Proof of Legendre conjecture by implication

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Abstract A proof by implication method is introduced for proving the Legendre conjecture.

Keywords proof by implication; Legendre conjecture proof

Introduction

Legendre's conjecture, proposed by Adrien-Marie Legendre, states that there is a prime number between n^2 and $(n+1)^2$ for every positive integer n. [1] The conjecture is one of Landau's problems (1912) on prime numbers, and is one of many open problems on the spacing of prime numbers.

In this research a proof by implication method will be used to prove the Legendre conjecture.
A proof by implication is a method of proving a statement of the form “If a then b”.

Proof by implication

Legendre conjecture implies that the number of primes up to $N^2||N \geq 2$ is greater than or equal to N and N is a postive integer. Algebraically:

$$\pi(N^2) \geq N|| : N \geq 2 \tag{1}$$

This also implies that

$$\pi((N+1)^2) \geq N+1|| : N \geq 2 \tag{2}$$

It also imples

$$\pi((N+1)^2) - \pi(N^2) \geq 1 \tag{3}$$

Proof

By the prime number theorem:

$$\pi(N^2) \approx \frac{N^2}{2\ln N} \tag{4}$$

Therefore the above implication means that

$$\frac{N^2}{2\ln N} \geq N \tag{5}$$

or

$$\frac{N}{2\ln N} \geq 1 \tag{6}$$

For $N \geq 2$ we note that:

$$\frac{N}{2\ln N} > 1|| : N \geq 2 \tag{7}$$

This result implies that there is more than one prime number in between two consecutive positive square integers Q.E.D.

Summary and conclusion

A proof by implication method exists for proving the Legendre conjecture. There exists more than one prime number in between two consecutive positive perfect square integers.

References

[1]Legendre, Adriene Marie (1808). Essai sur la Théorie des Nombres (in French) (2 ed.). Paris: Chez Courcier. pp. 405–406.