A Study of Continuous Frames and Operators in Quaternionic Hilbert Spaces

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Abstract

The aim of this work is to study continuous frame theory in quaternionic Hilbert spaces. We provide a characterization of continuous frames in these spaces through the associated operators. Additionally, we examine continuous frames of the form $LF : \Omega \to \mathcal{H}$, where (Ω, μ) is a measure space, $L : \mathcal{H} \to \mathcal{H}$ is a right $\mathbb{H}$-linear bounded operator and $F : \Omega \to \mathcal{H}$ is a continuous frame for $\mathcal{H}$.

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1 Introduction and preliminaries

Frames in quaternionic Hilbert spaces offer a powerful way to analyze and rebuild signals in higher-dimensional spaces. These frames extend the classic theory of frames, helping with efficient data representation and processing. The special features of quaternionic spaces open up new possibilities, especially in areas like signal processing and communication. This work aims to explore the theory of continuous frames in quaternionic Hilbert spaces. The quaternionic field is an extension of the real and complex numbers, made up of numbers called quaternions. Quaternions are used to represent rotations and orientations in three-dimensional space, making them very useful in computer graphics and robotics. Unlike real and complex numbers, quaternion multiplication does not follow the usual commutative rule, which makes their algebra more complex. However, quaternions provide a more efficient way to calculate things in three-dimensional space, which is helpful in fields like physics and engineering.

Definition 1 (The field of quaternions)**.** *The non-commutative field of quaternions $\mathbb{H}$ is a four-dimensional real algebra with unity. In $\mathbb{H}$, 0 denotes the null element and 1 denotes the identity with respect to multiplication. It also includes three so-called imaginary units, denoted by i, j, k. i.e.,*

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k : \ a_0, a_1, a_2, a_3 \in \mathbb{R}\},$$

where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. For each quaternion $q = a_0 + a_1 i + a_2 j + a_3 k$, we deifine the conjugate of q denoted by $\overline{q} = a_0 - a_1 i - a_2 j - a_2 k \in \mathbb{H}$ and the module of q denoted by $|q|$ as

$$|q| = (\overline{q}q)^{\frac{1}{2}} = (q\overline{q})^{\frac{1}{2}} = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}.$$

For every $q \in \mathbb{H}$, $q^{-1} = \dfrac{\overline{q}}{|q|^2}$.

Definition 2 (Right quaternionic vector space)**.** *A right quaternioniq vector space V is a linear vector space under right scalar multiplication over the field of quaternions $\mathbb{H}$, i.e., the right scalar multiplication*

$$\begin{aligned} V \times \mathbb{H} &\to V \\ (v, q) &\mapsto v.q, \end{aligned}$$

satisfies the following for all $u, v \in V$ and $q, p \in \mathbb{H}$:

1. $(v + u).q = v.q + u.q$,
2. $v.(p + q) = v.p + v.q$,
3. $v.(pq) = (v.p).q$.

Instead of $v.q$, we often use the notation vq.

Definition 3 (Right quaterninoic pre-Hilbert space)**.** *A right quaternionic pre-Hilbert space $\mathcal{H}$, is a right quaternionic vector space equipped with the binary mapping $\langle \cdot \mid \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{H}$ (called the Hermitian quaternionic inner product) which satisfies the following properties:*

(a) $\langle v_1 \mid v_2 \rangle = \overline{\langle v_2 \mid v_1 \rangle}$ *for all* $v_1, v_2 \in \mathcal{H}$,
(b) $\langle v \mid v \rangle > 0$ *if* $v \neq 0$,
(c) $\langle v \mid v_1 + v_2 \rangle = \langle v \mid v_1 \rangle + \langle v \mid v_2 \rangle$ *for all* $v, v_1, v_2 \in \mathcal{H}$,
(d) $\langle v \mid uq \rangle = \langle v \mid u \rangle q$ *for all* $v, u \in \mathcal{H}$ *and* $q \in \mathbb{H}$.

In view of Definition 3 , a right pre-Hilbert space $\mathcal{H}$ also has the property:

(i) $\langle vq \mid u \rangle = \overline{q}\langle v \mid u \rangle$ for all $v, u \in \mathcal{H}$ and $q \in \mathbb{H}$.

Let $\mathcal{H}$ be a right quaternionic pre-Hilbert space with the Hermitian inner product $\langle \cdot \mid \cdot \rangle$. Define the quaternionic norm $\| \cdot \| : \mathcal{H} \to \mathbb{R}^+$ on $\mathcal{H}$ by

$$\|u\| = \sqrt{\langle u \mid u \rangle}, \quad u \in \mathcal{H},$$

which satisfies the following properties:

1. $\|uq\| = \|u\||q|$, for all $u \in \mathcal{H}$ and $q \in \mathbb{H}$,
2. $\|u + v\| \leq \|u\| + \|v\|$,
3. $\|u\| = 0 \Longleftrightarrow u = 0$ for $u \in \mathcal{H}$.

Definition 4 (Right quaternionic Hilbert space)**.** *A right quaternionic pre-Hilbert space is called a right quaternionic Hilbert space if it is complete with respect to the quaternionic norm.*
Example 1. *Define*

$$L^2(\Omega, \mathbb{H}) := \left\{ g : \Omega \to \mathbb{H} \text{ measurable } : \int_\Omega |g(\omega)|^2 d\mu(\omega) < \infty \right\}.$$

$L^2(\Omega, \mathbb{H})$ under right multiplication by quaternionic scalars together with the quaternionic inner product defined as: $\langle g \mid h \rangle := \int_\Omega \overline{g(\omega)} h(\omega) d\mu(\omega)$ for all $g, h \in L^2(\Omega, \mathbb{H})$, is a right quaternionic Hilbert space.
Theorem 1 (The Cauchy-Schwarz inequality)**.** *[11] If $\mathcal{H}$ is a right quaternionic Hilbert space, then for all $u, v \in \mathcal{H}$,*

$$|\langle u \mid v \rangle| \leq \|u\|\|v\|.$$

Definition 5 (orthogonality)**.** *Let $\mathcal{H}$ be a right quaternionic Hilbert space and A be a subset of $\mathcal{H}$. Then, define the set:*

- $A^\perp = \{v \in \mathcal{H} : \langle v \mid u \rangle = 0 \ \forall\, u \in A\}$;
- *$\langle A \rangle$ as the right quaternionic vector subspace of $\mathcal{H}$ consisting of all finite right $\mathbb{H}$-linear combinations of elements of A.*

Property 1. *[11] Let $\mathcal{H}$ be a right quaternionic Hilbert space and A be a subset of $\mathcal{H}$. Then,*

1. $A^\perp = \langle A \rangle^\perp = \overline{\langle A \rangle}^\perp = \overline{\langle A \rangle^\perp}$.
2. $(A^\perp)^\perp = \overline{\langle A \rangle}$.
3. $\overline{A} \oplus A^\perp = \mathcal{H}$.

