Relativistic Kinematics of Photon-Electron Scattering in the Lab Frame

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Abstract

We present a unique derivation for photon-electron scattering that operates exclusively in the laboratory frame of reference. It considers the relativistic principle and angular dependence of photon and electron. A general equation emerges at the end which relates the energies of the photon before and after scattering through the application of energy and momentum conservation laws [4]. Such a laboratory-based methodology gets rid of the requirement for frame transformations into electron rest reference frames while providing an integrated approach to understanding both classical and relativistic regimes [3].

Keywords: Photon-electron scattering, Compton scattering, Inverse Compton scattering, Relativistic kinematics, Lab-frame derivation.

1. Introduction

Interaction between photons and electrons is a core concept in present-day physics because it enables our understanding of quantum mechanics alongside astrophysical radiation processes. In 1923 Arthur Compton proved light behaved as a particle by examining experiments demonstrating how photons transfer momentum as well as energy to electrons [1].In high-energy astrophysical jets and particle accelerators, where ultra-relativistic electrons produce substantial energy gain in photons through a process known as inverse Compton scattering [2]. A conventional approach involves transforming to an electron rest frame for simplifying calculation but it introduces difficulties in the lab frame analysis. In this work, throughout our complete analysis, we utilize an alternative method to establish the photon energy relation using only laboratory frames of reference.

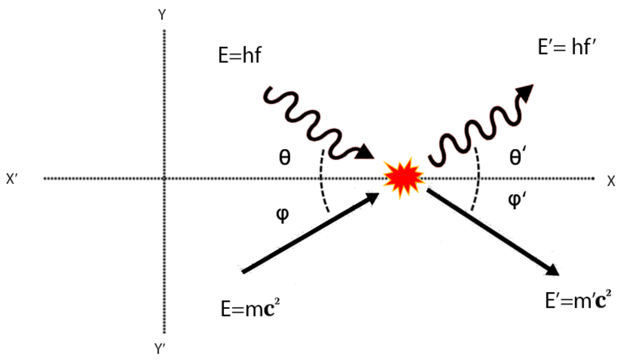

Figure 1: This diagram illustrates the scattering process of a photon interacting with a moving electron in the lab frame. The incoming photon approaches the electron at an angle $(\theta + \phi)$

2. Derivation

Conservation of Energy.

The total energy before and after an interaction remains conserved:

$$E_p + E_e = E'_p + E'_e \tag{1}$$

$$\frac{hc}{\lambda} + mc^2 = \frac{hc}{\lambda'} + m'c^2 \tag{2}$$

This equation represents the conservation of energy, where:

- $\frac{hc}{\lambda}$ and $\frac{hc}{\lambda'}$ are the energies of the incoming and scattered photons.
- mc^2 and $m'c^2$ are the energies of the electron before and after interaction.

Squaring Both Sides.
We square both sides to prepare for further manipulations:

$$\left(\frac{hc}{\lambda} + mc\right)^2 = \left(\frac{hc}{\lambda'} + m'c\right)^2 \tag{3}$$

Expanding the Squared Term.

$$\frac{h^2c^2}{\lambda^2} + 2\frac{hc}{\lambda}mc + m^2c^2 = \frac{h^2c^2}{\lambda'^2} + 2\frac{hc}{\lambda'}m'c + m'^2c^2 \tag{4}$$

Conservation of Momentum in the x-Direction.

Momentum in the x-direction must also be conserved:

$$P_{px} + P_{ex} = P'_{px} + P'_{ex} \tag{5}$$

Substituting the Momentum Terms.

$$\frac{h}{\lambda}\cos(\theta) + mv\cos(\phi) = \frac{h}{\lambda'}\cos(\theta') + m'v'\cos(\phi') \tag{6}$$

Squaring the Momentum Equation.

$$\left(\frac{h}{\lambda}\cos(\theta) + mv\cos(\phi)\right)^2 = \left(\frac{h}{\lambda'}\cos(\theta') + m'v'\cos(\phi')\right)^2 \tag{7}$$

Expanding the Squared Terms.

$$\frac{h^2}{\lambda^2}\cos^2(\theta) + 2\frac{h}{\lambda}mv\cos(\theta)\cos(\phi) + m^2v^2\cos^2(\phi) = \frac{h^2}{\lambda'^2}\cos^2(\theta') + 2\frac{h}{\lambda'}m'v'\cos(\theta')\cos(\phi') + m'^2v'^2\cos^2(\phi') \tag{8}$$

Conservation of Momentum in the y-Direction.

$$P_{py} + P_{ey} = P'_{py} + P'_{ey} \tag{9}$$

Substituting the Momentum Terms.

$$\frac{h}{\lambda}\sin(\theta) - mv\sin(\phi) = \frac{h}{\lambda'}\sin(\theta') - m'v'\sin(\phi') \tag{10}$$

here Y component of electron's momentum is negative.

