Computational Complexity of Quantum Error Correction

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Abstract
Errors on computations depend on their arguments and are, in general, different arguments by arguments. As linear superposition for quantum parallelism keeps the errors different, QEC (Quantum Error Correction) for N quantum parallel computations must correct N different errors, which requires O(N) correction information, which makes hardware complexity of QEC circuit O(N), which is no better than classical N parallel computations. Shor overlooked a fact that an initial environment state around a qubit is output from a QEC encoder and is entangled with all the qubits used to compute the qubit, which makes the errors depend on the qubits.

Keywords: Quantum Error Correction, Quantum Supremacy

1. Introduction

As unitary quantum operators are linear, instead of evaluating a unitary operator serially with N different input arguments one by one for N times, the operator with linear combination, or quantum superposition, of N different input arguments may be evaluated once to obtain a superpositioned evaluation result, which is, so called, quantum parallelism.

A problem is that, because errors on computations depend on their arguments and are, in general, different arguments by arguments, usual QEC (Quantum Error Correction) works only for quantum serial computations.

Quantum serial computations to obtain outputs of a quantum circuit with different input arguments, errors may be corrected by QEC. QEC is “quantum analog of error correcting codes” [1] and is not very different from CEC (Classical Error Correction). With CEC, during decoding, syndrome bits are computed to identify an error type. With QEC, during decoding, syndrome qubits are computed and observed to converge states into some error type which is identified by the observed result.

However, QEC for N parallel quantum parallel computation must correct N different errors, which makes usual QEC assuming only one error type not work. Though it may be possible to develop QEC to be able to correct N different errors, N times more syndrome qubits are necessary to identify

N error types (including cases without error), which requires O(N) hardware complexity, which is no better than classical N parallel computation, which denies quantum supremacy by QEC.

The argument above is almost purely computational and the only quantum mechanical knowledge required is that quantum superposition for quantum parallelism is linear. The linearity keeps N different errors of N quantum serial computations still different even with N quantum parallel computation.

In this paper, in section 2, it is explained that even with a simple error model of Shor, quantum entanglement makes errors depend on arguments. In section 3, simple examples are given on how Shor code fails when an input qubit of a Shor encoder is entangled with another qubit. Section 4 concludes the paper.

2. Error Model of Shor

Shor assumed simple error (called "decoherence" in [1]) model "Assuming that the decoherence process affects the different qubits in memory independently". That is, he assumes a qubit state degrades only by interaction with its environment state as follows:

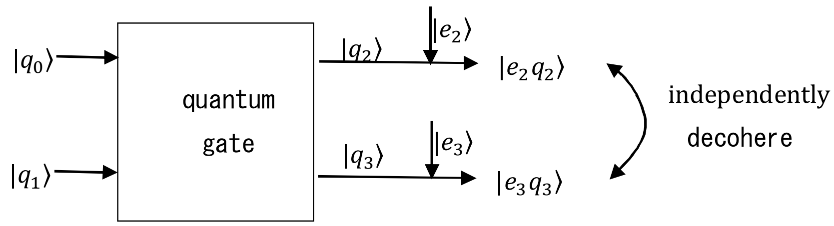
$$|e_0\rangle|0\rangle \rightarrow |a_0\rangle|0\rangle + |a_1\rangle|1\rangle$$
$$|e_0\rangle|1\rangle \rightarrow |a_2\rangle|0\rangle + |a_3\rangle|1\rangle$$

where $|e_0\rangle$ is the initial environment state and $|a_0\rangle$, $|a_1\rangle, |a_2\rangle$ and $|a_3\rangle$ are the environment states after the decoherence process.

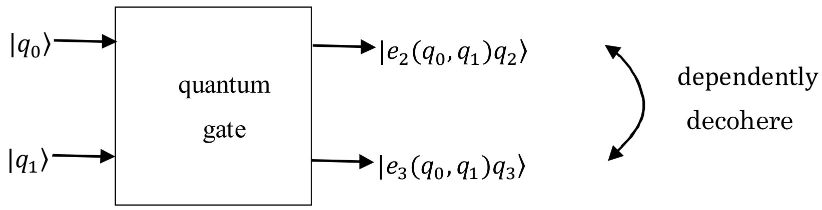
However, his assumption "The important thing to note is that the state of the environment is the same for corresponding vectors from the decoherence of the two quantum states encoding 0 and encoding 1." [1] as if the environment state around a qubit were supplied from an external source independent from the qubit (Fig. 1(a)) is inappropriate.

