Born-Infeld Theory Geometrized

Gerald Vones [1]

Abstract

A symmetry facilitated by Newton's constant inside a generalized metric brings together the concepts of spacetime and vector fields in a way reminding of the unification of space and time by Special Relativity, with a subtle difference though. This can be regarded as a geometrization of the historic "New Field Theory" of Born and Infeld. The underlying symmetry brings some insight in the fact that the cosmological term does not gravitate.

[1] Metahofgasse 9, 8020 Graz, Austria, European Union, mailto:gerald@vones.eu (retired, no affiliation)

1 Introduction and history

Special Relativity (SR) brought the important insight that time t does not exist on its own, rather spans a 4-dimensional metric manifold in union with what we experience as space. In addition to t this “spacetime” is parametrized by the three components $\vec{x}$ of what shall be called abstract space. When it gets physical, spacetime is inhabited by a number of (pointlike) objects. For any object called n comes a concrete worldline $\vec{x}_n(t)$ as a submanifold of abstract spacetime, with time as the only parameter remaining.

The nonrelativistic Lagrangian of a free object reads $\frac{m}{2}\dot{\vec{x}}^2$, where m is mass, and the overdot means derivative w.r.t. coordinate time t. SR modified this to be consistent with the Lorentz symmetry of spacetime. If the energies are well below the masses of any of the objects or its constituents involved, one can explicitly omit the rest term. Then the Lagrangian of any such object reads $mc^2\left[1-\sqrt{1-\left(\frac{\dot{\vec{x}}}{c}\right)^2}\right]$, where c is the velocity of light, which will be set unity in the following except where explicitly stated otherwise.

It does not request much effort to reckognize a close analogy between Lagrangians of (massless) fields and the nonrelativistic Lagrangian of objects, hence to conjecture a mechanism in close analogy to SR. It is more than remarkable that for the case of the electromagnetic field a corresponding result was published already in 1934 by born Born and Infeld [1], although they did not at all follow the above argumentation. If they had, it would have appeared as uncircumventable that the fundamental implications reach out to the entirety of meanwhile discoverd vector fields, thus the associated millenium problem.

The electromagnetic vector field A is abelian, massless and gauge invariant. Its currently used Lagrangian reads $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. ∂ is a partial derivative and indices everywhere in this paper are moved by means of the spacetime metric $g_{\mu\nu}$ which shall be abbreviated as g and whose determinant will be denoted $\det\left(g_{\mu\nu}\right)$. It is helpful to reckognize that this term has no individual forefactor comparable to mass in Special Relativity, rather this is absorbed in the field per definition. As will turn out later, such individual factors exist and discriminate among different vector fields.

The final version of Born and Infeld for the electromagnetic field action was

$$S = b^2 \int \left[\sqrt{-\det\left(g_{\mu\nu}\right)} - \sqrt{-\det\left(g_{\mu\nu} + \frac{1}{b}F_{\mu\nu}\right)} \right] \mathrm{d}^4x =$$

$$= b^2 \int \left[1 - \sqrt{1 + \frac{1}{2b^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{16b^4}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2} \right] \sqrt{-\det\left(g_{\mu\nu}\right)}\, \mathrm{d}^4x\,, \quad (1)$$

where b is a parameter with the same dimension as F, that is the square root from an energy density. $\tilde{F}$ is the Hodge dual of F. The entire formula is taken from reference [2], whose notation and conventions are applied. This clearly implies nonlinear field equations, but for a completly different reason than the internal gauge groups which are known today in the context of Yang-Mills fields.

It shall be clarified that - despite of the central role Newton's constant will play - this paper is not about gravity. Hence one can replace g by the Minkowski metric η in the above formula and everywhere else without missing the substance. In this case, there is no need to discriminate between partial and covariant derivatives in the appendix. But since Born and Infeld introduced this convincing way of coupling vector fields to the metric, this is retained here. It can be added that publications exist dealing with so called Born-Infeld gravity theories where essentially in place of the antisymmetric F comes the symmetric Ricci tensor, but these are not relevant here.