Theorem 2. *[11] Let $\mathcal{H}$ be a quaternionic Hilbert space and let N be a subset of $\mathcal{H}$ such that, for $z, z' \in N$, we have $\langle z \mid z' \rangle = 0$ if $z \neq z'$ and $\langle z \mid z \rangle = 1$. Then, the following conditions are equivalent:*

(a) *For every $u, v \in \mathcal{H}$, the series $\sum_{z\in N}\langle u \mid z\rangle\langle z \mid v\rangle$ converges absolutely and*

$$\langle u \mid v\rangle = \sum_{z\in N}\langle u \mid z\rangle\langle z \mid v\rangle;$$

(b) *For every $u \in \mathcal{H}$, $\|u\|^2 = \sum_{z\in N} |\langle z \mid u\rangle|^2$;*

(c) *$N^{\perp} = \{0\}$;*

(d) *$\langle N\rangle$ is dense in $\mathcal{H}$.*

Definition 6. *A subset N of $\mathcal{H}$ that satisfies one of the statements in Theorem 2 is called Hilbert basis or orthonormal basis for $\mathcal{H}$.*

Theorem 3. *[11] Every quaternionic Hilbert space has a Hilbert basis.*

2 Auxiliary results

In this section, we will present some interesting results on operator theory in quaternionic Hilbert spaces, which will be utilized in our study. The properties of the associated operators of a frame will also be provided. For more details on Quaternionic calculus on Hilbert spaces, the reder can refer to [1], [9], [11], [15].

Definition 7 (Right $\mathbb{H}$-linear operator)**.** *[11] Let $\mathcal{H}$ and $\mathcal{K}$ be two right quaternionic Hilbert spaces. Let $L : \mathcal{H} \to \mathcal{K}$ be a map.*

1. *L is said to be right $\mathbb{H}$-linear operator if $L(uq+vp) = L(u)q+L(v)p$ for all $u, v \in \mathcal{H}$ and $p, q \in \mathcal{H}$.*
2. *If L is a right $\mathbb{H}$-linear operator. L is continuous if and only if L is bounded; i.e., there exists $M > 0$ such that for all $u \in \mathcal{H}$,*

$$\|Lu\| \leq M\|u\|.$$

We denote $\mathbb{B}(\mathcal{H}, \mathcal{K})$ the set of all right $\mathbb{H}$-linear bounded operators from $\mathcal{H}$ to $\mathcal{K}$, and if $\mathcal{H} = \mathcal{K}$, we denote $\mathbb{B}(\mathcal{H})$ instead of $\mathbb{B}(\mathcal{H}, \mathcal{H})$.

3. *If L is a right $\mathbb{H}$-linear bounded operator, we define the norm of L as:*

$$\|L\| = \sup_{\|u\|=1} \|Lu\| = \inf\{M > 0 : \ \|Lu\| \leq M\|u\|, \ \forall u \in \mathcal{H}\}.$$

And we have for all $L, M \in \mathbb{B}(\mathcal{H})$, $\|L + M\| \leq \|L\| + \|M\|$ and $\|MN\| \leq \|L\|\|M\|$.

Theorem 4 (Quaternionic representation Riesz' theorem)**.** *[11] If $\mathcal{H}$ is a right quaternionic Hilbert space, the map*

$$v \in \mathcal{H} \mapsto \langle v \mid \cdot\rangle \in \mathcal{H}'$$

is well-posed and defines a conjugate-$\mathbb{H}$-linear isomorphism.

Theorem 5 (The uniform boundedness principle)**.** *[11] Let $\mathcal{H}$ be a right quaternionic Hilbert space and F be any subset F of $\mathbb{B}(\mathcal{H})$, if*

$$\sup_{L\in F} \|Lv\| < +\infty \quad \text{for every } v \in \mathcal{H},$$

then:

$$\sup_{L\in F} \|L\| < +\infty.$$

Theorem 6 (The open map theorem)**.** *[11] Let $\mathcal{H}$ be a right quaternionic Hilbert space. If $L \in \mathbb{B}(\mathcal{H})$ is surjective, then L is open. In particular, if L is bijective, then $L^{-1} \in \mathbb{B}(\mathcal{H})$.*

Theorem 7 (The closed graph theorem)**.** *[11] Let $\mathcal{H}$ be a right quaternionic Hilbert space and let $L : \mathcal{H} \to \mathcal{H}$ be a right $\mathbb{H}$-linear opeartor. If $Graph(L)$ is closed, then $L \in \mathbb{B}(\mathcal{H})$.*

Definition 8 (the adjoint operator)**.** *[11] Let $\mathcal{H}$ be a right quaternionic Hilbert space and $L \in \mathbb{B}(\mathcal{H})$. The adjoint operator of L, denoted L^*, is the unique operator in $\mathbb{B}(\mathcal{H})$ satisfying for all $u, v \in \mathcal{H}$:*

$$\langle Lu \mid v\rangle = \langle u \mid L^*v\rangle.$$

Definition 9. *[11] Let $\mathcal{H}$ be a right quaternionic Hilbert space and $L \in \mathbb{B}(\mathcal{H})$.*

1. *L is called normal if $LL^* = L^*L$.*
2. *L is called self-adjoint if $L = L^*$.*
3. *L is called isometric if $\|Lu\| = \|u\|$ for all $u \in \mathcal{H}$.*
4. *L is called unitary if $LL^* = L^*L = I$, where I is the identity opertaor of $\mathbb{B}(\mathcal{H})$. An operator is a unitary if and only if it is an isometric surjective operator.*
5. *L is called positive, and we write $L \geqslant 0$, if $\langle Lu \mid u\rangle \geqslant 0$ for all $u \in \mathcal{H}$.*

Proposition 1. *[11] Let $\mathcal{H}$ be a right quaternionic Hilbert space and $L \in \mathbb{B}(\mathcal{H})$. Then:*

1. *$\overline{R(L)}^{\perp} = ker(L^*)$.*
2. *$\overline{R(L^*)} = ker(L)^{\perp}$.*

Theorem 8 (Square root of an operator)**.** *[11] Let $\mathcal{H}$ be a right quaternionic Hilbert space and let $L \in \mathbb{B}(\mathcal{H})$. If $L \geq 0$, then there exists a unique operator in $\mathbb{B}(\mathcal{H})$, indicated by $\sqrt{L}$, such that $\sqrt{L} \geq 0$ and*

$$\sqrt{L}\sqrt{L} = L.$$

Furthermore, it turns out that $\sqrt{L}$ commutes with every operator that commutes with L.

