_i} - 3^{kt}}$$

Either for any $k \geq 2$ if:

$$q = \frac{3^{t-1} + C_1}{2^{e_1} - 3^t} = \frac{3^{kt-1} + C_k}{2^{e_k} - 3^{kt}}$$

Where $e_1 = \sum_{i=0}^{t-1}\alpha_i, C_1 = \sum_{i=0}^{t-2} 3^{t-2-i}.2^{\sum_{j=0}^{i}\alpha_j}, e_k = \sum_{i=0}^{kt-1}\alpha_i, C_k = \sum_{i=0}^{kt-2} 3^{kt-2-i}.2^{\sum_{j=0}^{i}\alpha_j}$, which gives:

$$\left(2^{e_k} - 3^{kt}\right)\left(3^{t-1} + C_1\right) = \left(2^{e_1} - 3^t\right)\left(3^{kt-1} + C_k\right)$$

And finally, either for any $k \geq 2$ if:

$$\mathbf{2^{e_1}A = 3^tB} \tag{5.1}$$

Where $A = 3^{kt-1} + C_k - 2^{e_k - e_1}.3^{t-1} - 2^{e_k - e_1}C_1$ is odd, and $B = C_k - 3^{kt-t}C_1$ is even.

First case $A = B = 0$
The equation 5.1 can have solutions if $A = B = 0$. However:

$$B = \sum_{i=0}^{kt-2} 3^{kt-2-i}.2^{\sum_{j=0}^{i}\alpha_j} - \sum_{i=0}^{t-2} 3^{kt-2-i}.2^{\sum_{j=0}^{i}\alpha_j} = \sum_{i=t-1}^{kt-2} 3^{kt-2-i}.2^{\sum_{j=0}^{i}\alpha_j} > 0$$

The term B is a series that tends to $+\infty$ when $k \longrightarrow +\infty$, and given that it is always strictly positive, this case is not possible.

Second case $A < 0$ and $B > 0$
If $A < 0$ then $2^{e_1}A < 3^tB$, this case is either not possible.

Third case $A > 0$ and $B > 0$
Since 2 and 3 are prime numbers and the decomposition of an integer into prime numbers is unique, according to equation 5.1 we have:

$$A = 3^t m \text{ and } B = 2^{e_1} m$$

where $m \in 2\mathbb{N}+1$, because A is odd and B even. Since we are considering the possibility that the sequence $\{v_l\}$ can have a cycle length of t, for any $i \in \{0, \ldots, t-1, \ldots, kt-2\}$, we will have $\alpha_i \in \{\alpha_0, \ldots, \alpha_{t-1}\}$, and the equation $B = 2^{e_1}m$ will have a solution if there exist $m \in 2\mathbb{N}+1$, $t \geq 4$ and $(\alpha_0, \ldots, \alpha_{t-1}) \in (\mathbb{N}^*)^t$, such that for any $k \geq 2$, $B = 2^{e_1}m$ and $A = 3^t m$.

It is clear that for $m = 1$, for any $t{\geq}4$ and $(\alpha_0, \ldots, \alpha_{t-1}) \in (\mathbb{N}^*)^t$, simply take $k = 2$ to get $B > 2^{e_1}m$. If there existed $m > 1$, $t{\geq}4$ and $(\alpha_0, \ldots, \alpha_{t-1}) \in (\mathbb{N}^*)^t$, such that for certain value of $k{\geq}2, B = 2^{e_1}m$, then, since we know that B tends toward infinity exponentially when $k \longrightarrow +\infty$ and that it is possible to take k as large as desired, it would be sufficient to take $k+1$ for that $B > 2^{e_1}m$. So, there is no $m{\in}2\mathbb{N}+1, t \geq 4$ and $(\alpha_0, \ldots, \alpha_{t-1}) \in (\mathbb{N}^*)^t$ such that for any $k{\geq}2, B = 2^{e_1}m$, and therefore the equation 5.1 has no solution.

This implies that for any $p{\in}2\mathbb{N}+1$, there is no cycle of length $t{\geq}4$ into which the sequence can enter from a certain rank, and confirms that the sequence does not have a cycle of length 2 or 3, since there is nothing to prevent t taking the value 2 or 3 in equation 5.1. Therefore, the sequence $\{v_l\}$ can have as its only cycle the cycle of length 1 and value (1). □

6 Stationarity of the sequence

We have demonstrated that for all odd value of p, the sequence $\{v_l\}$ is upper bounded, that it is also lower bounded by 1, and that consequently all its terms are in the finite set $V^p = \{x{\in}2\mathbb{N}+1 : x{\leq}v_{\omega^p}\}$, where v_{ω^p} was defined at the end of Section 3. It remains to be proved that for all odd value of p, all the terms in the sequence are equal to 1 from a certain rank.

