Lecture Notes on a quadratic inequality for gaps between consecutive primes and its extension to the proof of the Binary Goldbach conjecture [v3]

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Abstract

This research paper is in the form of lesson notes. In it an identity is established connecting to consecutive primes. Bertrand's postulate is used together with the identity to establish a quadratic inequality that can be used to establish minimum intervals containing at least three primes in between its limits. A generalization of the quadratic inequality is introduced to establish the minimum interval containining at least one pair of primes for Goldbach partitition. The concepts of Goldbach partition deviation and Goldbach partition interval are introduced by which it is shown that the minimum number of Goldbach partitions of a composite even number is 1.

Learning objectives

The learner should be able to

1. Derive an identity connecting two consecutive prime numbers

2. Obtain with the aid of Bertrand's postulate a quadratic inequality for solving the prime gap problem
3. Use the derived quadratic inequality to obtain intervals containing three primes
4. Define a Goldbach partition deviation and a Goldbach partition interval.
5. Extend the derived quadratic inequality to derive an interval containing at least one pair of primes for Goldbach partition
6. Derive inequalities by which the minimum number of Goldbach partitions of a composite even number is equal to 1
7. Establish a relationship between two Goldbach partition counting functions and show that the relationship implies that the minimum number of Goldbach partitions of a composite even number is equal to 1.
8. Estimate the number of Goldbach paritions using a suspected number of Goldbach partitions
9. To do a more through analysis of the prime gap problem and to establish establish the conditions under which the Oppermann's conjecture is met.
10. Obtain an exact prime gap relationship that accounts for the different conjecture on prime gaps
11. Explain the Riemann hypothesis Dimension of the prime prime gap problem.
12. Define and apply the concept of relative size of a prime gap.

Introduction

These lecture notes are derivatives from pages 19 to 26 of paper reference [1]. The expositions in this paper is meant to provide some insight on some cocepts in the paper that may not clear to its readers.

13. Explain how the Riemann hypothesis can be disproved
14. Prove Opperman's conjecture

Prime gaps upper bounds and justification for a quadratic inequality for their solution

Bertrand's postulated, proved in 1852, states that there is always a prime number between m and 2m (m is an integer greater or equal to 2) meaning that $p_{i+1} < 2p_i$.

This also means $g_n $. Hoheisel [6] in 1930, was the first to show that there exists a constant $\theta < 1$ such that

$$\pi(x + x^{\theta}) - \pi(x) \approx \frac{x^{\theta}}{\ln x} x \to \infty$$

hence showing that

$$g_i < p_i^{\theta}$$

for a sufficiently large p_i.

Ingham [4] showed that for a positive constant c, If

$$\zeta(\frac{1}{2} + it) = O(t^c)$$

Then

$$\pi(x + x^{\theta}) - \pi(x) \approx \frac{x^{\theta}}{\ln x} x \to \infty$$

for any $\theta > (1 + 4c)/(2 + 4c)$

A result due to Baker, Haman and Pintz [5] in 2001 shows that θ may be taken to be 0. 525. Thus the best proven bound on gap sizes is $g_i < p_i^{0.525}$ for i sufficienty large. It is observed that maximal gaps are significantly smaller than the above gap. There are hypothesis like the Opperman's conjecture that claim that θ can be reduced to $\theta = 0.5$.

Analysis of the claims of Opperman's conjecture

Opperman's conjecture implies

$$p_{i+1} < p_i + \sqrt{p_i} \land i > 30$$

This conjecture requires a deeper examination of quadratic inequalities of consecutive prime numbers.

Deriving an identity connecting to consectutive primes

Consider two numbers represented by two algebraic terms a and b. We can establish an identity connecting a and b through the steps below.

$$ab + (\frac{a-b}{2})^2 = \frac{4ab + a^2 - 2ab + b^2}{4} = \frac{a^2 + 2ab + b^2}{4} = (\frac{a+b}{2})^2 \tag{1}$$

Therefore

$$ab+(\frac{a-b}{2})^2=(\frac{a+b}{2})^2 \tag{2}$$

Now consider two consecutive primes p_i and p_{i+1} If we now set $a=\sqrt{p}_{i+1}$ and $b=\sqrt{p}_i$ then

$$\sqrt{p_i p_{i+1}}+(\frac{\sqrt{p}_{i+1}-\sqrt{p}_i}{2})^2=(\frac{\sqrt{p}_{i+i}+p_i}{2})^2 \tag{3}$$

