Solution to the problem on the existence and smoothness of the Navier–Stokes equations

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March 16, 2025

The Navier–Stokes equations are used to describe viscous incompressible fluid flow. It has been on the list of the Clay Mathematics Institute's millennium prize problems to decide whether or not physically reasonable solutions to the Navier–Stokes equations do in general exist. In this paper, the problem on the existence and smoothness of the Navier–Stokes equations is solved. It is proven that the Navier–Stokes equations are globally regular.

1. Introduction

The Navier–Stokes equations are thought to govern the motion of a fluid in $\mathbb{R}^3$, [1–6]. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^3$ be the fluid velocity and let $p = p(\mathbf{x}, t) \in \mathbb{R}$ be the fluid pressure, each dependent on position $\mathbf{x} \in \mathbb{R}^3$ and time $t \geqslant 0$. We take the externally applied force acting on the fluid to be identically zero. The fluid is assumed to be incompressible with constant viscosity $\nu > 0$ and to fill all of $\mathbb{R}^3$. The Navier–Stokes equations can then be written as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \nabla^2 \mathbf{u} - \nabla p, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

with initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^\circ \tag{3}$$

where $\mathbf{u}^\circ = \mathbf{u}^\circ(\mathbf{x}) \in \mathbb{R}^3$. In these equations

$$\nabla = \left(\frac{\partial}{\partial \mathbf{x}_1}, \frac{\partial}{\partial \mathbf{x}_2}, \frac{\partial}{\partial \mathbf{x}_3} \right) \tag{4}$$

is the gradient operator and

$$\nabla^2 = \sum_{i=1}^{3} \frac{\partial^2}{\partial {\mathbf{x}_i}^2} \tag{5}$$

is the Laplacian operator. When $\nu = 0$ equations (1), (2), (3) are called the Euler equations. Solutions of (1), (2), (3) are to be found with

$$\mathbf{u}^\circ(\mathbf{x} + e_i) = \mathbf{u}^\circ(\mathbf{x}) \tag{6}$$

for $1 \leqslant i \leqslant 3$ where e_i is the i^{th} unit vector in $\mathbb{R}^3$. The initial condition $\mathbf{u}^\circ$ is a given C^∞ divergence-free vector field on $\mathbb{R}^3$. A solution of (1), (2), (3) is then accepted to be physically reasonable [3] if

$$\mathbf{u}(\mathbf{x} + e_i, t) = \mathbf{u}(\mathbf{x}, t), \quad p(\mathbf{x} + e_i, t) = p(\mathbf{x}, t) \tag{7}$$

on $\mathbb{R}^3 \times [0, \infty)$ for $1 \leqslant i \leqslant 3$ and

$$\mathbf{u}, p \in C^\infty\left(\mathbb{R}^3 \times [0, \infty)\right). \tag{8}$$

2. Solution to the Navier–Stokes problem

Theorem 1. Take $\nu > 0$. Let $\mathbf{u}^\circ$ be any smooth, divergence-free vector field satisfying (6). Then there exist smooth functions $\mathbf{u}, p$ on $\mathbb{R}^3 \times [0, \infty)$ that satisfy (1), (2), (3), (7), (8).

Proof. We consider a generalised Navier–Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nu \diamond^\gamma \mathbf{u} - \nabla p \tag{9}$$

along with (2), (3), (6), (7) where $\gamma \geqslant 1$. For a Fourier series

$$\tilde{\mathbf{f}} = \sum_{\mathbf{L}} \mathbf{f}_{\mathbf{L}} \mathrm{e}^{\mathrm{i}c\mathbf{L}\cdot\mathbf{x}} \tag{10}$$

where $\mathbf{f}_{\mathbf{L}} = \mathbf{f}_{\mathbf{L}}(t) \in \mathbb{C}^3$, $\mathrm{i} = \sqrt{-1}$, $c \in \mathbb{R}$ is a constant, and $\sum_{\mathbf{L}}$ denotes the sum over all $\mathbf{L} \in \mathbb{Z}^3$, we have

$$\diamond^\gamma \tilde{\mathbf{f}} = \sum_{\mathbf{L}} |c\mathbf{L}|^\gamma \mathbf{f}_{\mathbf{L}} \mathrm{e}^{\mathrm{i}c\mathbf{L}\cdot\mathbf{x}}. \tag{11}$$