Theorem 9 (The closed range theorem)**.** *[11] Let $\mathcal{H}$ be a right quaternionic Hilbert space and let $L \in \mathbb{B}(\mathcal{H})$. If $R(L)$ is closed, then $R(L^*)$ is also closed.*

Proposition 2. *Let $\mathcal{H}$ be a right quaternionic Hilbert space and let $L \in \mathbb{B}(\mathcal{H})$. If L is bounded below, i.e. there exists $M > 0$ such that for all $u \in \mathcal{H}$, $M\|u\| \leq \|Lu\|$, and L^* is injective, then L is invertible.*

Proof. Since L is bounded below, then it is injective. Assume that L^* is injective, so $\text{Ker}(L^*) = \{0\}$, hence $\overline{R(L)} = \mathcal{H}$. Let us show that $R(L)$ is closed in $\mathcal{H}$. For this, let $y \in \overline{R(L)}$, so there exists a sequence $\{y_n\}_{n\geq 1}$ in $R(L)$ such that $\lim\limits_{n\to\infty} y_n = y$. For each $n \geq 1$, there exists $x_n \in \mathcal{H}$ such that $y_n = L(x_n)$. We have:

$$m\|x_n - x_m\| \leq \|L(x_n) - L(x_m)\| = \|y_n - y_m\|, \quad \forall n, m \geq 1.$$

This implies that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in $\mathcal{H}$, hence convergent, and let x be its limit. Since L is bounded, we have $y = \lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} L(x_n) = L(x)$. Thus, $y \in R(L)$. Therefore, $R(L)$ is closed, so $R(L) = \mathcal{H}$. Thus, L is surjective and therefore invertible. $\square$

Proposition 3. *Let $\mathcal{H}$ be a right quaternionic Hilbert space and let $L \in \mathbb{B}(\mathcal{H})$ be normal. Then, the following statements are equivalent:*

1. *L is invertible.*
2. *L is bounded below.*

Proof. It is clear that 1. implies 2. by taking $M = \dfrac{1}{\|L^{-1}\|}$ as the lower bound. Conversely, we use the fact that $\|Lu\| = \|L^*u\|$, $\forall u \in \mathcal{H}$. Then, we apply Proposition 2. $\square$

Proposition 4. *Let $\mathcal{H}, \mathcal{K}$ be two right quaternionic Hilbert spaces and let $L \in \mathbb{B}(\mathcal{K}, \mathcal{H})$. Then, the following statements are equivalent:*

1. *L is injective with closed range.*
2. *L is bounded below.*
3. *L^* is surjective.*

Proof. Proposition 1 and Theorem 9 together prove the equivalence 1. $\Longleftrightarrow$ 3. Assume that L is injective with closed range, then the operator $L : \mathcal{H} \to R(L)$ is invertible, and since $R(L)$ is closed, it follows, by the open map theorem 6, that $L^{-1} : R(L) \to \mathcal{H}$ is bounded. Now, assume for contradiction that L is not bounded below. Then, for every $n \geq 1$, there exists $u_n \in \mathcal{K}$ with $\|u_n\| = 1$ such that $\dfrac{1}{n} \geq \|L(u_n)\|$, hence $Lu_n \to 0$, which implies $u_n = L^{-1}(L(u_n)) \to 0$. This is a contradiction since $\|u_n\| = 1$, $\forall n \geq 1$. Conversely, assume that L is bounded below, thus, there exists a strictly positive constant M such that for all $x \in \mathcal{K}$, $M\|x\| \leq \|Lx\|$. It is clear that L is injective. Let $\{x_n\}_{n\geq 1} \in \mathcal{K}$ be such that $Lx_n \to y \in \mathcal{H}$ and let's show that $y \in R(L)$. We have

$$\alpha\|x_n - x_m\| \leq \|L(x_n - x_m)\| = \|L(x_n) - L(x_m)\|,$$

thus $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in $\mathcal{K}$, which means it converges to some $x\in\mathcal{K}$. Since L is bounded, $Lx_n\to L(x)$, and by the uniqueness of limits, we conclude that $y=L(x)$ and therefore $y\in R(L)$. Hence, $R(L)$ is closed. $\square$

Let $L\in\mathbb{B}(\mathcal{K},\mathcal{H})$ be a right $\mathbb{H}$-linear bounded operator with closed range, and let $R_L=\mathrm{Im}(L)$ and $N_L=\mathrm{Ker}(L)$. The restriction of L on $N_L^{\perp}$, denoted $L_{|N_L^{\perp}}$, is injective. Indeed, if $x\in N_L^{\perp}$, we have

$$L_{|N_L^{\perp}}(x)=0 \implies x\in N_L^{\perp}\cap N_L=0.$$

On the other hand, we have $L_{|N_L^{\perp}}(N_L^{\perp})=L(\mathcal{K})=R_L$. Therefore:

$$L_{|N_L^{\perp}}:N_L^{\perp}\to R_L$$

is invertible. And since $N_L^{\perp}$ and R_L are closed, it follows, by the open map theorem 6, that:

$$(L_{|N_L^{\perp}})^{-1}:R_L\to N_L^{\perp}$$

is bounded.

Definition 10 (Pseudo-inverse of an operator with closed range)**.** *Let $\mathcal{H}$ and $\mathcal{K}$ be two right quaternionic Hilbert spaces and $L\in\mathbb{B}(\mathcal{K},\mathcal{H})$ be with closed range R_L and kernel N_L. The pseudo-inverse of L, denoted by $L^{\dagger}$, is the right $\mathbb{H}$-linear bounded operator extending $(L_{|N_L^{\perp}})^{-1}:R_L\to N_L^{\perp}$ to $\mathcal{H}$ by the property ker$(L^{\dagger})=R(L^{\perp})$, i.e.,*

$$L^{\dagger}=(L_{|N_L^{\perp}})^{-1}P_{R_L},$$

where P_{R_L} is orthogonoal projection onto R_L.

Theorem 10. *Let $L\in\mathbb{B}(\mathcal{K},\mathcal{H})$ be with closed range R_L, and let $v\in R_L$. The equation*

$$Lx=v$$

admits a unique solution of minimal norm. This solution is exactly $L^{\dagger}(v)$.

Proof. Let us first show that $L^{\dagger}(v)$ is a solution. We have $v\in R_L$, thus $L^{\dagger}(v)=(L_{|N_L^{\perp}})^{-1}(v)$, which gives us $L(L^{\dagger}(v))=v$. Let $g\in\mathcal{K}$ be a solution to this equation. There exist $g_1\in N_L$ and $g_2\in N_L^{\perp}$ such that $g=g_1+g_2$. Let us show that $g_2=L^{\dagger}(v)$: We have $L(g)=L(g_2)$, and since g is a solution to the equation, we get $L(g_2)=v$. Because g_2 and $L^{\dagger}(v)$ belong to $N_L^{\perp}$ and $L_{|N_L^{\perp}}$ is invertible, it follows that $g_2=L^{\dagger}(v)$. Thus, we have $g=g_1+L^{\dagger}(v)$. Consequently,

$$\|g\|^2=\|g_1\|^2+\|L^{\dagger}(v)\|^2,$$

which implies that:

$$\|g\|\geq\|L^{\dagger}(v)\|.$$

Furthermore, we have:

$$\|g\| = \|L^\dagger(v)\| \iff \|g_1\| = 0 \iff g_1 = 0 \iff g = L^\dagger(v).$$

$\square$

Definition 11 (Frames)**.** *Let (Ω, μ) be a measure space and $F : \Omega :\to \mathcal{H}$ be a a weakly measurable mapping, i.e., for all $u \in \mathcal{H}$, $\omega \mapsto \langle u, F\omega \rangle$ is measurable. F is said to be Continuous Frame for $\mathcal{H}$ if there exist $0 < A \leq B < \infty$ such that for all $u \in \mathcal{H}$, the following inequality holds:*

$$A\|u\|^2 \leq \int_\Omega |\langle F\omega, u\rangle|^2 d\mu(\omega) \leq B\|u\|^2.$$

1. *If only the upper inequality holds, F is called a Bessel mapping for $\mathcal{H}$.*
2. *If $A = B = 1$, F is called a Parseval continuous frame for $\mathcal{H}$.*