For the purpose of achieving a quadratic inequality, the above identity will be re-arranged to a more covenient form. That is:

$$\sqrt{p}_{i+1}+\sqrt{p}_i=2\sqrt{((\frac{\sqrt{p}_{i+1}-\sqrt{p}_i}{2})^2+\sqrt{p_i p_{i+1}})} \tag{4}$$

It also means that

$$\sqrt{p}_{i+1}+\sqrt{p}_i=2\sqrt{((\frac{\sqrt{p}_{i+1}-\sqrt{p}_i}{2\sqrt[4]{p_i p_{i+1}}})^2+1)} \tag{5}$$

This also means

$$\sqrt{p}_{i+1}+\sqrt{p}_i=2\sqrt{(p_i p_{i+1})^{\frac{1}{2}}(\frac{p_i-p_{i+1}}{2})^2((\frac{2}{p_i-p_{i+2}})^2+\frac{1}{p_i p_{i+1}})} \tag{6}$$

That is

$$\sqrt{p}_{i+1}+\sqrt{p}_i=2(p_i p_{i+1})^{\frac{1}{4}}\frac{\sqrt{p}_{i+1}-\sqrt{p}_i}{2}\sqrt{(\frac{2}{\sqrt{p}_{i+1}-\sqrt{p}_i})^2+\frac{1}{(p_i p_{i+1})^{\frac{1}{2}}}} \tag{7}$$

Using Bertrand's postulate in a rearranged form to obtain a quadratic inequality for solving the prime gap problem

Bertrand's postulate postulate requres $p_{i+1}<2p_i$. Therefore. Therefore substituting $p_{i+1}=2p_i$

$$\frac{\sqrt{p}_{i+1}-\sqrt{p}_i}{2}\sqrt{(\frac{2}{\sqrt{p}_{i+1}-\sqrt{p}_i})^2+\frac{1}{(p_i p_{i+1})^{\frac{1}{2}}}}<\frac{\sqrt{p}_i(\sqrt{2}-1)}{2}\sqrt{(\frac{2}{\sqrt{2p_i}-\sqrt{p}_i})^2+\frac{1}{\sqrt{2}p_i}} \tag{8}$$

Therefore

$$\frac{\sqrt{p}_{i+1}-\sqrt{p}_i}{2}\sqrt{(\frac{2}{\sqrt{p}_{i+1}-\sqrt{p}_i})^2+\frac{1}{(p_ip_{i+1})^{\frac{1}{2}}}}<\sqrt{1+\frac{(\sqrt{2}-1)^2}{4\sqrt{2}}}=1.015 \tag{9}$$

Now because

$$\sqrt{p}_{i+1}+\sqrt{p}_i>2\sqrt{p_{i+1}p_i} \tag{10}$$

$$1<\frac{\sqrt{p}_{i+1}-\sqrt{p}_i}{2}\sqrt{(\frac{2}{\sqrt{p}_{i+1}-\sqrt{p}_i})^2+\frac{1}{(p_ip_{i+1})^{\frac{1}{2}}}}<1.015 \tag{11}$$

The function

$$f(p_i)=1.05^{\frac{1}{p_i}}\wedge p_i>3 \tag{12}$$

lies within the interval $(1, 1.05)$ Therefore intervals containing three primes are determined by solving the quadratic inequality below.

$$\sqrt{p}_{i+1}+\sqrt{p}_i<2\times 1.05^{\frac{1}{p_i}}\sqrt[4]{p_{i+1}p_i} \tag{13}$$

The important result about the above quadratic inequality is that primes greater that 9500 achieve the gap result

$g_i < p_i 0.525$

Using the quadratic inequality (13) to obtain intervals containing three primes

Example 1 Find the integer inteval centered around $p_i = 7$ containing at least three primes

Solution

$$\sqrt{x}+\sqrt{7}<\sqrt[4]{7x}\times 1.05^{\frac{1}{7}}$$

calculator step

$$4\leq x\leq 11$$

In this interval the three primes are 5, 7 and 11.

Example 2 Use the inequality above to find at least 3 primes centering around 23.

Solution

$$\sqrt{x} + \sqrt{2}3 < \sqrt[4]{23x} \times 1.05^{\frac{1}{7}}$$

Calculator step. The integer interval is:

$$17 \leq x \leq 29$$

In the above interval the primes are 17, 19, 23 and 29. The limitations of inequality (13) is that it cannot account for the observable gaps $p_i < p_i$. There is still need to come up with an approach that takes care of these gaps.

