# A simple proof of Legendre's conjecture

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## DECLARATION

I declare that "A simple proof of Legendre's conjecture" is our own original work and that all sources that I have used or quoted have been acknowledged by means of complete references.


## Abstract

A simple proof of Legendre's conjecture is presented

Introduction

The properties of prime numbers have been studied for many centuries. Euclid gave the first proof of infinity of primes. Euler gave a proof which connected primes to the zeta function. Then there was the Gauss and Legendre's formulation of the prime number theorem and its proof by Hadamard and de la Vallee Poussin. Riemann further came with some hypothesis about the roots of the Riemann-zeta function.

Many others have contributed towards prime number theory.

Legendre's conjecture, proposed by Adrien-Marie Legendre states that there is a prime number between $n^2$ and $(n+1)^2$ for every positive integer n. The conjecture is one of Landau's problems (1912) on prime numbers. The conjecture has not been proved to the time of writing of this paper.

The Goldbach conjecture was proved in a paper entitled "An NP complete proof of Goldbach conjecture".

In this research a method will be presented of proving Legendre's conjecture.

## Methodology

Carl Friedrich Gauss on investigating the properties of primes came up with a prime counting function that approximate the number of primes between the numbers 0 and x given by:

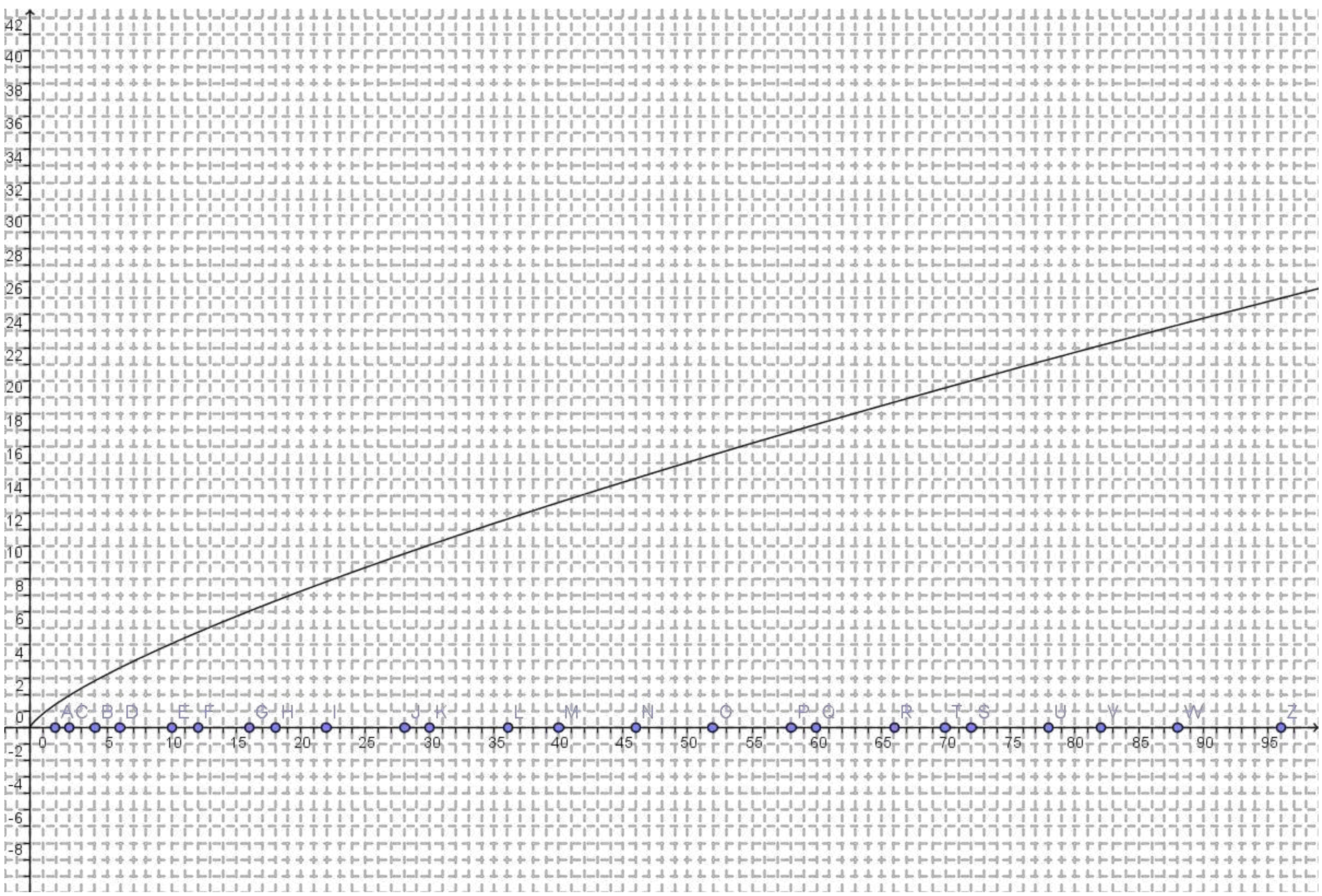
$$\pi(x) = \frac{x}{\ln x} \qquad 1$$

For the purpose of seeking for a proof of Legendre conjecture we will investigate the minimum number of primes between 1 and x.