In practice, the initial environment state around a qubit is output from a QEC decoder as the final environment state around the qubit output from the last quantum gate of the QEC encoder. As such, as shown in Fig. 1(b), the environment state ($|e_2\rangle$ or $|e_3\rangle$ of Fig. 1(b)) is affected by input qubits ($|q_0\rangle$ and $|q_1\rangle$) of the quantum gate to be $|e_2(q_0, q_1)\rangle$ or $|e_3(q_0, q_1)\rangle$.

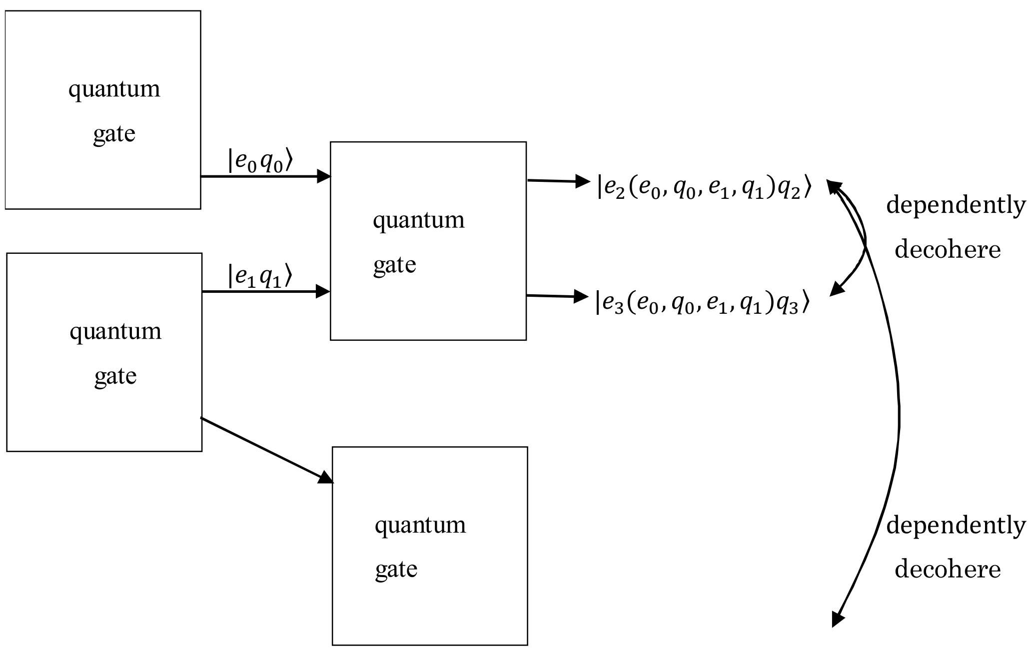
Moreover, as shown in Fig. 1(c), for the last quantum gate, considering not only input qubit states



(a) Qubits and their initial environment states assumed by Shor



(b) Qubits and their initial environment states output from a quantum gate



(c) Qubits and their environment states output from multi-stage quantum circuit

Fig 1. Qubit states $|q_i\rangle$ and their environment states $|e_j\rangle$

($|q_0\rangle$ and $|q_1\rangle$) but also initial environment states around the input qubits ($|e_0\rangle$ and $|e_1\rangle$) as the final environment states around output qubits of the previous quantum gates, the final quantum states of the output environment states of the last quantum gate is affected by and entangled with $|e_0 q_0\rangle$ and $|e_1 q_1\rangle$ to be $|e_2(e_0, q_0, e_1, q_1)q_2\rangle$ and $|e_3(e_0, q_0, e_1, q_1)q_3\rangle$.

Recursive applications of such arguments make an environment state around a qubit depends on and entangled with all the input qubit states of all the quantum gates used to compute the qubit, which, in turn, makes a state of the qubit after decoherence depends on all the input qubit states of all the quantum gates used to compute the qubit. In other words, errors on computations depend on their arguments and are, in general, different arguments by arguments.

3. A Simple Example of How Shor Code Fails

In this section, with a simple quantum circuit composed of a CNOT gate followed by a Shor code encoder and a Shor code decoder (Fig. 2), it is shown that Shor code decoder fails to decode the original state if an environment state $|e_0(q_0,\ q_1)\rangle$ around an input qubit $|q_2\rangle$ of a Shor code encoder is entangled with $|q_0\rangle$. According to the discussion in the previous section, $|e_1\rangle$ depends on $|e_0(q_0,\ q_1)\rangle$ and $|q_2\rangle$ as $|e_1(e_0(q_0,\ q_1), q_2)\rangle$. But, as $|e_0(q_0,\ q_1)\rangle$ and $|q_2\rangle$, then, depend on $|q_0\rangle$ and $|q_1\rangle$, notation of $|e_1(q_0, q_1)\rangle$ is used for simplicity.

