About the Riemann hypothesis

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Abstract

Here is proposed to confirm the Riemann hypothesis.

1. Introduction

Bernhard Riemann made the hypothesis, that is here proposed to confirm, that the complex xi (ξ) function zeros are real [2] (p.139). The eta (η) Dirichlet function will also be proposed to be used for a proof that the Riemann zeta function nontrivial zeros, which are linked to the xi zeros, have real part equal to $\frac{1}{2}$.

2. The zeros of ξ

Theorem 2.1. *There exists a real sequence* $(a_n)_{n\in\mathbb{N}}$ *such that the Riemann* ξ *function can be written such as, for* $t \in \mathbb{C} : \xi(t) = \sum_{n=0}^{\infty}(-1)^n|a_n|t^{2n}$.

Proof. According to Riemann [2] (p.138), for $t \in \mathbb{C}$:

$$\xi(t) = 4\int_1^{\infty} \frac{d(x^{\frac{3}{2}}\psi'(x))}{dx} x^{-\frac{1}{4}} \cos(\frac{1}{2}t\log(x))\,dx.$$

So, as Riemann typed: “Diese Function... ...lässt sich nach Potenzen von tt in eine... ...convergirende Reihe entwickeln.”, which can be translated as “This function... ...allows itself to be developed in powers of tt... ...as a converging series.”, the Riemann ξ function can be such as, for $t \in \mathbb{C}$: $\xi(t) = \sum_{n=0}^{\infty} a_n(t^2)^n$, where for $n \in \mathbb{N}$:

$$a_n = 4\,\frac{(-1)^n}{2^{2n}(2n)!}\int_1^{\infty} \frac{d(x^{\frac{3}{2}}\psi'(x))}{dx} x^{-\frac{1}{4}} (\log(x))^{2n}\,dx.$$

The function $\dfrac{d(x^{\frac{3}{2}}\psi'(x))}{dx} = \pi x^{\frac{1}{2}} \sum_{n=1}^{\infty} \left(n^2\pi x - \frac{3}{2}\right) n^2 e^{-n^2\pi x}$ being positive on $(1;+\infty)$, and having a finite limit at 1^+, for all $n \in \mathbb{N}$, $(-1)^n a_n > 0$, which gives the theorem 2.1. □

Noting for $z \in \mathbb{C}$, $\operatorname{Arg}(z)$ as the principal argument of z being in $(-\pi;\pi]$, $(u_n(t))_{n\in\mathbb{N}} = (a_n t^{2n})_{n\in\mathbb{N}}$, follows this theorem:

Theorem 2.2. *For any t in $\mathbb{C}$ such that $\Re(t) \neq 0$, $\Im(t) \in (-\frac{1}{2}, \frac{1}{2})$ and $\xi(t) = 0$, t is real.*

Proof. Let be $t = a + ib \in \mathbb{C}$ such that $a \neq 0, b \in (-\frac{1}{2}; \frac{1}{2})$, and $\xi(t) = 0$. Thus,

$$\sum_{n=0}^{\infty} u_{n+1} = -a_0. \tag{1}$$

Let us name $z = \sum_{n=0}^{\infty} |u_{n+1}(t)| e^{i \operatorname{Arg}(u_n(t))}$ (convergence confirmed below). We have:

$$\begin{aligned}
\Im(z) &= \Im\left(\sum_{n=0}^{\infty} |u_{n+1}(t)| e^{i\frac{1}{2}\operatorname{Arg}(u_n^2(t))}\right) \\
&= \Im\left(\sum_{n=0}^{\infty} |u_{n+1}(t)| e^{i\frac{1}{2}\operatorname{Arg}\left(\left(\frac{u_n(t)}{u_{n+1}(t)}\right)^2 u_{n+1}^2(t)\right)}\right) \\
&= \Im\left(\sum_{n=0}^{\infty} |u_{n+1}(t)| e^{i\frac{1}{2}\operatorname{Arg}\left(t^{-4}u_{n+1}^2(t)\right)}\right) \\
&= \Im\left(\sum_{n=0}^{\infty} |u_{n+1}(t)| e^{i\operatorname{Arg}\left(t^{-2}u_{n+1}(t)\right)}\right) \\
&= \Im\left(\sum_{n=0}^{\infty} u_{n+1}(t) e^{i\operatorname{Arg}\left(t^{-2}\right)}\right) \\
&= \Im\left(-a_0 e^{i\operatorname{Arg}(t^{-2})}\right) \text{ from the equation (1)}
\end{aligned}$$

$$\Im(z) = -a_0 \sin(\operatorname{Arg}(t^{-2})). \tag{2}$$

And:

$$
\begin{aligned}
\Im(z) &= \Im\left(\sum_{n=0}^{\infty} |u_{n+1}(t)| e^{i\,\mathrm{Arg}(u_n(t))}\right) \\
&= \Im\left(\sum_{n=0}^{\infty} u_{n+1}(t) e^{i\,\mathrm{Arg}\left(\frac{u_n(t)}{u_{n+1}(t)}\right)}\right) \\
&= \Im\left(\sum_{n=0}^{\infty} u_{n+1}(t) e^{i\,\mathrm{Arg}(-t^{-2})}\right) \\
&= \Im\left(-a_0 e^{i\,\mathrm{Arg}(-t^{-2})}\right) \text{ from the equation (1)} \\
\Im(z) &= a_0 \sin(\mathrm{Arg}(t^{-2})). \qquad (3)
\end{aligned}
$$