The investigation will begin by introducing the suspected function:

$$\theta(x) = \frac{x}{\sqrt[4]{(2x)}} = \frac{x^{3/4}}{\sqrt[4]{2}} \qquad 3$$

The function gives a fairly good estimate of the number of primes for values of x up to around 200. Above for x greater than 200 it gives an under estimate of the number of primes. The graph of the function 3 is shown in the figure below:



$\theta(4) = 2.378143$

Actual number of primes from 1 to 4 is 2

$\theta(20) = 7.592707288$

Actual number of primes from 1 to 20 is 8

$\theta(50) = 15.8113383$

Actual number of primes from 1 to 50 is 15

$$\theta(70) = 20.35007579$$

Actual number of primes from 1 to 70 is 19

$$\theta(100) = 26.59147948$$

Actual number of primes from 1 to 100 is 25

The number of primes significantly increases above the number predicted by the function 3 as x increases in value. The significance of the above function 3 is we can use it to get a relationship about the minimum number of primes given by:

$$\theta \min \geq \frac{x}{\sqrt[4]{(2x)}} - 1 = \frac{x^{3/4}}{\sqrt[4]{2}} - 1 \qquad 4$$

From Legendre conjecture:

$$(n+1)^2 - n^2 = 2n+1 \qquad 5$$

The minimum number of primes between the two consecutive square numbers is given by:

$$\theta \min \geq \frac{2n+1}{\sqrt[4]{(4n+2)}} - 1$$

For case n = 1, that is primes between 1 and 4, the minimum number of primes is:

$$\theta \min \geq 0.91683$$

The actual number of primes is 2

The minimum number of primes between 4 and 9 is given by:

$$\theta \min \geq 1.8117$$

The actual number of primes is 2.

The minimum number of primes between 9 and 25 is given by:

$$\theta \min \geq 3.369$$

The actual number of primes is 4

This therefore proves the Legendre's conjecture.

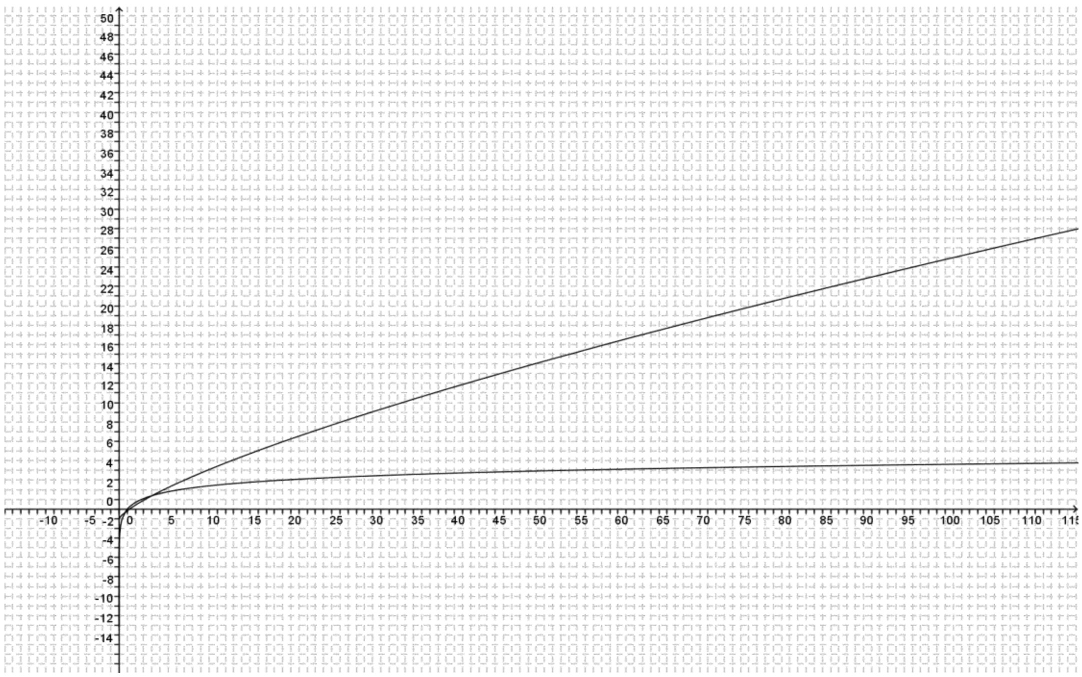


Figure 2: comparing the minimum number of primes to that of the prime counting function.

To further reinforce the proof of Legendre conjecture we will represent the two square numbers graphically.