Example 2. *Let $(\Omega, \mu) := (\mathbb{R}, \lambda)$ the Lebesgue measure space and $\mathcal{H} := \ell^2(\mathbb{Z}, \mathbb{H})$. Let $\{e_n\}_{n\in\mathbb{Z}}$ be the standard Hilbert basis of $\ell^2(\mathbb{Z}, \mathbb{H})$ and define $F : \mathbb{R} \to \ell^2(\mathbb{Z}, \mathbb{H})$ as follows: For all $\omega \in \mathbb{R}$, $F(\omega) = e_{\lfloor \omega \rfloor}$. Then F is a Parseval continuous frame for $\ell^2(\mathbb{Z}, \mathbb{H})$. In fact, for all $u \in \ell^2(\mathbb{Z}, \mathbb{H})$, we have:*

$$\begin{aligned} \int_{\mathbb{R}} |\langle F(\omega), u\rangle|^2 d\lambda(\omega) &= \sum_{n\in\mathbb{Z}} \int_{[n,n+1]} |\langle F(\omega), u\rangle|^2 d\lambda(\omega) \\ &= \sum_{n\in\mathbb{Z}} |\langle e_n, u\rangle|^2 \\ &= \|u\|^2. \end{aligned}$$

Hence, F is a Parseval continuous frame with respect to the Lebesgue measure space.

Thanks to the definition of a Bessel mapping, the following definition is well justified.

Definition 12. *Let (Ω, μ) be a measure space and $F : \Omega \to \mathcal{H}$ be a Bessel mapping for $\mathcal{H}$. The transform operator of F is the right $\mathbb{H}$-linear bounded operator denoted by θ and defined as follows:*

$$\begin{aligned} \theta : \mathcal{H} &\to L^2(\Omega, \mathbb{H}) \\ u &\mapsto \theta(u), \end{aligned}$$

where $\theta(u)\omega := \langle F\omega, u\rangle$ for all $\omega \in \Omega$.

Let $u \in \mathcal{H}$ and $g \in L^2(\Omega, \mathbb{H})$, we have:

$$\begin{aligned}
\langle \theta(u), g\rangle &= \int_\Omega \overline{\theta(u)} g d\mu \\
&= \int_\Omega \overline{\langle F\omega, u\rangle} g(\omega) d\mu(\omega) \\
&= \int_\Omega \langle u, F\omega\rangle g(\omega) d\mu(\omega) \\
&= \int_\Omega \langle u, F(\omega) g(\omega)\rangle d\mu(\omega).
\end{aligned}$$

By the quaternionic represention Riesz' theorem 4, there exists a unique vector in $\mathcal{H}$, denoted by $\displaystyle\int_\Omega F(\omega)g(\omega)d\mu(\omega)$, such that for all $u \in \mathcal{H}$, we have:

$$\int_\Omega \langle u, F(\omega)g(\omega)\rangle d\mu(\omega) = \left\langle u, \int_\Omega F(\omega)g(\omega)d\mu(\omega)\right\rangle. \tag{1}$$

Hence, the adjoint operator of θ is defined as follows:

$$\begin{aligned}
\theta^* : \mathcal{H} &\to L^2(\Omega, \mathbb{H}) \\
u &\mapsto \int_\Omega F(\omega)g(\omega)d\mu(\omega).
\end{aligned}$$

Definition 13 (The pre-frame operator)**.** *Let (Ω, μ) be a measure space and $F : \Omega \to \mathcal{H}$ be a Bessel mapping for $\mathcal{H}$. The pre-frame operator of F, denoted by T, is the adjoint of its transform operator, i.e., $T = \theta^*$.*

Definition 14 (The frame operator)**.** *Let (Ω, μ) be a measure space and $F : \Omega \to \mathcal{H}$ be a Bessel mapping for $\mathcal{H}$. The frame operator of F, denoted by S, is the composite of its pre-frame operator and its transform operator, i.e., $S = T\theta$. By respecting the previous notation in* (1)*, S is defined as follows:*

$$\begin{aligned}
S : \mathcal{H} &\to \mathcal{H} \\
u &\mapsto \int_\Omega F(\omega)\langle F(\omega), u\rangle d\mu(\omega).
\end{aligned}$$

Remark 1. *Let (Ω, μ) be a measure space and $F : \Omega \to \mathcal{H}$ be a Bessel mapping for $\mathcal{H}$. Then, for all $u \in \mathcal{H}$, we have:*

$$\int_\Omega |\langle F(\omega), u\rangle|^2 = \langle Su, u\rangle = \|\theta u\|^2.$$

The following proposition quickly follows from the definition of a frame and Remark 1.

Proposition 5. *Let (Ω, μ) be a measure space and $F : \Omega \to \mathcal{H}$ be a continuous frame of $\mathcal{H}$. Then:*

1. θ is a right $\mathbb{H}$-linear bounded injective operator with closed range.

2. T is a right $\mathbb{H}$-linear bounded surjective operator.
3. S is a right $\mathbb{H}$-linear bounded, positive, and invertible operator.

3 Frame coefficients

In this section, let $\mathcal{H}$ be a right quaternionic Hilbert space, $\mathbb{B}(\mathcal{H})$ be the set of all right $\mathbb{H}$-linear bounded operators on $\mathcal{H}$, and (Ω, μ) be a measure space. We will show that any vector in a right quaternionic Hilbert space can be written (in a generally non-unique way) as a combination (in the integral sens) of the values of a given continuous frame. This vector also has a natural representation with an interesting property.
Proposition 6. *Let $F : \Omega \to \mathcal{H}$ be a continuous frame for $\mathcal{H}$. Then, for all $u \in \mathcal{H}$, we have:*

$$u = \int_\Omega F(\omega)\langle F(\omega), S^{-1}u\rangle d\mu(\omega).$$

- *$\omega \in \Omega \mapsto \langle F(\omega), S^{-1}u\rangle$ is called the frame coefficients for u.*
- *The expression $u = \int_\Omega F(\omega)\langle F(\omega), S^{-1}u\rangle d\mu(\omega)$ is often called the natural representation of u.*

Proof. Let $u \in \mathcal{H}$, we have $u = SS^{-1}u = \int_\Omega F(\omega)\langle F(\omega), S^{-1}u\rangle$. □

Remark 2. *In general, the representation of a vector in a right quaternionic Hilbert space with respect to a continuous frame is not unique, as shown in the following example: Let $(\Omega, \mu) = (\mathbb{N}, card)$ be a measure space and $(v_i)_{i\geq 1}$ be a Hilbert basis of a seperable Hilbert space $\mathcal{H}$. Then the sequence $(v_1, v_1, v_2, v_3, v_4, \ldots)$ is a frame of $\mathcal{H}$. If we set $u = 2v_1$, we can see that u can also be expressed as $u = v_1 + v_1$, which means that $(1, 1, 0, 0, \ldots)$ and $(2, 0, 0, \ldots)$ are two different representations of the same vector u in the frame $(v_1, v_1, v_2, v_3, v_4, \ldots)$.*

The following theorem shows the particularity of the frame coefficients.
Theorem 11. *Let $F : \Omega \to \mathcal{H}$ be a continuous frame and let $u \in \mathcal{H}$ such that $u = \int_\Omega F(\omega)g(\omega)d\mu(\omega)$ where $g \in L^2(\Omega, \mathbb{H})$. Then:*

$$\int_\Omega |g(\omega)|^2 d\mu(\omega) = \int_\Omega \left|\langle F(\omega), S^{-1}u\rangle\right|^2 d\mu(\omega) + \int_\Omega \left|\langle F(\omega), S^{-1}u\rangle - g(\omega)\right|^2 d\mu(\omega).$$

In particular, the frame coefficients $\omega \in \Omega \mapsto \langle F(\omega), S^{-1}u\rangle$ of u with respect to the continious frame F represent the representation with minimal $L^2(\Omega, \mathbb{H})$-norm.