From the second line of equation (1) it is obvious that the desired approximation works if b^2 is sufficiently large. [1] conjectured $|b|$ to be order of magnitude $\frac{m_-^2}{|e|^3}$, where m_- is the electron mass and e is the elementary charge, since their considerations focused on the field energy of the electron. This was a quite straightforward idea in those early days of particle physics. But today, this is falsified from both theory and observation. There is a vast variety of elementary particles and picking the electron appears as quite random. The limits from observation recently were estimated by [3, 4]. Rather, one should acknowledge that the expansion of the square root in the second line of equation (1) yields a dominant contribution $-b^2\sqrt{-\det g}\,\mathrm{d}^4x$, which acts as cosmological term. It is derived on a "classical" route, furthermore it is omitted to be in line with observation. Nevertheless, it has to be consistent with the ideas of quantum physics which specifies the vacuum by a cutoff of momentum integrals at Planckian energies. So b should be of Planckian orders of magnitude.

In the context of string theory, the Born-Infeld action has gained renewed attention, but nowhere the main idea of this paper is developed, namely that fields span extra dimensions attached to spacetime, which in the case of vectors become paired with the degrees of spacetime.

2 Motivation

The most fundamental symmetries of nature are those mediated by the elements of the Planckian set of units: the velocity of light, Planck's constant $\hbar$ and Newton's constant G. The velocity of light appears inside the line element of spacetime in a very transparent way which has made it the superstar among all constants, even reckognized by the broad public. Statistical physics in union with quantization teaches that physical quantities are to be expressed as pure numbers in a uniquely defined way by means of the phase space volume form $\omega = \pm \frac{\mathrm{d}p \wedge \mathrm{d}x}{\hbar}$, where x is position and p is conjugate momentum, respectively. In contrast, the current role of G is way less comprehensible. It may be worth trying to put it inside some generalized metric like c and $\hbar$. Since it has dimension $\frac{length}{momentum}$ or equivalent - what remarkably only is the case in our number of dimensions - it could well find its place inside some modification of ω.

As another historic fact, it was again Max Born who proposed a so called reciprocity symmetry relating coordinates and momenta. In the essence, in addition to the skew-symmetric structure there is the separate conservation of a symmetric line element where G acts a conversion factor between space and conjugate momentum [5]. In this paper, a quite different route will be demonstrated to lead to the geometrization of the Born-Infeld theory. First, to ω is added a pure spatial term $\hat{\omega} := \omega + \frac{\mathrm{d}x \otimes \mathrm{d}x}{G\hbar}$, where the relative sign is not important. In appropriate coordinates this reads

$$\hat{\omega} = \begin{pmatrix} 1 & -G \\ G & 0 \end{pmatrix} . \tag{2}$$

Second, this result is interpreted as the generalized metric of a 2-dimensional space, which however no longer acts as phase space of an object (in one dimension). Rather, momentum will aquire a new role.

The natural dimension of a bosonic field is $\frac{1}{\sqrt{momentum \cdot length^3}}$ [6], since its square essentially counts the probability density of quanta. For historic reasons there usually is a factor of $\hbar$ involved in electrodynamics, but here the natural dimension shall be retained for clarity. In high energy physics, this field usually is multiplied by $\sqrt{\hbar^3}$ what assigns to it the dimension of *momentum*. If multiplied by the respective dimensionless coupling constant, this becomes a quantity with immediate physical meaning, namely it is the additional momentum aquired by an elementary charge immersed in the field. This is the appropriate quantity to work with in the space defined above. In place of phase space dynamics of an object with such momentum comes differential geometry applied to the field. In the following, the field times the coupling constant times $\sqrt{\hbar^3}$ shall be denoted as $\mathscr{A}$.

3 The geometric framework

[1] deals with electrodynamics what yet is sufficient to comprehend the mechanism. Like in SR the action of the object is derived from the embedding of the concrete 1-dimensional worldline in abstract 4-dimensional spacetime, here the action of electrodynamics will be derived from embedding a concrete 4-dimensional "field manifold" called $\mathscr{M}$ in an abstract 8-dimensional space. This embedding space is spanned by 4-spacetime x^μ and the abstract vector field $\mathscr{A}^\mu$ as defined above. Its generalized metric is the tensor product

$$U = g \otimes \hat{\omega} \,, \tag{3}$$

where the overall sign is not important. Whether numerical factors order of magnitude unity are involved is not decisevely clear and these shall be included in the definition of G. The signs of the off-diagonal elements could be flipped, what means a change of the sign of charge and has no influence on the result. The sign convention of spacetime can be left unspecified. That G appears is the consequence of the Planckian scale. Like c acts as a metric conversion factor in SR, so G now does at the level of field physics. Spacetime and vector fields living in it are unified like space (or spatial positions of objects) and time are unified in SR. This is the central innovation brought forward by this paper, which up to now has not been introduced anywhere, neither constructively nor in the context of no-go theorems.

