New Representation of the Euler Constant "e"

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Summary:

This article reminds us that certain fundamental constants are associated with functions, in particular the base of the exponential function. We use certain remarkable identities associated with the exponential function and Lambert's function W (also called Product log) to obtain a new representation of the Euler constant e = exp(1) using a function.

Demonstration:

According to Euler's equation

$$e^{i\pi} = -1 = i^2$$

we deduce that

$$\pi = -2i \ln i$$

And [i] $W_0\left(-\frac{\pi}{2}\right) = i\,\frac{\pi}{2}$

therefore, relationship (1)

$$i = \frac{-\pi/2}{W_0\left(-\frac{\pi}{2}\right)} = \frac{2}{\pi} W_0\left(-\frac{\pi}{2}\right)$$

with W_0 the Lambert function (the main branch of the Product log).

It is also accepted that

$$i^i = e^{-\frac{\pi}{2}}$$

therefore, relation (2) :

$$e = \left(e^{-\frac{\pi}{2}}\right)^{-\frac{2}{\pi}} = i^{\frac{-2i}{\pi}} = i^{\frac{-2i}{-2i\ln i}} = i^{\frac{1}{\ln i}}$$

Since pi and e are related to i, we deduce that e is related to pi: (1) combined with (2)

$$e = -\frac{\pi}{2W_0\left(-\frac{\pi}{2}\right)}^{\frac{1}{\ln -\frac{\pi}{2W_0\left(-\frac{\pi}{2}\right)}}}$$

In addition, we can see (graphically, or by technically complex calculation based on the derivation properties of the W and ln functions) that

$$\forall x \in \mathbb{R}, \qquad \frac{d}{dx}\left(-\frac{x}{2W_0\left(-\frac{x}{2}\right)}^{\frac{1}{\ln\left(-\frac{x}{2W_0\left(-\frac{x}{2}\right)}\right)}}\right) = 0$$

so

$$\boxed{\forall x \in \mathbb{R}, f(x) = -\frac{x}{2W\left(-\frac{x}{2}\right)}^{\frac{1}{\ln\left(-\frac{x}{2W\left(-\frac{x}{2}\right)}\right)}} = e}$$

to be compared, for example, with long-established trigonometric functions such as[ii]

$$g(x) = \cos x^2 + \sin x^2 = 1, \forall x \in \mathbb{R}$$

$$h(x) = \tan^{-1}\frac{1}{x} + \tan^{-1} x = \frac{\pi}{2}, \forall x \in \mathbb{R}^+$$

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[i] Lambert W function - Wikipedia (special values)
[ii] Tangent arc - Wikipedia (wikipedia.org)