Proof. By proposition 6, we have that $\int_\Omega F(\omega)\langle F(\omega), S^{-1}u\rangle = \int_\Omega F(\omega)g(\omega)$. Then, by multiplying both terms by $S^{-1}u$ in the left (in the sens of the inner product), we obtain:

$$\begin{aligned}
&\left\langle S^{-1}u, \int_\Omega F(\omega)\langle F(\omega), S^{-1}u\rangle \right\rangle = \left\langle S^{-1}u, \int_\Omega F(\omega)g(\omega)d\mu(\omega) \right\rangle \\
\Longrightarrow &\int_\Omega \left\langle S^{-1}u, F(\omega)\langle F(\omega), S^{-1}u\rangle \right\rangle = \int_\Omega \langle S^{-1}u, F(\omega)\rangle g(\omega)d\mu(\omega) \\
\Longrightarrow &\int_\Omega |\langle F(\omega), S^{-1}u\rangle|^2 d\mu(\omega) = \int_\Omega \overline{\langle F(\omega), S^{-1}u\rangle} g(\omega)d\mu(\omega).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_\Omega \left|\langle F(\omega), S^{-1}u\rangle - g(\omega)\right|^2 d\mu(\omega) \\
= &\int_\Omega |\langle F(\omega), S^{-1}u\rangle|^2 d\mu(\omega) - 2Re\int_\Omega \overline{\langle F(\omega), S^{-1}u\rangle} g(\omega)d\mu(\omega) + \int_\Omega |g(\omega)|^2 d\mu(\omega).
\end{aligned}$$

Hence,

$$\int_\Omega |g(\omega)|^2 d\mu(\omega) = \int_\Omega \left|\langle F(\omega), S^{-1}u\rangle\right|^2 d\mu(\omega) + \int_\Omega \left|\langle F(\omega), S^{-1}u\rangle - g(\omega)\right|^2 d\mu(\omega).$$

□

Lemma 1. *Let $F : \Omega \to \mathcal{H}$ be a continuous frame. Then, for all $u \in \mathcal{H}$, we have:*

$$Tu(\omega) = \langle F(\omega), S^{-1}u\rangle \quad (\text{ for all } \omega \in \Omega).$$

i.e., $T^\dagger = \theta S^{-1}$.

Proof. T is a surjective right $\mathbb{H}$-linear bounded operator, then $R(T) = \mathcal{H}$. Let $u \in \mathcal{H}$ and consider the following equation $(E) : Tx = u$. By Theorem 10, The equation (E) has a unique solution with minimal norm which is exactly $T^\dagger u$. On the other hand, $g \in L^2(\Omega, \mathbb{H})$ is a solution to (E), if and only if, $\int_\Omega F(\omega)g(\omega)d\mu(\omega) = u$, then the unique solution with minimal norm of (E) is, by Theorem 11, $\omega \mapsto \langle F(\omega), S^{-1}u\rangle$. Then, $T^\dagger u(\omega) = \langle F(\omega), S^{-1}u\rangle = \theta S^{-1}u(\omega)$ (for all $\omega \in \Omega$). Hence, $T^\dagger = \theta S^{-1}$. □

By definition of a continuous frame, the optimal frame bounds are given by:

$$A_{opt} := \inf_{\|u\|=1} \langle Su, u\rangle, \quad B_{opt} := \sup_{\|u\|=1} \langle Su, u\rangle.$$

The following theorem express the optimal frame bounds of a continuous frame using its assiciated operators.

Theorem 12. *Let $F : \Omega \to \mathcal{H}$ be a continuous frame, T and S be, respectively, its pre-frame operator and its frame operator. Let $A_{opt} \leq B_{opt}$ be the optimal frame bounds of F. Then:*

1. $A_{opt} = \dfrac{1}{\|S^{-1}\|} = \dfrac{1}{\|T^{\dagger}\|^2}$.

2. $B_{opt} = \|S\| = \|T\|^2$.

Proof. We have $\langle Su, u\rangle = \langle S^{\frac{1}{2}}u, S^{\frac{1}{2}}u\rangle = \|S^{\frac{1}{2}}u\|^2$, then $B_{opt} = \|S^{\frac{1}{2}}\|^2 = \|S\|$ and $A_{opt} = \dfrac{1}{\|S^{\frac{-1}{2}}\|^2} = \dfrac{1}{\|S^{-1}\|}$. On the other hand, we have $\|S\| = \|TT^*\| = \|T\|^2$ and since $T^{\dagger} = T^*S^{-1}$, then $\|T^{\dagger}\|^2 = \|T^{\dagger^*}T^{\dagger}\| = \|S^{-1}TT^*S^{-1}\| = \|S^{-1}SS^{-1}\| = \|S^{-1}\|$. $\square$

Let $F : \Omega \to \mathcal{H}$ be a continuous frame for $\mathcal{H}$ with the optimal frame bounds $A \leq B$. If $R \in \mathbb{B}(\mathcal{H})$ and $u \in \mathcal{H}$, the frame coefficients of Ru are $\omega \in \Omega \mapsto \langle F(\omega), S^{-1}Ru\rangle$. An interesting question arises: Can we determine the frame coefficients of Ru from those of u?

We consider the map defined as follows:

$$\begin{aligned} \Lambda : L^2(\Omega, \mathbb{H}) &\to L^2(\Omega, \mathbb{H}) \\ g &\mapsto \Lambda g, \end{aligned}$$

where $\Lambda g(\omega_0) = \displaystyle\int_{\Omega} \langle S^{-1}F(\omega_0), RF(\omega)\rangle g(\omega) d\mu(\omega)$ (for all $\omega_0 \in \Omega$).