Whereas Born and Infeld focused on invariancies inside spacetime alone, now the field action can be derived from pure geometry. The 4 field equations $\mathscr{A}^\mu(x^0,x^1,x^2,x^3)$ define the concrete 4-dimensional $\mathscr{M}$ embedded in the abstract 8-dimensional space. Its 4-volume can be derived by estimating the induced metric called γ by means of Gauss' formula

$$\gamma_{\mu\nu} = U_{ij} \frac{\partial X^i}{\partial x^\mu} \frac{\partial X^j}{\partial x^\nu} \,, \tag{4}$$

where X are the eight contravariant embedding space coordinates at the location of $\mathscr{M}$, and x are the four contravariant coordinates on $\mathscr{M}$ simply identified with part of the X and interpreted as coordinates in spacetime. ∂ is a partial derivative. This means $\frac{\partial X^\mu}{\partial x^\nu} = \delta^\mu_\nu$ for $\mu, \nu = 0,1,2,3$, while for $\mu = 4,5,6,7$ there is the identification $X^\mu = \mathscr{A}^{\mu-4}$.

In contrast to the coordinate functions x, the concrete $\mathscr{A}$ shall become a vector field living in spacetime. There is no good analogy in SR, since there one-dimensional time is the only parameter describing the worldline. Here, any $\mathscr{M}$ is parametrized by the entirety of spacetime. When its respective 4-volume for any concrete vector field is calculated, the transformation properties of the field in 4-dimensional spacetime have to be taken into account. In other words, the partial derivatives acting on the components of $\mathscr{A}$ have to be replaced by covariant ones associated with the first factor in equation (3), that is the metric of spacetime. The calculation is done in the appendix. The result is

$$\tilde{\gamma}_{\mu\nu} = g_{\mu\nu} \pm G(\partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu) \ , \tag{5}$$

where the tilde over γ shall indicate the said modification of Gauss' formula.

The mechanism explains where the derivatives in the action originate from in terms of differential geometry. The relevant element of 4-volume is proportional to the square root from the determinant of the absolute value of the modified induced metric $\tilde{\gamma}$. Hence, equation (5) immediately implies equation (1) if, for the concrete case of electrodynamics the replacedment $\mathscr{A} \to eA$ is performed, where the factors bear their historic units of $\sqrt{momentum \cdot length}$ and $\sqrt{\frac{momentum}{length}}$, respectively, and

$$|b| \to \frac{1}{|e|G} \ . \tag{6}$$

This value originates from those conjectured in [1] by the substitution $\left(\frac{e}{m_-}\right)^2 \to G$. b^2 is the Planckian energy density ρ_{pl} divided by the fine structure constant α understood as the low energy value

$$b^2 = \frac{1}{G^2 \hbar \alpha} \approx 137 \ \rho_{pl} \ , \tag{7}$$

where $\hbar$ is Plancks reduced constant.

4 Yang-Mills fields

As meanwhile is known, the electromagnetic field is not even a fundamental one, rather lies in a specific manner inside the SU(2)xU(1) symmetry. The fundamental fields which are all massless prior to the Higgs mechanism are 1 weak hypercharge, 3 weak isospin and 8 gluons. Like in SR there can live numerous objects in one and only spacetime, here live a dozen of vector fields in the one and only embedding space, each with its own $\mathscr{M}_n$ with n running from 1 to 12.