Equation (1) is recovered when $\gamma = 2$. Let the Fourier series of $\mathbf{u}, p$ be

$$\tilde{\mathbf{u}} = \sum_{\mathbf{L}} \mathbf{u}_{\mathbf{L}} \mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}}, \tag{12}$$

$$\tilde{p} = \sum_{\mathbf{L}} p_{\mathbf{L}} \mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}} \tag{13}$$

respectively. Here $\mathbf{u}_{\mathbf{L}} = \mathbf{u}_{\mathbf{L}}(t) \in \mathbb{C}^3$, $p_{\mathbf{L}} = p_{\mathbf{L}}(t) \in \mathbb{C}$, and $k = 2\pi$ to satisfy (7). The initial condition $\mathbf{u}^\circ$ is a Fourier series [2] of which is convergent for all $\mathbf{x} \in \mathbb{R}^3$. Substituting $\mathbf{u} = \tilde{\mathbf{u}}$, $p = \tilde{p}$ into (9) gives

$$\begin{aligned} &\sum_{\mathbf{L}} \frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} \mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}} + \sum_{\mathbf{L}} \sum_{\mathbf{M}} (\mathbf{u}_{\mathbf{L}} \cdot \mathrm{i}k\mathbf{M}) \mathbf{u}_{\mathbf{M}} \mathrm{e}^{\mathrm{i}k(\mathbf{L}+\mathbf{M})\cdot\mathbf{x}} \\ &= -\sum_{\mathbf{L}} \nu (k|\mathbf{L}|)^\gamma \mathbf{u}_{\mathbf{L}} \mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}} - \sum_{\mathbf{L}} \mathrm{i}k\mathbf{L} p_{\mathbf{L}} \mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}}. \end{aligned} \tag{14}$$

Equating like powers of the exponentials in (14) yields

$$\frac{\partial \mathbf{u_L}}{\partial t} + \sum_{\mathbf{M}} (\mathbf{u_{L-M}} \cdot \mathrm{i}k\mathbf{M})\mathbf{u_M} = -\nu(k|\mathbf{L}|)^{\gamma}\mathbf{u_L} - \mathrm{i}k\mathbf{L}p_{\mathbf{L}} \tag{15}$$

on using the Cauchy product type formula

$$\sum_{l=-\infty}^{\infty} a_l x^l \sum_{m=-\infty}^{\infty} b_m x^m = \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{l-m} b_m x^l \tag{16}$$

where a_l, b_l are independent of x. Substituting $\mathbf{u} = \tilde{\mathbf{u}}$ into (2) gives

$$\sum_{\mathbf{L}} \mathrm{i}k\mathbf{L} \cdot \mathbf{u_L} \mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}} = 0. \tag{17}$$

Equating like powers of the exponentials in (17) yields

$$\mathbf{L} \cdot \mathbf{u_L} = 0. \tag{18}$$

Applying $\mathbf{L}\cdot$ to (15) and noting (18) leads to

$$p_{\mathbf{L}} = -\sum_{\mathbf{M}} \left(\mathbf{u_{L-M}} \cdot \hat{\mathbf{L}}\right)\left(\mathbf{u_M} \cdot \hat{\mathbf{L}}\right) \tag{19}$$

where $p_{\mathbf{0}}$ is arbitrary and $\hat{\mathbf{L}} = \mathbf{L}/|\mathbf{L}|$ is the unit vector in the direction of $\mathbf{L}$. Then substituting (19) into (15) gives

$$\frac{\partial \mathbf{u_L}}{\partial t} = \sum_{\mathbf{M}} \hat{\mathbf{L}} \times \left(\hat{\mathbf{L}} \times ((\mathbf{u_{L-M}} \cdot \mathrm{i}k\mathbf{L})\mathbf{u_M})\right) - \nu(k|\mathbf{L}|)^{\gamma}\mathbf{u_L} \tag{20}$$

where $\mathbf{u_0} = \mathbf{u_0}(0)$. Here we have used the $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$ vector identity. Without loss of generality, we take $\mathbf{u_0} = \mathbf{0}$. This is due to the Galilean invariance property. The equations for $\mathbf{u_L}$ are to be solved for all $\mathbf{L} \in \mathbb{Z}^3$. For $\gamma = 1$, we can prove by mathematical induction that $\mathbf{u_L}$ has the form

$$\mathbf{u_L} = \sum_{l \in S} \mathbf{u}_{\mathbf{L},l} \mathrm{e}^{-\nu k l t} \tag{21}$$

