Solution to the problem on the existence and smoothness of the Navier–Stokes equations

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The Navier–Stokes equations are a set of nonlinear partial differential equations used to describe viscous incompressible fluid flow. It has been on the list of the Clay Mathematics Institute's millennium prize problems to decide whether or not physically reasonable solutions to the Navier–Stokes equations do in general exist. In this paper, the problem on the existence and smoothness of the Navier–Stokes equations is solved. It is proven that the Navier–Stokes equations are globally regular.

1. Introduction

The Navier–Stokes equations are thought to govern the motion of a fluid in $\mathbb{R}^3$, [1]. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^3$ be the fluid velocity and let $p = p(\mathbf{x}, t) \in \mathbb{R}$ be the fluid pressure, each dependent on position $\mathbf{x} \in \mathbb{R}^3$ and time $t \geqslant 0$. We take the externally applied force acting on the fluid to be identically zero. The fluid is assumed to be incompressible with constant viscosity $\nu > 0$ and to fill all of $\mathbb{R}^3$. The Navier–Stokes equations can then be written as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \nabla^2 \mathbf{u} - \nabla p, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

with initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^\circ \tag{3}$$

where $\mathbf{u}^\circ = \mathbf{u}^\circ(\mathbf{x}) \in \mathbb{R}^3$. In these equations

$$\nabla = \left(\frac{\partial}{\partial \mathbf{x}_1}, \frac{\partial}{\partial \mathbf{x}_2}, \frac{\partial}{\partial \mathbf{x}_3} \right) \tag{4}$$

is the gradient operator and

$$\nabla^2 = \sum_{i=1}^{3} \frac{\partial^2}{\partial {\mathbf{x}_i}^2} \tag{5}$$

is the Laplacian operator. When $\nu = 0$ equations (1), (2), (3) are called the Euler equations. Solutions of (1), (2), (3) are to be found with

$$\mathbf{u}^\circ(\mathbf{x} + e_i) = \mathbf{u}^\circ(\mathbf{x}) \tag{6}$$

for $1 \leqslant i \leqslant 3$ where e_i is the i^{th} unit vector in $\mathbb{R}^3$. The initial condition $\mathbf{u}^\circ$ is a given C^∞ divergence-free vector field on $\mathbb{R}^3$. A solution of (1), (2), (3) is then accepted to be physically reasonable [2,3,6,8] if

$$\mathbf{u}(\mathbf{x}+e_i,t)=\mathbf{u}(\mathbf{x},t),\quad p(\mathbf{x}+e_i,t)=p(\mathbf{x},t) \tag{7}$$

on $\mathbb{R}^3\times[0,\infty)$ for $1 \leqslant i \leqslant 3$ and

$$\mathbf{u},p\in C^\infty\left(\mathbb{R}^3\times[0,\infty)\right). \tag{8}$$

2. Solution of the Navier–Stokes problem

Theorem 1. Take $\nu > 0$. Let $\mathbf{u}^\circ$ be any smooth, divergence-free vector field satisfying (6). Then there exist smooth functions $\mathbf{u}, p$ on $\mathbb{R}^3\times[0,\infty)$ that satisfy (1), (2), (3), (7), (8).
Proof. Let the Fourier series of $\mathbf{u}, p$ be

$$\tilde{\mathbf{u}}=\sum_{\mathbf{L}=-\infty}^{\infty}\mathbf{u}_{\mathbf{L}}\mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}}, \tag{9}$$

$$\tilde{p}=\sum_{\mathbf{L}=-\infty}^{\infty}p_{\mathbf{L}}\mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}} \tag{10}$$

respectively. Here $\mathbf{u}_{\mathbf{L}}=\mathbf{u}_{\mathbf{L}}(t)\in\mathbb{C}^3$, $p_{\mathbf{L}}=p_{\mathbf{L}}(t)\in\mathbb{C}$, $\mathrm{i}=\sqrt{-1}$, $k=2\pi$, and $\sum_{\mathbf{L}=-\infty}^{\infty}$ denotes the sum over all $\mathbf{L}\in\mathbb{Z}^3$. The initial condition $\mathbf{u}^\circ$ is a Fourier series [2] of which is convergent for all $\mathbf{x}\in\mathbb{R}^3$. Substituting $\mathbf{u}=\tilde{\mathbf{u}}$, $p=\tilde{p}$ into (1) gives

$$\sum_{\mathbf{L}=-\infty}^{\infty}\frac{\partial\mathbf{u}_{\mathbf{L}}}{\partial t}\mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}}+\sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}(\mathbf{u}_{\mathbf{L}}\cdot\mathrm{i}k\mathbf{M})\mathbf{u}_{\mathbf{M}}\mathrm{e}^{\mathrm{i}k(\mathbf{L}+\mathbf{M})\cdot\mathbf{x}}$$
$$=-\sum_{\mathbf{L}=-\infty}^{\infty}\nu k^2|\mathbf{L}|^2\mathbf{u}_{\mathbf{L}}\mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}}-\sum_{\mathbf{L}=-\infty}^{\infty}\mathrm{i}k\mathbf{L}p_{\mathbf{L}}\mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}}. \tag{11}$$