The currently used action for such fields is (the index of the field denoted as upper index) with a running through a subset of the n

$$L = -\frac{1}{4}\sum_a F^a_{\mu\nu} F^{a\,\mu\nu} \qquad \text{with} \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \tilde{\alpha} f^{abc} A^b_\mu A^c_\nu \,, \tag{8}$$

where $\tilde{\alpha}$ is the respective coupling constant. f are the totally antisymmetric structure constants of the internal gauge group, so this term is also antisymmetric in μ and ν. The parts of F containing partial derivatives are identical to those discussed in the context of electrodynamics, and their geometrization comes along exactly in the same way only with $\tilde{\alpha}$ replacing α in equation (7), again understood as the low energy value. The part containg the f can be regarded as originating from an interaction among the respective field manifolds, what is not a topic of this paper. But the term can be taken into account when equation (1) is evaluated. The Higgs mechanism then is another phenomenon not be regarded here. Despite of the said aspects left aside, the conclusion is clear: The current action used for Yang-Mills fields is not a correct basis for their description, rather modifications in line with the Born-Infeld theory are inevitable.

The antisymmetry of the dynamic parts (those containing derivatives) of the action is woven into equation (3), and this makes sense. This has directly to do with the spacetime properties of any vector field, projecting out the pure helicity 1 component. This would not work if the lower right entrance of $\hat{\omega}$ was unequal zero. In particular, the natural supression of this entrance by a factor of G^2 would lead to diagonal contributions to the determinant same order of magnitude as the off-diagonal ones.

One aspect could turn out as quite essential. Any $\mathscr{M}_n$ is a membrane, with an action resulting from embedding associated. Mathematically, string and membrane theory yet has reached an enormously advanced stage, but never has a membrane been given the physical interpretation as it is the case here.

5 Omitting the cosmological term

From the coupling to spinor fields it is known that gravity is to be formulated in terms of tetrads, which are four mutually coupled vector fields. This fits well to the ideas presented here, however shall be treated elsewhere because of the amount of discussion necessary. But yet from equation (3) emerges an argument why the cosmological term can be omitted as a source of gravity even if simply the Einstein-Hilbert curvature term is added as reference [1] suggests. $\hat{\omega}$ does not describe Lorentz symmetry as it is the case in Special Relativity. Rather, introducing the transformation matrix $K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and its transpose K^T (symbols not to be confused with similar ones used elsewhere), it is

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 & -G \\ G & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 & ab - Gad + Gbc \\ ab - Gbc + Gad & b^2 \end{pmatrix} . \tag{9}$$

If $\hat{\omega}$ shall be conserved by this transformation, the three restricting equations are $b = 0$, $a = d = \pm 1$. This leaves c as a free parameter. The determinant of the transformation matrix is unity in any case. If the positive sign is chosen for a, such transformation applied to a contravariant vector has the effect $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} x \\ \mathscr{A} \end{pmatrix} = \begin{pmatrix} x \\ \mathscr{A} + cx \end{pmatrix}$, where c has the inverse dimension of Newton's constant. This means only $\mathscr{A}$ is modified, while spacetime remains unchanged thus defining an absolute frame of reference. This is the crucial difference to Lorentz symmetry where no preferred direction of time exists.

To see how this can be extended to locally Minkowskian spacetime, it is sufficient to regard time plus one spatial dimension, i.e. an embedding space with metric $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & -G \\ G & 0 \end{pmatrix}$. The transformation matrix $K_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ c_1 & 1 & C & 0 \\ 0 & 0 & 1 & 0 \\ -C & 0 & c_2 & 1 \end{pmatrix}$, where c_1, c_2, C are free parameters, leaves this metric invariant. c_1 and c_2 are just the parameters of transformations inside the respective pair of x and $\mathscr{A}$ discussed above, whereas C intertwines different degrees of freedom of spacetime.

Omitting the cosmological term is to say, gravity feels the 4-volume (based on $\tilde{\gamma}$) of any $\mathscr{M}_n$ minus the 4-volume of the manifold $\mathscr{A}^\mu = const^\mu$, that simply is spacetime itself. Spacetime is intrinsically curved because the embedding space is while its embedding equations are trivial, whereas any $\mathscr{M}_n$ is intrinsically curved for both reasons, the curved embedding space and the nontrivial embedding equations. Topological aspects are determined by those of the embedding space.