Proposition 7. *Λ is a well defined right $\mathbb{H}$-linear bounded operator. i.e., $\Lambda \in \mathbb{B}(\mathcal{H})$.*

Proof. Let $g \in L^2(\Omega, \mathcal{H})$. It is clear that $\displaystyle\int_{\Omega} \langle S^{-1}F(\omega_0), RF(\omega)\rangle g(\omega) d\mu(\omega) \in \mathbb{H}$ (for all $\omega_0 \in \Omega$) since $\omega \mapsto \langle S^{-1}F(\omega_0), RF(\omega)\rangle \in L^2(\Omega, \mathbb{H})$ and $g \in L^2(\Omega, \mathbb{H})$. Then, we have:

$$\begin{aligned} &\int_{\Omega} \left| \int_{\Omega} \langle S^{-1}F(\omega_0), RF(\omega)\rangle \, g(\omega) d\mu(\omega) \right|^2 d\mu(\omega_0) \\ &= \int_{\Omega} \left| \int_{\Omega} \left\langle R^*S^{-1}F(\omega_0), F(\omega)g(\omega) \right\rangle d\mu(\omega) \right|^2 d\mu(\omega_0) \\ &= \int_{\Omega} \left| \left\langle R^*S^{-1}F(\omega_0), \int_{\Omega} F(\omega)g(\omega) d\mu(\omega) \right\rangle \right|^2 d\mu(\omega_0) \\ &= \int_{\Omega} |\langle R^*S^{-1}F(\omega_0), Tg\rangle|^2 d\mu(\omega_0) \\ &= \int_{\Omega} \left|\langle F(\omega_0), S^{-1}RTg\rangle\right|^2 d\mu(\omega_0) \\ &\leq B\|S^{-1}RTg\|^2 \\ &\leq B\|S^{-1}\|^2\|R\|^2\|T\|^2\|g\|^2 = \left(\frac{B\|R\|}{A}\right)^2 \|g\|^2 . \end{aligned}$$

Hence, Λ is well defined, clearly right $\mathbb{H}$-linear and bounded operator. Moreover:

$$\|\Lambda\| \leq \frac{B\|R\|}{A}.$$

$\square$

In the following proposition, we present an answer to the above question.

Proposition 8. *For all $u \in \mathcal{H}$, the frame coefficients of Ru are obtained from those of u via the right $\mathbb{H}$-linear bounded operator Λ.*

Proof. Let $u \in \mathcal{H}$ and denote $g : \omega \in \Omega \mapsto \langle F(\omega), S^{-1}u \rangle$. For all $\omega_0 \in \Omega$, we have:

$$\begin{aligned}
\Lambda g(\omega_0) &= \int_\Omega \langle S^{-1}F(\omega_0), RF(\omega) \rangle \langle F(\omega), S^{-1}u \rangle d\mu(\omega) \\
&= \int_\Omega \left\langle S^{-1}F(\omega_0), RF(\omega)\langle F(\omega), S^{-1}u \rangle \right\rangle d\mu(\omega) \\
&= \int_\Omega \left\langle R^* S^{-1}F(\omega_0), F(\omega)\langle F(\omega), S^{-1}u \rangle \right\rangle d\mu(\omega) \\
&= \left\langle R^* S^{-1}F(\omega_0), \int_\Omega F(\omega)\langle F(\omega), S^{-1}u \rangle \right\rangle \\
&= \langle R^* S^{-1}F(\omega_0), u \rangle \\
&= \langle F(\omega_0), S^{-1}Ru \rangle.
\end{aligned}$$

□

4 Continuous frames and operators on quaternionic Hilbert spaces

In this section, let $\mathcal{H}$ be a right quaternionic Hilbert space, $\mathbb{B}(\mathcal{H})$ be the set of all right $\mathbb{H}$-linear bounded operators on $\mathcal{H}$, and (Ω, μ) be a measure space. We will characterize continuous frames in a right quaternionic Hilbert space by their associated operators. Specifically, we study continuous frames of the form $\{LF\}$, where $L \in \mathbb{B}(\mathcal{H})$ and $F : \Omega \to \mathcal{H}$ is a continuous frame for $\mathcal{H}$.
We begin by giving a characterization of continuous frames using the pre-frame operator.
Theorem 13. *Let $F : \Omega \to \mathcal{H}$ be a weakly measurable mapping and T be its pre-frame operator. Then, the following statements are equivalent:*

1. *F is a frame for $\mathcal{H}$.*
2. *T is well defined, bounded and surjective.*

Proof. We have already seen that 1. implies 2. Conversely, since T is surjective, then, by Proposition 4, T^* (which is well defined and bounded since T is bounded) is bounded below. The fact that $\int_\Omega |\langle F(\omega), u \rangle|^2 d\mu(\omega) = \|T^*u\|^2$ completes the proof. □

Now, we characterize frames by the frame operator.
Theorem 14. *Let $F : \Omega \to \mathcal{H}$ be a mapping and S be its frame operator. Then, the following statements are equivalent:*

1. *$F : \Omega \to \mathcal{H}$ is a continuous frame for $\mathcal{H}$.*
2. *S is well defined and surjective.*

Proof. It is well known that 1. implies 2.. Assume that:

$$\begin{aligned} S : \mathcal{H} &\to \mathcal{H} \\ u &\mapsto \int_\Omega F(\omega)\langle F(\omega), u\rangle d\mu(\omega), \end{aligned}$$

is well defined and surjective. Let $\{w_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ such that $w_n \to w \in \mathcal{H}$ and $Sw_n \to w' \in \mathcal{H}$. Since S is surjective, then there exists $v \in \mathcal{H}$ such that $w' = Sv$. Then, $Sw_n \to Sv$ and $w_n \to w \implies \langle S(w_n - v), w_n - v\rangle \to 0$ (Cauchy-Schwarz inequality), then $\int_\Omega |\langle F(\omega), w_n - v\rangle|^2 d\mu(\omega)$ converges to 0 (by definition of S), then $\omega \mapsto \langle F(\omega), w_n\rangle$ converges in $L^2(\Omega, \mathbb{H})$ to $\omega \mapsto \langle F(\omega), v\rangle$. thus, There exists a subsequence $\langle F(\omega), w_{n_k}\rangle$ wich converge for a.e $\omega \in \Omega$ to $\langle F(\omega), v\rangle$. Then, for a.e $\omega \in \Omega$, $\langle F(\omega), w\rangle = \langle F(\omega), v\rangle$ since $w_n \to w$. Then, $\int_\Omega F(\omega)\langle F(\omega), w\rangle d\mu(\omega) = \int_\Omega F(\omega)\langle F(\omega), v\rangle d\mu(\omega)$, thus $S(w) = S(v) = w'$. Then, the graph of S is closed, hence, by the closed graph theorem 7, S is bounded. Since S is positive and surjective, then it is invertible. Thus, $S^{\frac{1}{2}}$ is positive and invertible. Hence the existence of $0 < A \leq B$ such that for all $u \in \mathcal{H}$, $A\|u\|^2 \leq \|S^{\frac{1}{2}}u\|^2 \leq B\|u\|^2$. The fact that $\|S^{\frac{1}{2}}u\|^2 = \langle Su, u\rangle = \int_\Omega |\langle F(\omega), u\rangle|^2 d\mu(\omega)$ completes the proof. □

Let L be a right $\mathbb{H}$-linear bounded operator and $F : \Omega \to \mathcal{H}$ be a continuous frame for $\mathcal{H}$. In what follows, we study the mapping LF.

In general, the image of a continous frame under a right $\mathbb{H}$-linear bounded operator (even if it is injective) is not a continuous frame, as illustrated by the following example: Consider $(\Omega, \mu) := (\mathbb{N}, \text{card})$, $\{v_i\}_{i\geqslant 1}$, a Hilbert basis of a separable right quaternionic Hilbert space $\mathcal{H}$, and the following injective bounded linear operator:

$$\begin{aligned} L : \mathcal{H} &\to \mathcal{H} \\ x &\mapsto \sum_{i=1}^{+\infty} v_{i+1}\langle v_i, x\rangle. \end{aligned}$$

We have $\{L(v_i)\}_{i\geqslant 1} = \{v_i\}_{i\geqslant 2}$, which is not even complete.