Thus, thanks to the equations (2) and (3): $a_0 \sin(\mathrm{Arg}(t^{-2})) = 0$. Then, $t^2 = a^2 - b^2 + i2ab$ is real, $b = 0$ because $a \neq 0$, and t is real. □

Therefore, as Riemann typed [2] (p.138) "...so kann die Function $\xi(t)$ nur verschwinden, wenn der imaginäre Theil von t zwischen $\frac{1}{2}i$ und $-\frac{1}{2}i$ liegt.", which can be translated as "...it follows that the function $\xi(t)$ can only vanish if the imaginary part of t lies between $\frac{1}{2}i$ and $-\frac{1}{2}i$", with the lemma 2.1 is proposed that the Riemann hypothesis is confirmed.

Lemma 2.1. *For any t in $\mathbb{C}$ such that $\Re(t) = 0$ and $\Im(t) \in (-\frac{1}{2}, \frac{1}{2})$, $\xi(t)$ is not null.*

Proof. We proceed by contradiction. Let be t in $\mathbb{C}$ such that $\Re(t) = 0$, $\Im(t) \in (-\frac{1}{2}, \frac{1}{2})$ and $\xi(t) = 0$. Then, $\xi(t) = \sum_{n=0}^{\infty} |a_n| (\Im(t))^n$. Because of the symmetry of the zeros about the real axis (which can be proved by saying that $\xi(\bar{t}) = 0$ using the polar form of the xi expression of the theorem 2.1), $a_0 = 0$. But a_0 is positive. We have the lemma 2.1. □

3. The nontrivial zeros of ζ

In 1859, Riemann wrote: $\Gamma(\frac{s}{2}+1)(s-1)\pi^{\frac{-s}{2}}\zeta(s) = \xi(t)$ where $s = \frac{1}{2}+it$. Let us now propose to use the Dirichlet eta (η) function, present in an analytic continuation expression of ζ, on $0 < \Re(s) < 1$, where are its nontrivial zeros.

Lemma 3.1. *For any a in $(0,\infty)$, any n in $\mathbb{N}$, any s in $\mathbb{C}$, if $\Re(s)$ is in $(0,1)$ and $\zeta(s)=0$, then* $\displaystyle\int_0^\infty \frac{x^{s-1}}{e^{xa^{-4n}}+1}\,dx = 0$.

Proof. Let be a a positive real number, n in $\mathbb{N}$, and s in $\mathbb{C}$, such that $\Re(s)\in(0,1)$ and $\zeta(s)=0$. Lang [1] (p.157) and Spiegel [3] (line 15.82) give, for $\Re(s)>0$:

$$\zeta(s)=\frac{\eta(s)}{1-2^{1-s}} \quad \text{and} \quad \eta(s)=\frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x+1}\,dx.$$

Then, $\zeta(s)=0$ implies that $\displaystyle\int_0^\infty \frac{x^{s-1}}{e^x+1}\,dx=0$, and the variable change $x=ua^{-4n}$ gives the lemma 3.1. □

Theorem 3.1. *For any s in $\mathbb{C}$, if $\Re(s)$ is in $(0,1)$, $\Im(s)\neq 0$ and $\zeta(s)=0$, then $\Re(s)=\frac{1}{2}$.*

Proof. Let be $s=\sigma+it$ a nontrivial zero, with σ in $(0,1)$ and t a positive real number, and let be $a=\exp\left(\dfrac{\pi}{2t}\right)$. The lemma 3.1 gives that:

$$\int_0^1 \frac{x^{s-1}}{e^{xa^{-4n}}+1}\,dx = -\int_1^\infty \frac{x^{s-1}}{e^{xa^{-4n}}+1}\,dx$$

n going to infinity, the dominated convergence theorem makes the left term converge to $\frac{1}{2s}$. Thus,

$$\begin{aligned}\Re\left(\frac{1}{2s}\right)+\underset{n\to\infty}{o(1)} &= \Re\left(-\int_1^{+\infty}\frac{x^{s-1}}{e^{xa^{-4n}}+1}\,dx\right)\\ &= -\frac{1-\sigma}{\sigma}\int_1^\infty \frac{x^{-\sigma}\cos\left(t\,\frac{1-\sigma}{\sigma}\ln(x)\right)}{1+\exp\left(a^{-4n}\,x^{\frac{1-\sigma}{\sigma}}\right)}\,dx, \text{with the variable change } x=u^{\frac{1-\sigma}{\sigma}}\\ &= \frac{1-\sigma}{\sigma}\int_0^1 \frac{x^{-\sigma}\cos\left(t\,\frac{1-\sigma}{\sigma}\ln(x)\right)}{1+\exp\left(a^{-4n}\,x^{\frac{1-\sigma}{\sigma}}\right)}\,dx\end{aligned}$$

with the same variable change applied to $\Re\left(\displaystyle\int_0^\infty \frac{x^{s-1}}{e^{xa^{-4n}}+1}\,dx\right)=0$.