Squaring the Momentum Equation in the y-Direction.

$$\left(\frac{h}{\lambda}\sin(\theta) - mv\sin(\phi)\right)^2 = \left(\frac{h}{\lambda'}\sin(\theta') - m'v'\sin(\phi')\right)^2 \tag{11}$$

Expanding the Squared Terms.

$$\frac{h^2}{\lambda^2}\sin^2(\theta) - 2\frac{h}{\lambda}mv\sin(\theta)\sin(\phi) + m^2v^2\sin^2(\phi) = \frac{h^2}{\lambda'^2}\sin^2(\theta') - 2\frac{h}{\lambda'}m'v'\sin(\theta')\sin(\phi') + m'^2v'^2\sin^2(\phi') \tag{12}$$

Adding the x and y Components.

Adding the squared equations for the x- and y-momentum:

$$\left(\frac{h}{\lambda}\cos(\theta) + mv\cos(\phi)\right)^2 + \left(\frac{h}{\lambda}\sin(\theta) - mv\sin(\phi)\right)^2$$
$$= \left(\frac{h}{\lambda'}\cos(\theta') + m'v'\cos(\phi')\right)^2 + \left(\frac{h}{\lambda'}\sin(\theta') - m'v'\sin(\phi')\right)^2 \tag{13}$$

$$\frac{h^2}{\lambda^2}(\cos^2\theta+\sin^2\theta)+m^2v^2(\cos^2\phi+\sin^2\phi)+2\cdot\frac{h}{\lambda}mv(\cos\theta\cos\phi-\sin\theta\sin\phi) \tag{14}$$

$$=\frac{h^2}{\lambda'^2}(\cos^2\theta'+\sin^2\theta')+m'^2v'^2(\cos^2\phi'+\sin^2\phi')+2\cdot\frac{h}{\lambda'}m'v'(\cos\theta'\cos\phi'-\sin\theta'\sin\phi') \tag{15}$$

Using the trigonometric identity:

$$\cos^2 x+\sin^2 x=1 \tag{16}$$

This simplifies to:

$$\frac{h^2}{\lambda^2}+m^2v^2+2\cdot\frac{h}{\lambda}mv(\cos\theta\cos\phi-\sin\theta\sin\phi) \tag{17}$$

$$=\frac{h^2}{\lambda'^2}+m'^2v'^2+2\cdot\frac{h}{\lambda'}m'v'(\cos\theta'\cos\phi'-\sin\theta'\sin\phi') \tag{18}$$

Using

$$\cos(\theta)\cos(\phi)-\sin(\theta)\sin(\phi)=\cos(\theta+\phi)$$

, we simplify:

$$\frac{h^2}{\lambda^2}+m^2v^2+2\frac{h}{\lambda}mv\cos(\theta+\phi)=\frac{h^2}{\lambda'^2}+m'^2v'^2+2\frac{h}{\lambda'}m'v'\cos(\theta'+\phi')$$

Subtracting eq 18) from eq (4).

We start with:

$$\left(\frac{h^2}{\lambda^2}+m^2c^2+2\cdot\frac{h}{\lambda}mc\right)-\left(\frac{h^2}{\lambda^2}+m^2v^2+2\cdot\frac{h}{\lambda}mv\cos(\theta+\phi)\right)$$
$$=\left(\frac{h^2}{\lambda'^2}+m'^2c^2+2\cdot\frac{h}{\lambda'}m'c\right)-\left(\frac{h^2}{\lambda'^2}+m'^2v'^2+2\cdot\frac{h}{\lambda'}m'v'\cos(\theta'+\phi')\right)$$

$$m^2c^2-m^2v^2+2\frac{h}{\lambda}mc-2\frac{h}{\lambda}mv\cos(\theta+\phi)=m'^2c^2-m'^2v'^2+2\frac{h}{\lambda'}m'c-2\frac{h}{\lambda'}m'v'\cos(\theta'+\phi') \tag{20}$$

Taking common terms:

$$m^2(c^2-v^2)+2\frac{h}{\lambda}m(c-v\cos(\theta+\phi))=m'^2(c^2-v'^2)+2\frac{h}{\lambda'}m'(c-v'\cos(\theta'+\phi')) \tag{21}$$

Substituting $m = m_0\gamma$ and $m' = m_0\gamma'$:

$$\left(\frac{m_0^2c^2}{c^2-v^2}\right)(c^2-v^2)+2\frac{h}{\lambda}m(c-v\cos(\theta+\phi)) = \left(\frac{m_0^2c^2}{c^2-v'^2}\right)(c^2-v'^2)+2\frac{h}{\lambda'}m'(c-v'\cos(\theta'+\phi')) \tag{22}$$