The following proposition expresses the frame operator of LF whenever it forms a continuous frame (or only a Bessel sequence).

Proposition 9. *Let $L \in \mathbb{B}(\mathcal{H})$ and $F : \Omega \to \mathcal{H}$ be a continuous frame for $\mathcal{H}$.If LF is a continuous frame, then its frame operator is LSL^*.*

Proof. Denote by S_L the frame operator for LF. For all $u \in \mathcal{H}$, we have:

$$\begin{aligned} \langle S_L u, u\rangle = \int_\Omega |\langle LF(\omega), u\rangle|^2 d\mu(\omega) &= \int_\Omega |\langle F(\omega), L^*u\rangle|^2 d\mu(\omega) \\ &= \langle SL^*u, L^*u\rangle \\ &= \langle LSL^*u, u\rangle. \end{aligned}$$

Hence, $S_L = LSL^*$. $\square$

The next theorem presents a necessary and sufficient condition on $L \in \mathbb{B}(\mathcal{H})$ for it to transform one continous frame into another.

Theorem 15. *Let $L \in \mathbb{B}(\mathcal{H})$ and $F : \Omega \to \mathcal{H}$ be a continous frame for $\mathcal{H}$ with frame bounds $A \leq B$. Then, LF is a continuous frame for $\mathcal{H}$ if and only if L is surjective.*

Proof. Assume that LF is a continuous frame for $\mathcal{H}$ with frame operator S and let $v \in \mathcal{H}$. Then, $v = \displaystyle\int_\Omega LF(\omega)\langle LF(\omega), S^{-1}v\rangle d\mu(\omega)$. Let $u \in \mathcal{H}$, we have:

$$\begin{aligned}
\left\langle \int_\Omega LF(\omega)\langle LF(\omega), S^{-1}v\rangle d\mu(\omega),\, u \right\rangle &= \int_\Omega \left\langle LF(\omega)\langle LF(\omega), S^{-1}v\rangle, u\right\rangle d\mu(\omega) \\
&= \int_\Omega \left\langle F(\omega)\langle LF(\omega), L^*S^{-1}v\rangle, u\right\rangle d\mu(\omega) \\
&= \left\langle \int_\Omega F(\omega)\langle LF(\omega), S^{-1}v\rangle d\mu(\omega),\, L^*u \right\rangle \\
&= \left\langle L\left(\int_\Omega F(\omega)\langle LF(\omega), S^{-1}v\rangle d\mu(\omega)\right),\, u \right\rangle.
\end{aligned}$$

Then, by uniqueness, we have:

$$\int_\Omega LF(\omega)\langle LF(\omega), S^{-1}v\rangle d\mu(\omega) = L\left(\int_\Omega F(\omega)\langle LF(\omega), S^{-1}v\rangle d\mu(\omega)\right).$$

Thus, $v = Lw$, where $w = \displaystyle\int_\Omega F(\omega)\langle LF(\omega), S^{-1}v\rangle d\mu(\omega) \in \mathcal{H}$. Hence, L is surjective. Conversely, by Proposition 4, L is surjective if and only if L^* is bounded below. Then there exists $M > 0$ such that for all $u \in \mathcal{H}$, $M\|u\| \leq \|L^*u\|$. Let $u \in \mathcal{H}$, we have:

$$\int_\Omega |\langle LF(\omega), u\rangle|^2 d\mu(\omega) = \int_\Omega |\langle F(\omega), L^*u\rangle|^2 d\mu(\omega).$$

Then,

$$A\|L^*u\|^2 \leq \int_\Omega |\langle LF(\omega), u\rangle|^2 d\mu(\omega) \leq B\|L^*u\|^2.$$

Thus,

$$AM^2\|u\|^2 \leq \int_\Omega |\langle LF(\omega), u\rangle|^2 d\mu(\omega) \leq B\|L\|^2\|u\|^2.$$

Hence, LF is a continuous frame for $\mathcal{H}$ with frame bounds AM^2 and $B\|L\|^2$. $\square$

Corollary 1. *Let $F : \Omega \to \mathcal{H}$ be a continuous frame for $\mathcal{H}$ with frame bounds $A \leq B$ and let S be its frame operator. Then:*

1. *$S^{-1}F$ is a frame for $\mathcal{H}$ with frame bounds $\frac{1}{B}$ and $\frac{1}{A}$. $S^{-1}F$ is called the canonical dual frame for F.*
2. *$\S^{\frac{-1}{2}}F$ is a Parseval continuous frame for $\mathcal{H}$.*

Proof. 1. By Proposition 9, Theorem 12 and Theorem 15, we deduce easily that $S^{-1}F$ is a frame with the optimal frame bounds $\dfrac{1}{B_{opt}}$ and $\dfrac{1}{A_{opt}}$ where A_{opt} and B_{opt} are the optimal frame bounds of F. Next, we can deduce the same result for arbitrary frame bounds.

2. In view of Proposition 9, Theorem 12 and Theorem 15 again, we deduce that $S^{\frac{-1}{2}}F$ is a frame for $\mathcal{H}$ with the optimal frame bounds 1.

$\square$

The following result shows that unitary right $\mathbb{H}$-linear bounded operators transform continuous frames to other continuous frames with the same frame bounds.

Proposition 10. *Let $F : \Omega \to \mathcal{H}$ be a frame for $\mathcal{H}$, S be its frame operator and let $U \in \mathbb{B}(\mathcal{H})$ be a unitary right $\mathbb{H}$-linear bounded operator. Then UF is a continuous frame for $\mathcal{H}$ with the same frame bounds.*

Proof. By Theorem 15, UF is a continuous frame for $\mathcal{H}$. It sufficies to show that F and UF have the same optimal frame bounds. i.e. in view of Theorem 12 and Proposition 9, we will show that $\|USU^*\| = \|S\|$ and $\|(USU^*)^{-1}\| = \|S^{-1}\|$. We have:

$$\begin{aligned}
\|USU^*\| &= \sup_{\|u\|=1} \|USU^*(u)\| \\
&= \sup_{\|u\|=1} \|SU^*(u)\| \\
&= \|SU^*\| = \|US\| \\
&= \sup_{\|u\|=1} \|US(u)\| \\
&= \sup_{\|u\|=1} \|S(u)\| = \|S\|.
\end{aligned}$$

And:

$$\begin{aligned}
\|(USU^*)^{-1}\| = \|US^{-1}U^*\| &= \sup_{\|u\|=1} \|US^{-1}U^*(u)\| \\
&= \sup_{\|u\|=1} \|S^{-1}U^*(u)\| \\
&= \|S^{-1}U^*\| = \|US^{-1}\| \\
&= \sup_{\|u\|=1} \|US^{-1}(u)\| \\
&= \sup_{\|u\|=1} \|S^{-1}(u)\| = \|S^{-1}\|.
\end{aligned}$$

$\square$

In the following result, we study PF where $F : \Omega \to \mathcal{H}$ is a continuous frame for $\mathcal{H}$ and P is an orthogonal projection of $\mathcal{H}$.