n going to infinity, the dominated convergence theorem makes this last term converge to

$$\frac{1}{2}\frac{1-\sigma}{\sigma}\int_0^1 x^{-\sigma}\cos\left(t\,\frac{1-\sigma}{\sigma}\ln(x)\right)dx,$$

which equals $\dfrac{1}{2}\dfrac{1-\sigma}{\sigma}\dfrac{(1-\sigma)^2}{\sigma}\dfrac{1}{(1-\sigma)^2+t^2}$. Then, because this term equals $\Re\left(\frac{1}{2s}\right)$:

$$\frac{1-\sigma}{\sigma}\frac{(1-\sigma)^2}{\sigma}\frac{1}{(1-\sigma)^2+t^2}=\frac{\sigma}{\sigma^2+t^2}$$

$$\Leftrightarrow \frac{1-\sigma}{\sigma}(1-\sigma)^2\,(\sigma^2+t^2)=\sigma^2((1-\sigma)^2+t^2) \quad \text{multiplying by } \sigma((1-\sigma)^2+t^2)(\sigma^2+t^2)$$

$$\Leftrightarrow t^2\left((1-\sigma)^2\left(\frac{1-\sigma}{\sigma}\right)-\sigma^2\right)=\left(1-\frac{1-\sigma}{\sigma}\right)\sigma^2(1-\sigma)^2 \quad \text{factorizing by } t^2 \text{ and } \sigma^2(1-\sigma)^2$$

$$\Leftrightarrow \left(\frac{t}{1-\sigma}\right)^2\left(\left(\frac{1-\sigma}{\sigma}\right)^3-1\right)=\left(1-\frac{1-\sigma}{\sigma}\right) \quad \text{dividing by } \sigma^2(1-\sigma)^2$$

$$\Leftrightarrow \left(\frac{t}{1-\sigma}\right)^2\left(\frac{1-\sigma}{\sigma}-1\right)\left(1+\frac{1-\sigma}{\sigma}+\left(\frac{1-\sigma}{\sigma}\right)^2\right)=\left(1-\frac{1-\sigma}{\sigma}\right) \quad \text{factorizing by } \left(\frac{1-\sigma}{\sigma}-1\right)$$

$$\Leftrightarrow \left(\frac{1-\sigma}{\sigma}-1\right)\left(\left(\frac{t}{1-\sigma}\right)^2\left(1+\frac{1-\sigma}{\sigma}+\left(\frac{1-\sigma}{\sigma}\right)^2\right)+1\right)=0 \quad \text{factorizing by } \left(\frac{1-\sigma}{\sigma}-1\right)$$

$$\Leftrightarrow \left(\frac{1-\sigma}{\sigma}-1\right)=0 \quad \text{the term } \left(\frac{t}{1-\sigma}\right)^2\left(1+\frac{1-\sigma}{\sigma}+\left(\frac{1-\sigma}{\sigma}\right)^2\right)+1 \text{ being positive}$$

$$\Leftrightarrow \sigma=\frac{1}{2}.$$

The zeta nontrivial zeros being symmetric about the real axis, which can be proved saying that $\zeta(\bar{s})=0$ using $\eta(\bar{s})=\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n^{\bar{s}}}$, with the lemma 3.2 is proposed that the Riemann hypothesis is confirmed. □

Lemma 3.2. *For any s in $\mathbb{C}$ such that $\Im(s)=0$, and $\Re(s)$ in $(0,1)$, $\zeta(s)$ is not null.*

Proof. We proceed by contradiction. Let be $s\in\mathbb{C}$ such that $\Re(s)=\sigma\in(0;1), \Im(s)=0$, and $\zeta(s)=0$. Then $\eta(s)=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^{\sigma}}=0$ and the series $\left(\sum_{n=0}^{m}(2n+2)^{-\sigma}-(2n+1)^{-\sigma}\right)_{m\in\mathbb{N}^*}$ converges to 0, but is a sum of only positive terms. We have the lemma 3.2. □

References

[1] S. Lang. *Algebraic Number Theory.* Addison Wesley, 1970.

[2] B. Riemann. Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse. *Monatsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin*, **7**:136–139, November 1859.

[3] M. R. Spiegel. *Mathematical Handbook of Formulas and Tables*. Schaum. Mac Graw Hill, January 1968.