Having estimated where the geometrized Born-Infeld theory differs from SR, namely what regards the constant (mass there, cosmological here) term, one can summarize the similarities. For this sake, half of the antisymmetrized gradient of $\mathscr{A}$ here shall be identified with velocity there. The found role of the coupling constant is very plausible, since it is the only constant associated with a field other than the mass of its quanta. τ is proper time of any object and $\tilde{V}_4$ is the 4-volume

theory	mapping	action proportional to	global constant	individual constant
Special Relativity	1 time →3-position	$\int_{worldline} \mathrm{d}\tau$	c	m
Born-Infeld	4-spacetime →4-vector field	$\int_{\mathscr{M}} \mathrm{d}\tilde{V}_4$	$\frac{1}{G}$	$\frac{1}{\alpha}$

of any $\mathscr{M}$, where the tilde shall indicate that it is derived from $\tilde{\gamma}$. α shall mean the respective coupling constant (which, as a matter of definition, can be multiplied by $\hbar$).

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Appendix: Calculating the induced metric

Gauss'formula for the induced metric reads

$$\gamma_{\mu\nu} = U_{ij} \frac{\partial X^i}{\partial x^\mu} \frac{\partial X^j}{\partial x^\nu} \,, \tag{10}$$

where the X are the contravariant embedding space coordinates of the embedded manifold, x are contravariant coordinates inside the embedded manifold and U is the metric of the embedding space. Here U is given by equation (3)

$$U = \pm g \otimes \begin{pmatrix} 1 & -G \\ G & 0 \end{pmatrix} \,, \tag{11}$$

where G is Newton's constant with dimension $\frac{length}{momentum}$ and the overall sign is not important. The contravariant X^i are grouped as an 8-vector in embedding space as $\mathrm{col}(x^0\ x^1\ x^2\ x^3\ \mathscr{A}^0\ \mathscr{A}^1\ \mathscr{A}^2\ \mathscr{A}^3)$. The x are coordinates in spacetime - hence g is the metric of spacetime - and have dimension of *length* while the $\mathscr{A}$ have dimension of *momentum*.

To apply Gauss' formula, partial differentiation w.r.t. the x^μ (symbolized by comma) yields the four 8-vectors

$$\begin{aligned}
&\mathrm{col}(1\ 0\ 0\ 0\ \mathscr{A}^0_{,0}\ \mathscr{A}^1_{,0}\ \mathscr{A}^2_{,0}\ \mathscr{A}^3_{,0})\\
&\mathrm{col}(0\ 1\ 0\ 0\ \mathscr{A}^0_{,1}\ \mathscr{A}^1_{,1}\ \mathscr{A}^2_{,1}\ \mathscr{A}^3_{,1})\\
&\mathrm{col}(0\ 0\ 1\ 0\ \mathscr{A}^0_{,2}\ \mathscr{A}^1_{,2}\ \mathscr{A}^2_{,2}\ \mathscr{A}^3_{,2})\\
&\mathrm{col}(0\ 0\ 0\ 1\ \mathscr{A}^0_{,3}\ \mathscr{A}^1_{,3}\ \mathscr{A}^2_{,3}\ \mathscr{A}^3_{,3})
\end{aligned} \tag{12}$$

To get the 4 covariant 8-vectors, the above have to be multiplied by

$$U = \begin{pmatrix}
g_{00} & g_{01} & g_{02} & g_{03} & -Gg_{00} & -Gg_{01} & -Gg_{02} & -Gg_{03}\\
g_{10} & g_{11} & g_{12} & g_{13} & -Gg_{10} & -Gg_{11} & -Gg_{12} & -Gg_{13}\\
g_{20} & g_{21} & g_{22} & g_{23} & -Gg_{20} & -Gg_{21} & -Gg_{22} & -Gg_{23}\\
g_{30} & g_{31} & g_{32} & g_{33} & -Gg_{30} & -Gg_{31} & -Gg_{32} & -Gg_{33}\\
Gg_{00} & Gg_{01} & Gg_{02} & Gg_{03} & 0 & 0 & 0 & 0\\
Gg_{10} & Gg_{11} & Gg_{12} & Gg_{13} & 0 & 0 & 0 & 0\\
Gg_{20} & Gg_{21} & Gg_{22} & Gg_{23} & 0 & 0 & 0 & 0\\
Gg_{30} & Gg_{31} & Gg_{32} & Gg_{33} & 0 & 0 & 0 & 0
\end{pmatrix} \tag{13}$$