It is assumed that only one output qubit $|q_3\rangle$ of the Shor code decoder suffers from an error of a bit flip error with probability ε only if its environment state is $|e_1(1,0)\rangle$. That is, only $|q_3\rangle$ suffers from errors as:

$$|e_1(0,0)\ \rangle|0\rangle \rightarrow |a_0\rangle|0\rangle$$
$$|e_1(0,0)\rangle|1\rangle \rightarrow |a_3\rangle|1\rangle$$
$$(|a_0\rangle| = |a_3\rangle| = 1)$$

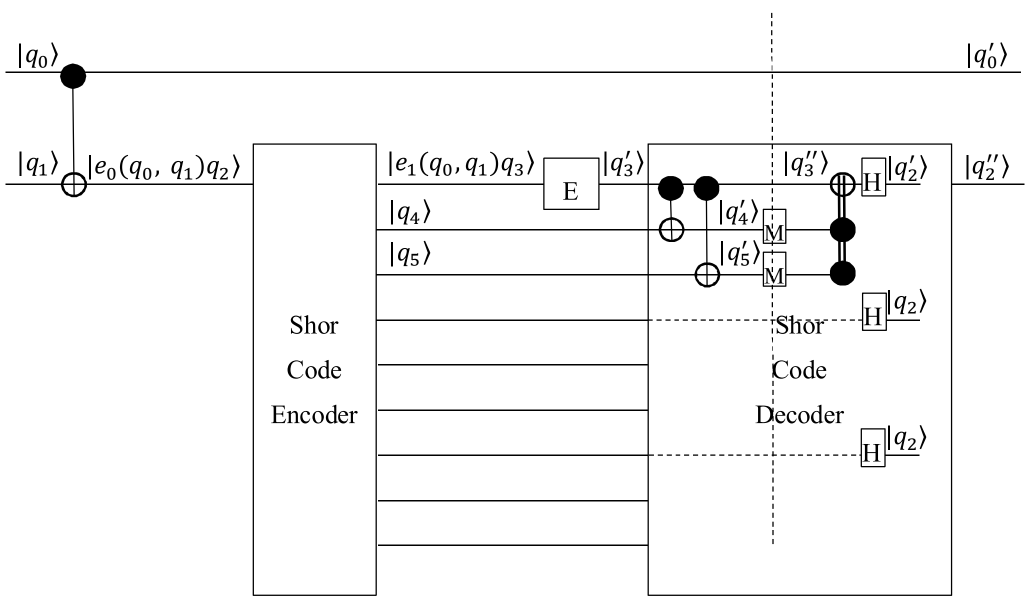


Fig 2. Simple Circuit with a CNOT gate followed by Shor Code Encoder/Decoder

and

$$|e_1(1,0)\rangle|0\rangle \to |a_0\rangle|0\rangle + |a_1\rangle|1\rangle$$
$$|e_1(1,0)\rangle|1\rangle \to |a_2\rangle|0\rangle + |a_3\rangle|1\rangle$$

$(|a_0\rangle| = |a_3\rangle| = \sqrt{1-\varepsilon},\ |a_1\rangle| = |a_2\rangle| = \varepsilon)$

Then, as shown in Table 1 (A column labelled with "M" shows observation results), when q_0 and q_2 are not entangled, that is, $|q_0q_1\rangle = |00\rangle$ or $|q_0q_1\rangle = |10\rangle$, a bit flip error, if any, on q_2 is properly corrected. That is, $|q_0q_2\rangle = |q_0'q_2''\rangle$.

However, with a superpositioned state of them, that is, if $|q_0q_1\rangle = (|00\rangle + |10\rangle)/\sqrt{2}$, q_2 is entangled with q_0 as $|q_0q_2\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ and error difference on q_3 will make

$$|q_0q_3'q_4'q_5'\rangle = (|0000\rangle + |0100\rangle + \sqrt{1-\varepsilon}(|1000\rangle - |1100\rangle) + \sqrt{\epsilon}(|1111\rangle - |1011\rangle))/2$$

which is an entangled state involving q_0. As such, observations on q_3' and q_4' at a vertical dashed line in Fig. 2 have non-local effect involving q_0, which causes quantum error correction fail. As shown by boxes enclosed by fat border lines in Table 1, the original state $|q_0q_2\rangle$ ($(|00\rangle + |11\rangle)/\sqrt{2}$) and the resulting state $|q_0'q_2''\rangle$ ($(|00\rangle + \sqrt{1-\varepsilon}|11\rangle)/\sqrt{2-\varepsilon}$ or $|11\rangle$) are significantly different.