The first square number will be represented by the function:

$$f(n) = (n+1)^2$$

The second square number will be represented by the function:

$$g(n) = n^2$$

The natural number n will be represented as:

$$x = n$$

And the prime number p will be represented by the relationship:

$h(n) = p$

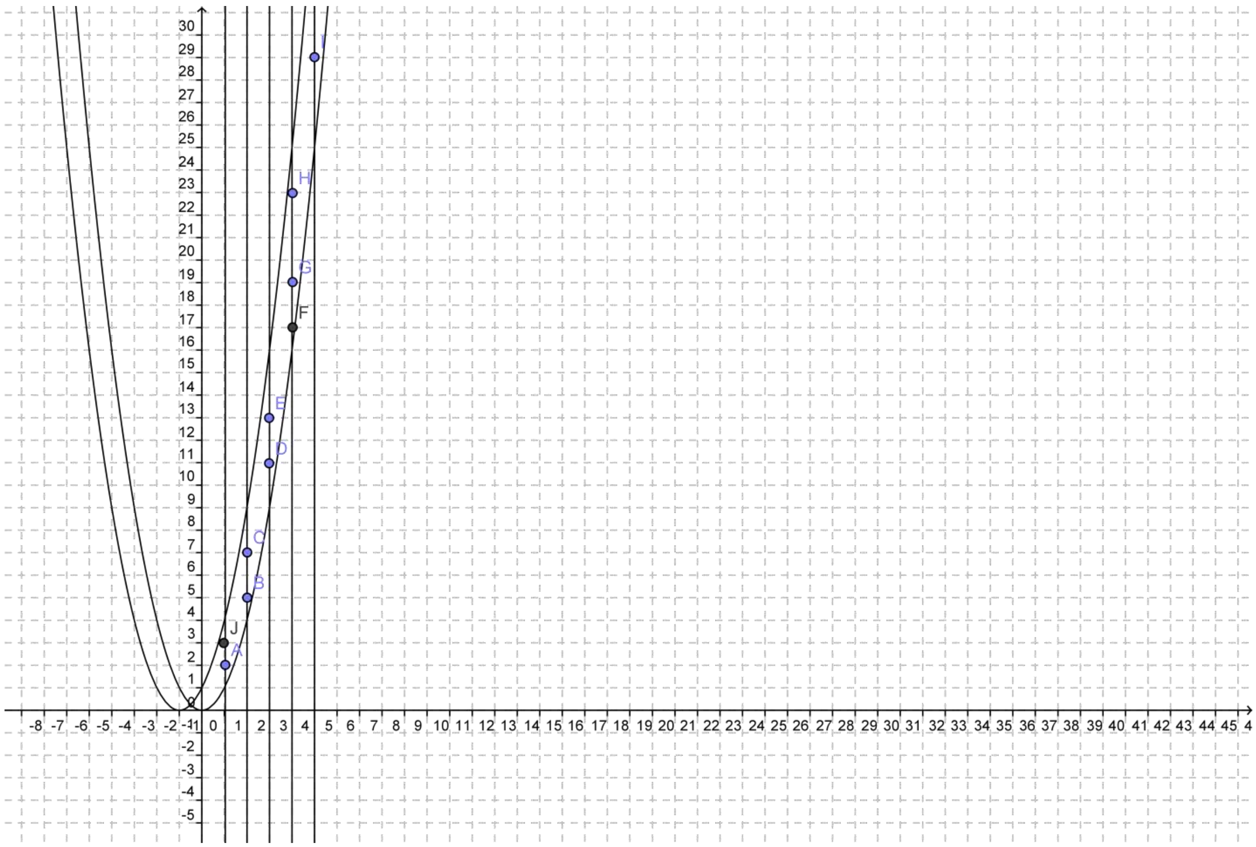



Figure 3: number of primes between two square numbers

Notice in the above diagram all the prime numbers are arranged in between the two curves representing the two square numbers. Specifically the prime numbers in the graph are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29 ….

Each n greater or equal to one (represented by a vertical line parallel to the x axis) has at least a prime number lying between the two curves. The prime numbers arrange themselves along the vertical lines but are bounded in between the two curves. The set of equations that defines, the curves, lines and points (containing prime numbers) of figure 3 is given by:

$$f(n) = (n+1)^2$$
$$g(n) = n^2$$
$$x = n$$
$$h(n) = p$$
$$g(n) < h(n) < f(n)$$

( $h(n) = p$ ) represents horizontal lines parallel to the x axis and containing prime numbers

All prime numbers are represented in the set of points $(n, p)$ in between the two curves ( $f(n) = (n+1)^2$ and $g(n) = n^2$ ). Each coordinate number coordinate n has at least one prime coordinate p between f (n) and g (n). This is another way of reframing Legendre conjecture.

## Conclusion

All prime numbers are in between two curves representing consecutive square numbers.

For a given n there is at least one prime number between two consecutive square numbers.

The Legendre's conjecture is proved

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