Dividing by $m_0^2c^2$ on both side :

$$2\frac{h}{\lambda}m(c-v\cos(\theta+\phi)) = 2\frac{h}{\lambda'}m'(c-v'\cos(\theta'+\phi')) \tag{23}$$

Dividing by 2h on both side and simplifying:

$$\frac{\lambda'}{\lambda} = \frac{m'c}{mc} \cdot \frac{1-\frac{v'}{c}\cos(\theta'+\phi')}{1-\frac{v}{c}\cos(\theta+\phi)} \tag{24}$$

Eq (23) relates the wavelength of the photon before and after scattering. **Using $m'c = P_e'$ and $mc = P_e$:**

$$\frac{\lambda'}{\lambda} = \frac{P_e'}{P_e} \cdot \frac{1-\frac{v'}{c}\cos(\theta'+\phi')}{1-\frac{v}{c}\cos(\theta+\phi)}$$

here P and P' is the relativistic momentum of electron before and after scattering.

Writing equation in terms of energy.

taking eq (24) and by putting the value of m and m', eq (24) can be simplified.

$$\frac{\lambda'}{\lambda} = \frac{\gamma'}{\gamma} \cdot \frac{1-\frac{v'}{c}\cos(\theta'+\phi')}{1-\frac{v}{c}\cos(\theta+\phi)} \tag{26}$$

Multiply and divide by hc on the left hand side

$$\frac{\lambda'}{\lambda} \times \frac{hc}{hc} = \frac{\gamma'}{\gamma} \cdot \frac{1-\frac{v'}{c}\cos(\theta'+\phi')}{1-\frac{v}{c}\cos(\theta+\phi)} \tag{27}$$

Since $E = \frac{hc}{\lambda}$ (energy of a photon), we substitute:

$$\frac{hc/\lambda}{hc/\lambda'} = \frac{\gamma'}{\gamma} \cdot \frac{1-\frac{v'}{c}\cos(\theta'+\phi')}{1-\frac{v}{c}\cos(\theta+\phi)} \tag{28}$$

Simplify Using Energy Relation.

$$\frac{E}{E'} = \frac{\gamma'}{\gamma} \cdot \frac{1-\frac{v'}{c}\cos(\theta'+\phi')}{1-\frac{v}{c}\cos(\theta+\phi)} \tag{29}$$

Final Expression in Terms of Energy.

$$E' = E \cdot \frac{\gamma}{\gamma'} \cdot \frac{1-\frac{v}{c}\cos(\theta+\phi)}{1-\frac{v'}{c}\cos(\theta'+\phi')} \tag{30}$$

eq (30) relates the energy of the photon before and after scattering.

3. Small-Angle Scattering

Small-angle scattering occurs when the photon scatters at a shallow angle relative to its original trajectory. In this regime, the angles $\theta+\phi$ (incoming photon direction) and $\theta'+\phi'$ (outgoing photon direction) are close to zero, and the cosine terms approach unity:

$$\cos(\theta+\phi) \approx 1, \quad \cos(\theta'+\phi') \approx 1.$$

Additionally, the electron velocity and Lorentz factor remain approximately unchanged during the interaction:

$$v' \approx v, \quad \gamma' \approx' \gamma.$$

Substituting these approximations into the general equation for photon-electron scattering:

$$E' = E \cdot \frac{\gamma}{\gamma'} \cdot \frac{1 - \frac{v}{c}\cos(\theta+\phi)}{1 - \frac{v'}{c}\cos(\theta'+\phi')},$$

We simplify the terms as follows:

1. For small angles, $\cos(\theta+\phi) \approx 1$ and $\cos(\theta'+\phi') \approx 1$, so:

$$1 - \frac{v}{c}\cos(\theta+\phi) \approx 1 - \frac{v}{c}, \quad 1 - \frac{v'}{c}\cos(\theta'+\phi') \approx 1 - \frac{v'}{c}.$$

2. Since $v' \approx v$, the numerator and denominator cancel out:

$$\frac{1 - \frac{v}{c}}{1 - \frac{v'}{c}} \approx 1.$$

3. Similarly, since $\gamma' \approx \gamma$, the prefactor $\frac{\gamma}{\gamma'} \approx 1$.

Thus, the final result for small-angle scattering is:

$$\boxed{E' \approx E.}$$

The photon sustains almost all of its initial energy while showing minimal energy transfer between an electron. Here the photon trajectory changes slightly due to its weak interaction with an electron.

4. Conclusion

This research introduces an innovative outlook to examine photon-electron scattering phenomena. By leveraging conservation laws, we derive a general equation that captures the interplay between photon energy, electron motion, and scattering angles [4]. Future work could explore specific cases like ultra-relativistic limits and their applications in astrophysics and high-energy physics. Recalling a classic problem with a novel approach, this work underscores the enduring richness of photon-electron interactions.

References

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