Short proofs of the Fermat theorem and trancendental character of e and π.

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0.1 Proofs.

We first show that e, π are transcendental numbers over $\mathbb{Q}$; suppose that $z \notin \mathbb{Q}$ and $p(x)$ is the minimal polynomial over $\mathbb{Z}$ such tht $p(z) = 0$. Then, by means of the algebraic completeness of $\mathbb{C}$, we can write

$$p(x) = a(x - z)(x - \alpha_2) \ldots (x - \alpha_n)$$

and an easy argument shows that p is minimal for all α_j. Indeed, suppose that there exists a q of degree less than n for α_2, then Eulidean division gives that

$$p = sq + r$$

with the degree of r strictly less than the degree of q. Therefore, $r(\alpha_2) = 0$ and therefore $r = 0$ by minimality of q for α_2. Hence, $p = sq$ wich contradicts the minimality of p regarding z in case s is of nontrivial degree. Hence, the conclusion; π is defined as the first nontrivial zero of $\sin(x)$ and all zero's are given by $m\pi$ where $m \in \mathbb{Z}$. Therefore $p(x) | \sin(x)$ and therefore $p(x) = (x - \pi)(x - k_2\pi) ldots (x - k_n\pi)$. Therefore, the existence of π should entail a series of integer numbers $1, k_2, \ldots k_n$. There is no logical ground for this; hence $p(x)$ does not exist and π is trancendental. A similar comment applies to $\ln(x) - 1$ and the minimal polynomial $q(x)$ for e. This should contain comlex conjugated pairs $a, \overline{a}$ of numbers with a nontrivial imaginary part; there is again no reason for this to be so and hence e must be transcendental.

We prove the extended Fermat theorem which asserts that for $n > 2$ it holds that $x^n + y^n = z^n$ has no solutions in $\mathbb{Q}$. Proof: $x^n + y^n = z^n$ is equivalent to

$$1 + (\frac{x}{y})^n = (\frac{z}{y})^n$$

supposing that $x < y$. The Taylor expansion of

$$(1 + q)^{\frac{1}{n}} = 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1(n-1)(2n-1)\ldots((k-1)n-1)}{n^k k!} q^k$$

implies that

$$\frac{z}{y} = 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1(n-1)(2n-1)\ldots((k-1)n-1)}{n^k k!} (\frac{x}{y})^{nk}.$$

Therefore, demanding $a = \frac{z}{y} = \frac{p}{q}$ with $p, q \in \mathbb{N}_0$ and $gcd(p, q) = 1$ to be rational implies that

$$\frac{p^{nk}(n-1)(2n-1)\ldots((k-1)n-1)}{q^{nk} n^k k!}$$

must contain at most a finite fixed number of prime factors of limied power in the denominator uniformly over k. This can only be when the nominator

$(n-1)(2n-1)\dots((k-1)n-1)$ almost cancels the denominator $k!$ up to a finite power of a finite number of prime factors for any k. It is cleary so that n cannot be an odd integer $2m+1$ since otherwise the nominator would be of the kind $(k-1)!m^{k-1}$ so that a factor of $\frac{1}{q^{(2m+1)k}k(2m+1)^k}$ survives which cannot be put on one denominator by choice of p. For the case $n = 2m$ with $m > 1$, one obtains gaps in the denominator between $1\dots 2(m-1), 2m, 4m-2, ...$ so that each time have a gap with $m-1$ odd numbers implying we have $(k-1)(m-1)$ odd "gap" numbers. So, we have to prove that there exists an infinite number of k such that p^{2mk} is not a product of numbers of the form $l2m-1$ or equivalently that there exists an infinite number of primes which are not as such. They simply are of the form $p_1\dots p_s + 1$ where p_j is the j'th prime number. Therefore, only $m = 1$ or equivalently $n = 2$ is allowed for. QED

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