Equating like powers of the exponentials in (11) in accordance with Theorem B of [5] yields

$$\frac{\partial\mathbf{u}_{\mathbf{L}}}{\partial t}+\sum_{\mathbf{M}=-\infty}^{\infty}(\mathbf{u}_{\mathbf{L}-\mathbf{M}}\cdot\mathrm{i}k\mathbf{M})\mathbf{u}_{\mathbf{M}}=-\nu k^2|\mathbf{L}|^2\mathbf{u}_{\mathbf{L}}-\mathrm{i}k\mathbf{L}p_{\mathbf{L}} \tag{12}$$

on using the Cauchy product type formula [4]

$$\sum_{l=-\infty}^{\infty}a_lx^l\sum_{m=-\infty}^{\infty}b_mx^m=\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}a_{l-m}b_mx^l. \tag{13}$$

Substituting $\mathbf{u} = \tilde{\mathbf{u}}$ into (2) gives

$$\sum_{\mathbf{L}=-\infty}^{\infty} \mathrm{i}k\mathbf{L} \cdot \mathbf{u}_{\mathbf{L}} \mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}} = 0. \tag{14}$$

Equating like powers of the exponentials in (14) in accordance with Theorem B of [5] yields

$$\mathbf{L} \cdot \mathbf{u}_{\mathbf{L}} = 0. \tag{15}$$

Applying $\mathbf{L}\cdot$ to (12) and noting (15) leads to

$$p_{\mathbf{L}} = -\sum_{\mathbf{M}=-\infty}^{\infty} \left(\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \hat{\mathbf{L}}\right)\left(\mathbf{u}_{\mathbf{M}} \cdot \hat{\mathbf{L}}\right) \tag{16}$$

where $p_{\mathbf{0}}$ is arbitrary and $\hat{\mathbf{L}} = \mathbf{L}/|\mathbf{L}|$ is the unit vector in the direction of $\mathbf{L}$. Then substituting (16) into (12) gives

$$\frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} = \sum_{\mathbf{M}=-\infty}^{\infty} \hat{\mathbf{L}} \times \left(\hat{\mathbf{L}} \times ((\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \mathrm{i}k\mathbf{L})\mathbf{u}_{\mathbf{M}})\right) - \nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}} \tag{17}$$

where $\mathbf{u}_{\mathbf{0}} = \mathbf{u}_{\mathbf{0}}(0)$. Without loss of generality [2], we take $\mathbf{u}_{\mathbf{0}} = \mathbf{0}$. This is due to the Galilean invariance property of solutions to the Navier–Stokes equations. The equations for $\mathbf{u}_{\mathbf{L}}$ are to be solved for all $\mathbf{L} \in \mathbb{Z}^3$. From (17) we find that $\tilde{\mathbf{u}}$ turns out to be of the form

$$\tilde{\mathbf{u}} = \sum_{l=0}^{\infty} \mathbf{a}_l(\mathbf{x}) \mathrm{e}^{-\nu k^2 l t} \tag{18}$$

of which is identical to a Maclaurin series

$$\tilde{\mathbf{u}} = \sum_{l=0}^{\infty} \mathbf{a}_l(\mathbf{x}) \tau^l \tag{19}$$

in the variable

$$\tau = \mathrm{e}^{-\nu k^2 t}. \tag{20}$$

We are given that $\tilde{\mathbf{u}}$ converges for all $\mathbf{x} \in \mathbb{R}^3$ at $t = 0$. This implies that $\tilde{\mathbf{u}}$ converges for all $\mathbf{x} \in \mathbb{R}^3$ at $\tau = 1$. Therefore $\tilde{\mathbf{u}}$ converges for all $\mathbf{x} \in \mathbb{R}^3$ and for all $0 \leqslant |\tau| \leqslant 1$ due to Taylor's theorem [7]. Thus $\tilde{\mathbf{u}}$ converges for all $\mathbf{x} \in \mathbb{R}^3$ and for all $t \in [0, \infty)$. We then conclude that the Navier–Stokes equations are globally regular. □

References

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