To process the result one has to take into account the relation

$$g_{\mu\alpha}\mathscr{A}^\alpha_{,\nu} = (g_{\mu\alpha}\mathscr{A}^\alpha)_{,\nu} - g_{\mu\alpha\,,\nu}\mathscr{A}^\alpha = \mathscr{A}_{\mu,\nu} - g_{\mu\alpha,\nu}\mathscr{A}^\alpha \equiv \mathscr{A}_{\mu,\nu} - g_{\alpha\mu,\nu}\mathscr{A}^\alpha\,. \tag{14}$$

This yields (only the first two written out, since this is sufficient to see the mechanism)

$$\begin{pmatrix}
g_{00} - G\mathscr{A}_{0,0} + Gg_{0\mu,0}\mathscr{A}^\mu\\
g_{10} - G\mathscr{A}_{1,0} + Gg_{1\mu,0}\mathscr{A}^\mu\\
g_{20} - G\mathscr{A}_{2,0} + Gg_{2\mu,0}\mathscr{A}^\mu\\
g_{30} - G\mathscr{A}_{3,0} + Gg_{3\mu,0}\mathscr{A}^\mu\\
Gg_{00}\\
Gg_{10}\\
Gg_{20}\\
Gg_{30}
\end{pmatrix}
\begin{pmatrix}
g_{01} - G\mathscr{A}_{0,1} + Gg_{0\mu,1}\mathscr{A}^\mu\\
g_{11} - G\mathscr{A}_{1,1} + Gg_{1\mu,1}\mathscr{A}^\mu\\
g_{21} - G\mathscr{A}_{2,1} + Gg_{2\mu,1}\mathscr{A}^\mu\\
g_{31} - G\mathscr{A}_{3,1} + Gg_{3\mu,1}\mathscr{A}^\mu\\
Gg_{01}\\
Gg_{11}\\
Gg_{21}\\
Gg_{31}
\end{pmatrix} \tag{15}$$

To arrive at the induced metric, the (transposes of the) 4 contravariant vectors (equation (12)) have to be contracted by the 4 covariant ones (equation (15)). For example zeroth contravariant vector times zeroth covariant vector yields $g_{00} - G\mathscr{A}_{0,0} + Gg_{\alpha 0,0}\mathscr{A}^{\alpha} + G\mathscr{A}_{0,0} - Gg_{\alpha 0,0}\mathscr{A}^{\alpha} = g_{00}$. Zeroth contravariant vector times first covariant vector yields $g_{01} - G\mathscr{A}_{0,1} + Gg_{\alpha 0,1}\mathscr{A}^{\alpha} + G\mathscr{A}_{1,0} - Gg_{\alpha 1,0}\mathscr{A}^{\alpha}$. And so on. So, again substituting $\mathscr{A} \to eA$ the result is

$$\gamma_{\mu\nu} = g_{\mu\nu} \pm G(\partial_\mu \mathscr{A}_\nu - \mathscr{A}^\alpha \partial_\mu g_{\alpha\nu} - \partial_\nu \mathscr{A}_\mu + \mathscr{A}^\alpha \partial_\nu g_{\alpha\mu}) = g_{\mu\nu} \pm eG(F_{\mu\nu} + \delta F_{\mu\nu}) \,, \tag{16}$$

where F is the usual expression for the electromagnetic field as it appears in particular in the Born-Infeld theory, whereas δF is a peculiar extra term.

If one expresses the derivatives of the metric in terms of the Christoffel symbols Γ, one gets

$$\delta F_{\mu\nu} = A^\alpha(\Gamma_{\mu\nu\alpha} - \Gamma_{\nu\mu\alpha}) \,. \tag{17}$$

In Einsteinian gravity, the Christoffel symbols are symmetric in the last two indices, but not in the first two. So this term vanishes if g is the Minkowski metric, but not in general - rather has to be subtracted explicitely.

To get rid of this additional term, one has to put covariant derivatives in place of the partial ones. This means taking into account the behaviour of the concrete vector field under transformations in spacetime as discussed in the main text. Then the contravariant index is simply lowered in equation (14), i.e. $g_{\mu\alpha}\mathscr{A}^{\alpha}_{;\nu} = \mathscr{A}_{\mu;\nu}$.

In the final result, the covariant derivatives can be re-substituted by the partial ones because of the skew-symmetry of the expression and the symmetry of the Christoffel symbols.

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