Proposition 11. *Let $F : \Omega \to \mathcal{H}$ be a frame of $\mathcal{H}$ with frame bounds A and B, and let P be an orthogonal projection of $\mathcal{H}$. Then PF is a continuous frame of $P(\mathcal{H})$ with bounds A and B.*

Proof. Let $u \in P(\mathcal{H})$, then $Pu = u$ and $\langle PF(\omega), u\rangle = \langle F(\omega), Pu\rangle = \langle F(\omega), u\rangle$. $\square$

Corollary 2. *An orthogonal projection P transforms a Parseval continuous frame for $\mathcal{H}$ into a Parseval continuous frame for $P(\mathcal{H})$.*

The following theorem provides conditions under which one continuous frame is the image of another under a right $\mathbb{H}$-linear bounded operator.

Theorem 16. *Let $F : \Omega \to \mathcal{H}$ and $G : \Omega \to \mathcal{H}$ be two continuous frames of $\mathcal{H}$ with pre-frame operators T_1 and T_2, respectively. We define the relation L by: $L\left(\int_\Omega F(\omega)g(\omega)d\mu(\omega)\right) = \int_\Omega G(\omega)g(\omega)d\mu(\omega)$ for all $g \in L^2(\Omega,\mathbb{H})$. Then, the following statements are equivalent:*

1. L is well defined as a right $\mathbb{H}$-linear bounded opeartor on $\mathcal{H}$.
2. $\mathrm{Ker}(T_1) \subset \mathrm{Ker}(T_2)$.

Proof. Since for all $v \in \mathcal{H}$, $v = \int_\Omega F(\omega)\langle F(\omega), S^{-1}v\rangle d\mu(\omega)$, where S is the frame operator of F, then it makes sense to consider L is a relation on $\mathcal{H}$.

Assume that L is well a right $\mathbb{H}$-linear bounded operator on $\mathcal{H}$. Let $g \in L^2(\Omega,\mathbb{H})$, we have:

$$\begin{aligned} T_1(q) = 0 &\Longrightarrow \int_\Omega F(\omega)g(\omega)d\mu(\omega) = 0 \\ &\Longrightarrow L\left(\int_\Omega F(\omega)g(\omega)d\mu(\omega)\right) = 0 \\ &\Longrightarrow \int_\Omega G(\omega)g(\omega)d\mu(\omega) = 0 \\ &\Longrightarrow T_2(g) = 0. \end{aligned}$$

Hence, $\mathrm{Ker}(T_1) \subset \mathrm{Ker}(T_2)$. Conversely, assume that $\mathrm{Ker}(T_1) \subset \mathrm{Ker}(T_2)$. Let's show, first, that L is well defined. Let $g, h \in L^2(\Omega,\mathbb{H})$, then we have:

$$\begin{aligned} \int_\Omega F(\omega)g(\omega)d\mu(\omega) = \int_\Omega F(\omega)h(\omega)d\mu(\omega) &\Longrightarrow T_1(g) = T_1(h) \\ &\Longrightarrow T_1(g-h) = 0 \\ &\Longrightarrow T_2(g-h) = 0 \\ &\Longrightarrow T_2(g) = T_2(h) \\ &\Longrightarrow \int_\Omega G(\omega)g(\omega)d\mu(\omega) = \int_\Omega G(\omega)h(\omega)d\mu(\omega) \end{aligned}$$

Hence, L is well defined. It is clear that L is right $\mathbb{H}$-linear. Let's show, now, that L is bounded. Let $T = T_1$ or T_2, it is clear that $T_{|kert(T)^\perp} : Ker(T)^\perp \to \mathcal{H}$ is invertible since $T : L^2(\Omega,\mathbb{H}) \to \mathcal{H}$ is sujective. Then, by the open map theorem 6, $(T_{|kert(T)^\perp})^{-1}$ is also bounded, and then for all $g \in ker(T)^\perp$, we have:

$$\alpha\|g\| \leq \|T_{|ker(T)^\perp}q\| \leq \beta\|g\|,$$

where $\alpha = \dfrac{1}{\|(T_{|ker(T)^\perp})^{-1}\|}$ and $\beta = \|T_{|ker(T)^\perp}\|$. Let, now, $g \in L^2(\Omega,\mathbb{H})$ and denote $g_1 := Proj_{ker(T_1)^\perp}g$ and $g_2 := Proj_{ker(T_2)^\perp}g$, where $Proj_F$ is the orthogonal projection onto F for $F \subset L^2(\Omega, \mathbb{H})$ closed. We have:

$$\begin{aligned}
\left\| L\left(\int_\Omega F(\omega)g(\omega)d\mu(\omega)\right)\right\| &= \|\int_\Omega G(\omega)g(\omega)d\mu(\omega)\| \\
&= \|T_{2_{|ker(T_2)^\perp}}(g_2)\| \\
&\leq \beta_2\|g_2\| \\
&\leq \beta_2\|g_1\| \quad \text{since } ker(T_1) \subset ker(T_2) \\
&\leq \frac{\beta_2}{\alpha_1}\|T_{1_{ker(T_1)^\perp}}(g_1)\| \\
&= \frac{\beta_2}{\alpha_1}\|T_1(g)\| \\
&= \frac{\beta_2}{\alpha_1}\left\|\int_\Omega F(\omega)g(\omega)d\mu(\omega)\right\|
\end{aligned}$$

Then L is bounded.

$\square$

Definition 15. *Let $F : \Omega \to \mathcal{H}$ and $G : \Omega \to \mathcal{H}$ be two mappings. We say that F and G are similar, and we write $F \approx G$, if there exists a right $\mathbb{H}$-linear bounded invertible operator $L : \mathcal{H} \to \mathcal{H}$ such that for almost every $\omega \in \Omega, LF(\omega) = G(\omega)$.*

Remark 3. *In view of the open map theorem 6, we have that:*

$$F \approx G \Longleftrightarrow G \approx F.$$

Example 3. *Each continuous frame on $\mathcal{H}$ is similar to a Parseval continuous frame. Indeed, we have already seen, in Corollary 1, that if F is a continuous frame for $\mathcal{H}$ with the frame operator S, then $S^{\frac{-1}{2}}F$ is a Parseval continuous frame for $\mathcal{H}$.*

The following theorem provides sufficient and necessary conditions under which two continuous frames are similar.

Theorem 17. *Let F and G be two continuous frames of $\mathcal{H}$ with pre-frame operators T_1 and T_2, respectively. Then the following statements are equivalent.*

1. $F \approx G$.
2. $ker(T_1) = ker(T_2)$.

Proof. Assume that $F \approx G$, then there exists a right $\mathbb{H}$-linear, bounded and invertible operator L such that for almost every $\omega \in \Omega$, $LF(\omega) = G(\omega)$. Then, by Theorem 16, $ker(T_1) \subset ker(T_2)$. By the open map theorem 6, L^{-1} is also a right $\mathbb{H}$-linear bounded operator and since $F = L^{-1}G$, then by Theorem 16, $ker(T_2) \subset ker(T_1)$. Hence $ker(T_1) = ker(T_2)$. Conversely, assume that $ker(T_1) = ker(T_2)$, then by Theorem 16, the mappings L_1 and L_2 verifying $L_1F = G$ and $L_2G = F$ (almost everywhere) are well defined and are two right $\mathbb{H}$-linear bounded operators. Moreover, $L_1L_2 = L_2L_1 = I$, where I is the identity operator of $\mathbb{B}(\mathcal{H})$. Hence F and G are similar. $\square$

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