where $\mathbf{u}_{\mathbf{L},l}$ is independent of t, and S denotes the set of sums of positive terms of the form $|Q_i|$ for $Q_i \in \mathbb{Z}^3$ where for each of these sums we have $\sum_i Q_i = \mathbf{L}$ but not every Q_i are in the same direction to each other unless there is only one. The basis step is

$$\mathbf{u_0} = \mathbf{0}. \tag{22}$$

The inductive step is to assume $\mathbf{u}_\mathbf{L}$ for all $\mathbf{L} \neq \mathbf{N}$ has the form (21). Then for $\mathbf{L} = \mathbf{N}$ we find that (20) with $\gamma = 1$ implies

$$\frac{\partial\left(\mathrm{e}^{\nu k|\mathbf{N}|t}\mathbf{u}_\mathbf{N}\right)}{\partial t} = \mathrm{e}^{\nu k|\mathbf{N}|t}\sum_\mathbf{M}\hat{\mathbf{N}}\times\left(\hat{\mathbf{N}}\times((\mathbf{u}_{\mathbf{N}-\mathbf{M}}\cdot \mathrm{i}k\mathbf{N})\mathbf{u}_\mathbf{M})\right). \tag{23}$$

The right hand side of (23) contains no $\mathbf{u}_\mathbf{N}$ since $\mathbf{u}_\mathbf{0} = \mathbf{0}$. The powers of the exponentials in the right hand side of (23) can not be zero due to the inductive assumption and the triangle inequality

$$|\mathbf{a}+\mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}| \tag{24}$$

where equality holds if $\mathbf{a}$ and $\mathbf{b}$ are in the same direction. Then (23) implies that

$$\mathbf{u}_\mathbf{N} = \mathrm{e}^{-\nu k|\mathbf{N}|t}\left(\int_0^t \mathrm{e}^{\nu k|\mathbf{N}|\tau}\sum_\mathbf{M}\hat{\mathbf{N}}\times\left(\hat{\mathbf{N}}\times((\mathbf{u}_{\mathbf{N}-\mathbf{M}}(\tau)\cdot \mathrm{i}k\mathbf{N})\mathbf{u}_\mathbf{M}(\tau))\right)d\tau + \mathbf{u}_\mathbf{N}(0)\right) \tag{25}$$

which proves (21). Then for $\gamma = 1$ we find $\tilde{\mathbf{u}}$ has the form

$$\tilde{\mathbf{u}} = \sum_\mathbf{L}\sum_{l\in S}\mathbf{u}_{\mathbf{L},l}\mathrm{e}^{-\nu klt}\mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}}. \tag{26}$$

Since e^x has a Maclaurin series $\mathrm{e}^x = \sum_{n=0}^\infty \frac{x^n}{n!}$ that converges for all x, we can write (26) as

$$\tilde{\mathbf{u}} = \sum_\mathbf{L}\sum_{l\in S}\mathbf{u}_{\mathbf{L},l}\sum_{n=0}^\infty\frac{(-\nu kl)^n t^n}{n!}\mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}}. \tag{27}$$

We have $\tilde{\mathbf{u}}$ converges as $t \to \infty$ for all $\mathbf{x} \in \mathbb{R}^3$, therefore $\tilde{\mathbf{u}}$ converges for all $t \geqslant 0$ and $\mathbf{x} \in \mathbb{R}^3$ and $\tilde{\mathbf{u}}$ has no finite-time singularity due to Taylor's theorem. Likewise the enstrophy [5] is bounded since

$$\sum_\mathbf{L}|\mathbf{L}|^2|\mathbf{u}_\mathbf{L}|^2 < \infty. \tag{28}$$

For the case $\gamma \geqslant 1$, we let

$$\mathbf{u}_\mathbf{L} = \mathbf{a}_\mathbf{L} + \mathrm{i}\mathbf{b}_\mathbf{L}, \tag{29}$$

$$p_\mathbf{L} = c_\mathbf{L} + \mathrm{i}d_\mathbf{L} \tag{30}$$

where $\mathbf{a}_\mathbf{L} = \mathbf{a}_\mathbf{L}(t) \in \mathbb{R}^3$, $\mathbf{b}_\mathbf{L} = \mathbf{b}_\mathbf{L}(t) \in \mathbb{R}^3$, $c_\mathbf{L} = c_\mathbf{L}(t) \in \mathbb{R}$, and $d_\mathbf{L} = d_\mathbf{L}(t) \in \mathbb{R}$. Substituting (29), (30) into (15) gives