Theorem 6.1. *For all $p \in 2\mathbb{N}+1$, such that $v_0 = \frac{3p+1}{2^{\alpha_0}}$, all the terms in the sequence $\{v_l\}$ are equal to 1 from a certain rank.*

Proof. Suppose that for any $l{\in}\mathbb{N}, v_l{\neq}1$, then, since the only cycle of length 1 is the cycle of value (1), there exist $c{\in}\mathbb{N}$ and $t{\geq}2$ such that $v_c = q = v_{c+t}$, where $q \in V^p \setminus \{1\}$. Indeed, since we have demonstrated that $\{v_l\}$ is an application from $\mathbb{N}$ to a finite subset of $\mathbb{N}$ (see Theorem 3.1), following the principle of drawers (known as the pigeonhole principle), we know that there are at least two elements of $\mathbb{N}$ which have the same image by the application $\{v_l\}$.

Therefore, there exist $c{\in}\mathbb{N}$ and $t{\geq}2$ such that:

$$v_{c+t} = q = v_c,$$

$$v_{c+t+1} = \frac{3q+1}{2^{\alpha_0}} = v_{c+1},$$

$$\ldots,$$

$$v_{c+2t} = \frac{3^{t-1}(3q+1)}{2^{\sum_{i=0}^{t-1}\alpha_i}} + \sum_{i=0}^{t-2} \frac{3^{t-2-i}}{2^{\sum_{j=i+1}^{l-1}\alpha_j}} = v_{c+t} = q,$$

$$\ldots$$

Thus forming the following cycle:

$$v_{c-1} \rightarrow \mathbf{v_c} \rightarrow v_{c+1} \rightarrow \ldots \rightarrow v_{c+t-1} \rightarrow \mathbf{v_{c+t}} = \mathbf{v_c} \rightarrow v_{c+t+1} = v_{c+1} \rightarrow \ldots$$

Now, we have just demonstrated that the sequence $\{v_l\}$ has no cycle of length $t{\geq}2$ (See Theorems 5.2 and 5.3), which implies that there exists $l{\in}\mathbb{N}$ such that $v_l = 1$. Therefore, for all $p{\in}2\mathbb{N}+1$, such that

$v_0 = \frac{3p+1}{2^{\alpha_0}}$, all the terms in the sequence $\{v_l\}$ will be equal to 1 from a certain rank, which corresponds for the sequence $\{u_n\}$ to repeating *ad vitam aeternam* the cycle $(1, 4, 2)$. □

7 Conclusion

First, we have made conjectures about the sums of the exponents α_i, whose accuracy has been proven, then we have demonstrated that the sequence $\{v_l\}$ is bounded, and finally we have determined its possible cycles. This allowed us to demonstrate that all the terms in the sequence are equal to 1 from a certain rank, in other words that the sequence becomes stationary. The main stages of the demonstration can be represented graphically as follows, where the black arrows correspond to the mathematical implications.