4. Colclusions

It is pointed out that Shor overlooked environment states around qubits are entangled with other qubits. Instead, we must assume errors on qubits depends on other qubits.

Then, it is demonstrated that such errors make Shor code fail to restore the original state against a single qubit error.

Worse, it is pointed out that, though it may be possible to construct an improved QEC, as *N* quantum parallel computations suffer from *N* different errors, hardware complexity of such QEC must be O(*N*), which is no better than complexity of classical parallel circuits, which denies quantum supremacy, at least with QEC.

Though Shor stated "The assumption that the qubits decohere independently is crucial" because "This assumption corresponds to independence of errors between different bits in classical information theory" [1], if entanglements are properly considered, the assumption of independent decoherence does not mean locality or independence of errors between different qubits.

References

[1] P. W. Shor, "Scheme for reducing decoherence in quantum computer memory", Phys. Rev. A, Oct. 1995.

Table 1. Quantum States of Fig. 2

<table>
<tr><th>$|q_0q_1\rangle$</th><th>$|q_0q_2\rangle$</th><th>$|q_0e_1(q_0,q_1)q_3\rangle$</th><th>$|q_0q_3'q_4q_5\rangle$</th><th>$|q_0q_3'q_4'q_5'\rangle$</th><th>M</th><th>$|q_0'q_3''\rangle$</th><th>$|q_0'q_2'q_2q_2\rangle$</th><th>$|q_0'q_2''\rangle$</th></tr>
<tr><td>$|00\rangle$</td><td>$|00\rangle$</td><td>$(|0e_1(0,0)0\rangle + |0e_1(0,0)1\rangle)/\sqrt{2}$</td><td>$(|0000\rangle + |0111\rangle)/\sqrt{2}$</td><td>$(|0000\rangle + |0100\rangle)/\sqrt{2}$</td><td>0
0</td><td>$(|00\rangle + |01\rangle)/\sqrt{2}$</td><td>$|0000\rangle$</td><td>$|00\rangle$</td></tr>
<tr><td rowspan="2">$|10\rangle$</td><td rowspan="2">$|11\rangle$</td><td rowspan="2">$(|1e_1(1,0)0\rangle - |1e_1(1,0)1\rangle)/\sqrt{2}$</td><td rowspan="2">$(\sqrt{1-\varepsilon}(|1000\rangle - |1111\rangle) + \sqrt{\epsilon}(|1100\rangle - |1011\rangle))/\sqrt{2}$</td><td rowspan="2">$(\sqrt{1-\varepsilon}(|1000\rangle - |1100\rangle) + \sqrt{\epsilon}(|1111\rangle - |1011\rangle))/\sqrt{2}$</td><td>0
0</td><td>$(|10\rangle - |11\rangle)/\sqrt{2}$</td><td>$|1111\rangle$</td><td>$|11\rangle$</td></tr>
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1</td><td>$(|10\rangle - |11\rangle)/\sqrt{2}$</td><td>$|1111\rangle$</td><td>$|11\rangle$</td></tr>
<tr><td rowspan="2">$(|00\rangle + |10\rangle)/\sqrt{2}$</td><td rowspan="2">$(|00\rangle + |11\rangle)/\sqrt{2}$</td><td rowspan="2">$(|0e_1(0,0)0\rangle + |0e_1(0,0)1\rangle + |1e_1(1,0)0\rangle - |1e_1(1,0)1\rangle)/2$</td><td rowspan="2">$(|0000\rangle + |0111\rangle + \sqrt{1-\varepsilon}(|1000\rangle - |1111\rangle) + \sqrt{\epsilon}(|1100\rangle - |1011\rangle))/2$</td><td rowspan="2">$(|0000\rangle + |0100\rangle + \sqrt{1-\varepsilon}(|1000\rangle - |1100\rangle) + \sqrt{\epsilon}(|1111\rangle - |1011\rangle))/2$</td><td>0
0</td><td>$(|00\rangle + |01\rangle + \sqrt{1-\varepsilon}(|10\rangle - |11\rangle))/\sqrt{4-2\varepsilon}$</td><td>$(|0000\rangle + \sqrt{1-\varepsilon}|1111\rangle)/\sqrt{2-\varepsilon}$</td><td>$(|00\rangle + \sqrt{1-\varepsilon}|11\rangle)/\sqrt{2-\varepsilon}$</td></tr>
<tr><td>1
1</td><td>$(|10\rangle - |11\rangle)/\sqrt{2}$</td><td>$|1111\rangle$</td><td>$|11\rangle$</td></tr>
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