$$\begin{aligned}&\frac{\partial\mathbf{a}_\mathbf{L}}{\partial t} + \mathrm{i}\frac{\partial\mathbf{b}_\mathbf{L}}{\partial t} + \sum_\mathbf{M}((\mathbf{a}_{\mathbf{L}-\mathbf{M}} + \mathrm{i}\mathbf{b}_{\mathbf{L}-\mathbf{M}})\cdot \mathrm{i}k\mathbf{M})(\mathbf{a}_\mathbf{M} + \mathrm{i}\mathbf{b}_\mathbf{M})\\ &= -\nu(k|\mathbf{L}|)^\gamma(\mathbf{a}_\mathbf{L} + \mathrm{i}\mathbf{b}_\mathbf{L}) - \mathrm{i}k\mathbf{L}(c_\mathbf{L} + \mathrm{i}d_\mathbf{L}).\end{aligned} \tag{31}$$

Equating real and imaginary parts in (31) yields

$$\frac{\partial \mathbf{a_L}}{\partial t} + \sum_{\mathbf{M}} (-(\mathbf{a_{L-M}} \cdot k\mathbf{M})\mathbf{b_M} - (\mathbf{b_{L-M}} \cdot k\mathbf{M})\mathbf{a_M}) = -\nu(k|\mathbf{L}|)^{\gamma}\mathbf{a_L} + k\mathbf{L}d_{\mathbf{L}}, \tag{32}$$

$$\frac{\partial \mathbf{b_L}}{\partial t} + \sum_{\mathbf{M}} ((\mathbf{a_{L-M}} \cdot k\mathbf{M})\mathbf{a_M} - (\mathbf{b_{L-M}} \cdot k\mathbf{M})\mathbf{b_M}) = -\nu(k|\mathbf{L}|)^{\gamma}\mathbf{b_L} - k\mathbf{L}c_{\mathbf{L}}. \tag{33}$$

Substituting (29) into (18) gives

$$\mathbf{L} \cdot (\mathbf{a_L} + \mathrm{i}\mathbf{b_L}) = 0. \tag{34}$$

Equating real and imaginary parts in (34) yields

$$\mathbf{L} \cdot \mathbf{a_L} = 0, \tag{35}$$

$$\mathbf{L} \cdot \mathbf{b_L} = 0. \tag{36}$$

From (32) and in light of (35) we have

$$\frac{\partial \mathbf{a_L}}{\partial t} \cdot \hat{\mathbf{a}}_{\mathbf{L}} + \sum_{\mathbf{M}} (-(\mathbf{a_{L-M}} \cdot k\mathbf{M})\mathbf{b_M} - (\mathbf{b_{L-M}} \cdot k\mathbf{M})\mathbf{a_M}) \cdot \hat{\mathbf{a}}_{\mathbf{L}} = -\nu(k|\mathbf{L}|)^{\gamma}\mathbf{a_L} \cdot \hat{\mathbf{a}}_{\mathbf{L}} \tag{37}$$

where $\hat{\mathbf{a}}_{\mathbf{L}} = \mathbf{a_L}/|\mathbf{a_L}|$ is the unit vector in the direction of $\mathbf{a_L}$. Then (37) implies

$$\frac{\partial |\mathbf{a_L}|}{\partial t} + \sum_{\mathbf{M}} (-(\mathbf{a_{L-M}} \cdot k\mathbf{M})\mathbf{b_M} - (\mathbf{b_{L-M}} \cdot k\mathbf{M})\mathbf{a_M}) \cdot \hat{\mathbf{a}}_{\mathbf{L}} = -\nu(k|\mathbf{L}|)^{\gamma}|\mathbf{a_L}|. \tag{38}$$