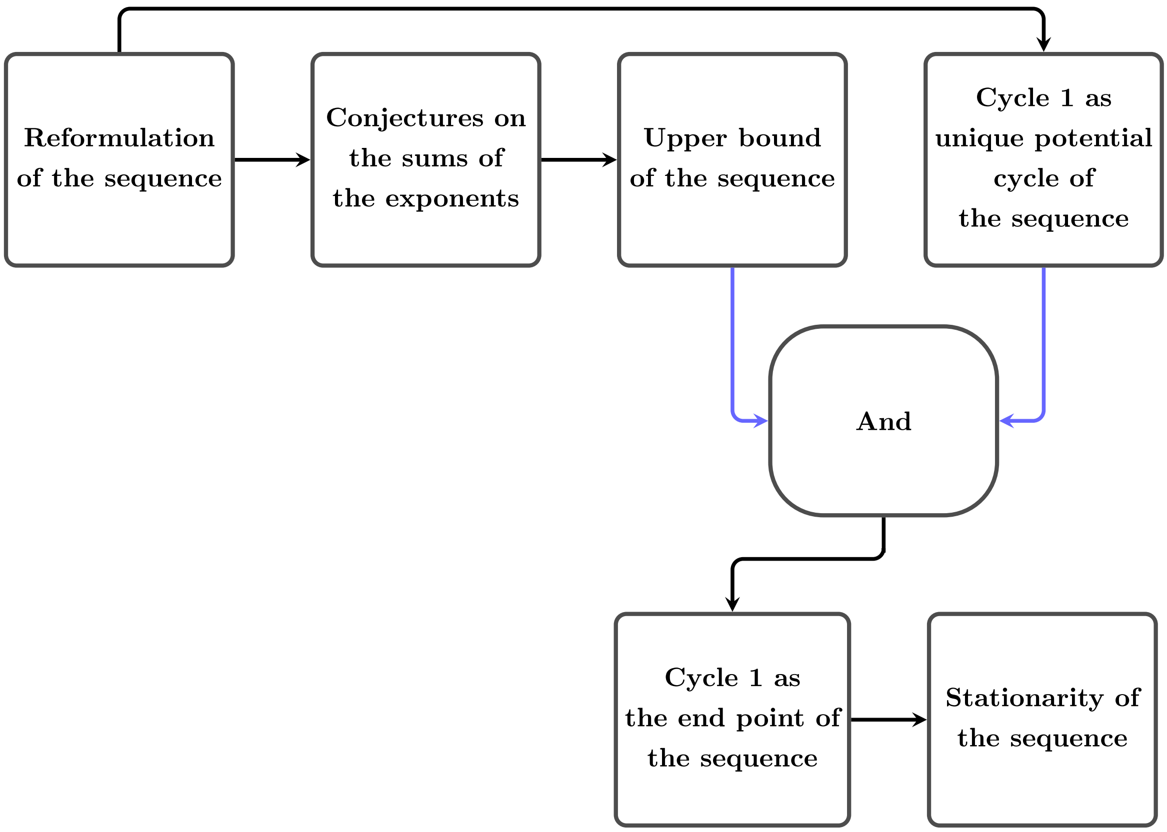

Figure 1. Diagram of the demonstration

Insofar as the sequence $\{v_l\}$ is equal to the sequence $\{u_n\}$ without the even-valued terms of the latter (see Section 1), we have demonstrated that for any $q \in \mathbb{N}^*$, such that $u_0 = q$, the sequence $\{u_n\}$ always ends up reaching the cycle $(1, 4, 2)$. Thus, we have proved that the Collatz conjecture is true.

8 Appendix

8.1 General formula for a term in a sequence $\{v_l\}$

We know that the formula is valid for the second term in the sequence. Suppose that for $l < L$, where $L \geq 2$:

$$v_l = \frac{3^l(3p+1) + \sum_{i=0}^{l-1}\left(3^{l-1-i} \cdot 2^{\sum_{j=0}^{i} \alpha_j}\right)}{2^{\sum_{i=0}^{l} \alpha_i}}$$

And calculate v_{l+1} as a function of v_l:

$$v_{l+1} = \frac{3v_l + 1}{2^{\alpha_{l+1}}} = \frac{3\left(\frac{3^l(3p+1)+\sum_{i=0}^{l-1}\left(3^{l-1-i} \cdot 2^{\sum_{j=0}^{i} \alpha_j}\right)}{2^{\sum_{i=0}^{l} \alpha_i}}\right) + 1}{2^{\alpha_{l+1}}}$$

Thus, we get:

$$v_{l+1} = \frac{3^{l+1}(3p+1) + \sum_{i=0}^{l}\left(3^{l-i} \cdot 2^{\sum_{j=0}^{i} \alpha_j}\right)}{2^{\sum_{i=0}^{l+1} \alpha_i}}$$

This shows that for all $l \in \mathbb{N}^*$:

$$v_l = \frac{3^l(3p+1) + \sum_{i=0}^{l-1}\left(3^{l-1-i} \cdot 2^{\sum_{j=0}^{i} \alpha_j}\right)}{2^{\sum_{i=0}^{l} \alpha_i}}$$

8.2 Probabilities of the exponent values

For each term in the sequence $\{v_l\}_{l \in \mathbb{N}^*}$, such that $2^{\gamma_l} v_{l-1} = 3v_{l-1} + 1$, we have:

$$\gamma_l = \frac{\ln\left(3 + \frac{1}{v_{l-1}}\right)}{\ln(2)}$$

Although the values of the exponents are not the result of chance, insofar as for any $i \in [0, l], \gamma_i \in]1, 2[$ and the exponents are strictly positive integers, the value of γ_i gives an indication of closeness. It is in fact clear that if γ_i is closer to 2 than 1, the chances of α_i being greater than or equal to 2, will be higher than the chances giving $\alpha_i = 1$. Thus, we can consider that the probability that $\alpha_i \geq 2$ is $\gamma_i - 1$, while the probability that $\alpha_i = 1$ is $1 - (\gamma_i - 1)$.

8.3 Irrational nature of $\beta(l+1)$

We will show that the term $\beta(l+1)$ introduced in Section 2 is irrational. Assume that $\beta = \frac{\ln(3)}{\ln(2)}$ is rational, then there exists $(p, q) \in \mathbb{N}^* \times \mathbb{N}^*$ such that:

$$\frac{\ln(3)}{\ln(2)} = \frac{p}{q} \iff \ln(3^q) = \ln(2^p) \iff e^{\ln(3^q)} = e^{\ln(2^p)} \iff 3^q = 2^p$$

This is obviously false, so β is irrational. And consequently, so is $\beta(l+1)$, because a rational multiplied by an irrational gives an irrational.