Equation (38) leads to

$$\frac{\partial |\mathbf{a_L}|}{\partial t} \leqslant \sum_{\mathbf{M}} (|\mathbf{a_{L-M}}|k|\mathbf{M}||\mathbf{b_M}| + |\mathbf{b_{L-M}}|k|\mathbf{M}||\mathbf{a_M}|) - \nu(k|\mathbf{L}|)^{\gamma}|\mathbf{a_L}| \tag{39}$$

on using the Cauchy–Schwarz inequality

$$|\mathbf{a} \cdot \mathbf{b}| \leqslant |\mathbf{a}||\mathbf{b}|. \tag{40}$$

From (33) and in light of (36) we have

$$\frac{\partial \mathbf{b_L}}{\partial t} \cdot \hat{\mathbf{b}}_{\mathbf{L}} + \sum_{\mathbf{M}} ((\mathbf{a_{L-M}} \cdot k\mathbf{M})\mathbf{a_M} - (\mathbf{b_{L-M}} \cdot k\mathbf{M})\mathbf{b_M}) \cdot \hat{\mathbf{b}}_{\mathbf{L}} = -\nu(k|\mathbf{L}|)^{\gamma}\mathbf{b_L} \cdot \hat{\mathbf{b}}_{\mathbf{L}} \tag{41}$$

where $\hat{\mathbf{b}}_{\mathbf{L}} = \mathbf{b_L}/|\mathbf{b_L}|$ is the unit vector in the direction of $\mathbf{b_L}$. Then (41) implies

$$\frac{\partial |\mathbf{b_L}|}{\partial t} + \sum_{\mathbf{M}} ((\mathbf{a_{L-M}} \cdot k\mathbf{M})\mathbf{a_M} - (\mathbf{b_{L-M}} \cdot k\mathbf{M})\mathbf{b_M}) \cdot \hat{\mathbf{b}}_{\mathbf{L}} = -\nu(k|\mathbf{L}|)^{\gamma}|\mathbf{b_L}|. \tag{42}$$

Equation (42) leads to

$$\frac{\partial|\mathbf{b_L}|}{\partial t} \leqslant \sum_{\mathbf{M}}(|\mathbf{a_{L-M}}|k|\mathbf{M}||\mathbf{a_M}| + |\mathbf{b_{L-M}}|k|\mathbf{M}||\mathbf{b_M}|) - \nu(k|\mathbf{L}|)^{\gamma}|\mathbf{b_L}| \tag{43}$$

on using the Cauchy–Schwarz inequality. We then have

$$\sum_{\mathbf{L}}|\mathbf{L}|^2|\mathbf{u_L}|^2 = \sum_{\mathbf{L}}|\mathbf{L}|^2\left(|\mathbf{a_L}|^2 + |\mathbf{b_L}|^2\right) \tag{44}$$

along with

$$\frac{\partial|\mathbf{a_L}|}{\partial t} \leqslant \sum_{\mathbf{M}}(|\mathbf{a_{L-M}}|k|\mathbf{M}||\mathbf{b_M}| + |\mathbf{b_{L-M}}|k|\mathbf{M}||\mathbf{a_M}|) - \nu(k|\mathbf{L}|)^{\delta}|\mathbf{a_L}|, \tag{45}$$

$$\frac{\partial|\mathbf{b_L}|}{\partial t} \leqslant \sum_{\mathbf{M}}(|\mathbf{a_{L-M}}|k|\mathbf{M}||\mathbf{a_M}| + |\mathbf{b_{L-M}}|k|\mathbf{M}||\mathbf{b_M}|) - \nu(k|\mathbf{L}|)^{\delta}|\mathbf{b_L}| \tag{46}$$

for any $\delta < \gamma$. We have already shown that $\sum_{\mathbf{L}}|\mathbf{L}|^2|\mathbf{u_L}|^2 < \infty$ when $\gamma = 1$. Therefore from (44), (45), (46) we find that $\sum_{\mathbf{L}}|\mathbf{L}|^2|\mathbf{u_L}|^2 < \infty$ when $\gamma \geqslant 1$. It then follows that the Navier–Stokes equations are globally regular. □

References

[1] Batchelor GK. 1967 *An introduction to fluid dynamics*. Cambridge, Cambridge University Press.
[2] Doering CR. 2009 The 3D Navier–Stokes problem. *Annu. Rev. Fluid. Mech* **41**.
[3] Fefferman CL. 2000 *Existence and smoothness of the Navier–Stokes equation*. Clay Mathematics Institute. Official problem description.
[4] Ladyzhenskaya OA. 1969 *The mathematical theory of viscous incompressible flows*. New York, Gordon and Breach.
[5] Robinson JC. 2020 The Navier–Stokes regularity problem. *Phil. Trans. R. Soc. A* **378**.
[6] Tao T. 2013 Localisation and compactness properties of the Navier–Stokes global regularity problem. *Analysis and PDE* **6**.