8.4 Existence of β^p

Let us take $L^p = \max\{L_0^p, L_1^p\}$ as defined in Section 3. The variables β_i and K_i, below, also depend on p but to simplify writing the exponent p has not been indicated. Let $\beta_0 \in \mathbb{R}_+^*$ such that $\beta < \beta_0$, it is then clear that for all $l \in \mathbb{N}^*$:

$$v_l^{\beta_0} = \frac{1}{2^{\beta_0}} \left(\frac{3^l(3p+1)}{2^{\beta_0(l)}} + \sum_{i=0}^{l-1} \frac{3^i}{2^{\beta_0(i)}} \right) < \frac{1}{2^{\beta}} \left(\frac{3^l(3p+1)}{2^{\beta(l)}} + \sum_{i=0}^{l-1} \frac{3^i}{2^{\beta(i)}} \right) = v_l^{\beta}$$

Assume that there exists $K_1 \geq L^p$ such that $v_{K_1}^{\beta_0} < v_{K_1} < v_{K_1}^{\beta}$ and remind that according to Conjectures 2.1 and 2.2 for all $l \geq L^P, v_l < v_l^{\beta}$, then, since it is possible to take β_0 as close to β as desired, and that the function $f_l : [\beta, \beta_0] \to \mathbb{R}_+^*$ defined by:

$$f_l(x) = \frac{1}{2^x} \left(\frac{3^l(3p+1)}{2^{x(l)}} + \sum_{i=0}^{l-1} \frac{3^i}{2^{x(i)}} \right)$$

is clearly continuous and strictly decreasing for all $l \in \mathbb{N}^*$, there exists $\beta_1 \in]\beta, \beta_0[$ such that $v_{K_1} \leq f_{K_1}(\beta_1) = v_{K_1}^{\beta_1} < v_{K_1}^{\beta}$, and so for all $l \in [L^P, K_2[$, if there exists K_2, or for all $l \geq L^P$, otherwise, $v_l \leq v_l^{\beta_1} < v_l^{\beta}$.

By induction, if there exists $K_i > K_{i-1} > K_1$ such that $v_{K_i}^{\beta_{i-1}} < v_{K_i} < v_{K_i}^{\beta}$, then, for the same reasons, there exists $\beta_i \in]\beta, \beta_{i-1}[$ such that $v_{K_i} \leq f_{K_i}(\beta_i) = v_{K_i}^{\beta_i} < v_{K_i}^{\beta}$, and so and so for all $l \in [L^P, K_{i+1}[$, if there exists K_{i+1}, or for all $l \geq L^P$, otherwise, $v_l \leq v_l^{\beta_i} < v_l^{\beta}$. Consequently, assuming that for any $i \in \mathbb{N}$ there exists $K_i > K_{i-1}$ such that $v_{K_i}^{\beta_{i-1}} < v_{K_i} < v_{K_i}^{\beta}$, for all $l \geq L^p$ there exists $\beta_{\phi(l)} \in]\beta, \beta_0]$ such that:

$$v_l \leq v_l^{\beta_{\phi(l)}} < v_l^{\beta}$$

Where $\phi(l) = \begin{cases} 0 & \text{if } l < K_1, \\ i & \text{if } l \in [K_i, K_{i+1}[. \end{cases}$

However, there cannot be an infinity of K_i such that $v_{K_i}^{\beta_{i-1}} < v_{K_i} < v_{K_i}^{\beta}$ and such that $\beta_i \in]\beta, \beta_{i-1}[$ to ensure that $v_{K_i} \leq v_{K_i}^{\beta_i}$, because considering that for all $i \in \mathbb{N}$:

$$v_{K_i}^{\beta_{i-1}} < v_{K_i} \leq v_{K_i}^{\beta_i} < v_{K_{i+1}} \implies v_{K_i} < v_{K_{i+1}}$$

the sub-sequence $\{v_{K_i}\}$ would diverge. Which would imply that the sequence $\{v_l\}$ also diverges, whatever the value of p, thus contradicting the fact that for many values of p the sequence $\{v_l\}$ eventually reaches the value 1, as demonstrated at the end of Section 4. The number of K_i is therefore finite and there exists $\beta^p > \beta$ such that for all $l \geq L^P$, $v_l \leq v_l^{\beta^p} < v_l^{\beta}$.

9 Conflict of interest

On behalf of the author, the corresponding author states that there is no conflict of interest.

10 References

The manuscript does not refer to any previous paper related to the Collatz conjecture, as the proof is based on a new approach, and no references are required for its completeness.