Goldbach partition deviation and interval

If $2m$ is a composite even number, we will define a Goldbach partition deviation as the ratio of m to the number of Goldbach partitions of 2m. If $R(2m)$ is the number of Goldbach partitions of 2m and d_g is Goldbach partition Goldbach partition deviation then

$$d_g = \frac{m}{R(2m)} \tag{14}$$

Thus by the above definition all composite even numbers wih having 1 Goldbach partition $d_g = m$ A Goldbach partition interval is an interval containing at least 1 Goldbach partition and its limits are defined as

$$m - d_g < i_g < m + d_g \tag{15}$$

The number 100 has 6 Goldbach partitions. This means $d_g = 8$. An interval containining primes for one Goldbach partition of 100 is $42, 58$. The Goldbach partition prime pairs in this are $(47, 53)$. Now we can construct an equation that determines this interval given

$$\sqrt{5}0 + \sqrt{x} < 2 \times 1.05^{\frac{1}{50}} \sqrt[4]{50x} \tag{16}$$

and we note that

$$41.9 < x < 59.7$$

In this interval the Goldbach partition pairs are $(41, 59)$ and $(47, 53)$. The above the length of the above interval is $2d_g$

Now Consider the composite even number 12.
The composite even number will have 1 Goldbach partition if $d_g = m = 6$ This would mean that the interval containing primes making up one Goldbach partition would be $(0, 12$. The quadratic inequality:

$$\sqrt{6} + \sqrt{x} < 2 \times 1.05^{\frac{1}{6}}$$

The interval from the solution of the above is $(3.6 < x < 10)$. The Goldbach partition primes pair in this interval is $(5, 7)$.

Extending the derived quadratic inequality to derive an interval containing at least one pair of primes for Goldbach partition

The interval containing one pair primes for Goldbach partition of a composite even number, 2m can be determining through solving the quadratic inequality below.

$$\sqrt{m}+\sqrt{x}<2\times 1.05^{\frac{1}{m}}\sqrt[4]{mx} \tag{17}$$

Laws governing the number of Goldbach partition

From the solution of the quadratic inequality, the length of the interval containing three primes is given

$$(\sqrt[4]{p}_i(1.05^{\frac{1}{p_i}})+\sqrt{(\sqrt[4]{p}_i(1.05^{\frac{1}{p_i}})^2-\sqrt{p}_i})^4-\sqrt[4]{p}_i(1.05^{\frac{1}{p_i}})-(\sqrt{(\sqrt[4]{p}_i(1.05^{\frac{1}{p_i}})^2-\sqrt{p}_i})^4 \tag{18}$$

The maximum length of interval containing a pair of Goldbach partition primes

$$(\sqrt[4]{m}(1.05^{\frac{1}{m}})+\sqrt{(\sqrt[4]{m}(1.05^{\frac{1}{m}})^2-\sqrt{m}})^4-(\sqrt[4]{m}(1.05^{\frac{1}{m}})-(\sqrt{(\sqrt{m}(1.05^{\frac{1}{m}}))^2-\sqrt{m}})^4) \tag{19}$$

It is observed that if

$$(\sqrt[4]{m}(1.05^{\frac{1}{m}})+\sqrt{(\sqrt[4]{m}(1.05^{\frac{1}{m}}))^2-\sqrt{m}})^4-\sqrt[4]{m}(1.05^{\frac{1}{m}})-(\sqrt{(\sqrt[4]{m}(1.05^{\frac{1}{m}}))^2-\sqrt{m}})^4\leq 2m \tag{20}$$

Then 2m has at least one Goldbach partition. It should be noted that 2m is the largest possible Goldbach partition interval, while m is the largest possible Goldbach partition deviation. It is also observed that If

$$(\sqrt[4]{m}(1.05^{\frac{1}{m}})+\sqrt{(\sqrt[4]{m}(1.05^{\frac{1}{m}})^2-\sqrt{m})})^4-(\sqrt[4]{m}(1.05^{\frac{1}{m}})-\sqrt{(\sqrt[4]{m}(1.05^{\frac{1}{m}}))^2-\sqrt{m}})^4\leq m \tag{21}$$

then 2m has at least 2 Goldbach partitions. Thus by the two equations above, composite even numbers less than 14 have at least one Goldbach partition. and those greater or equal to 14 have at least two Goldbach partitions. Thus the number of Goldbach partitions function $R(2m)$ is governed by the inequality

$$R(2m)\geq\frac{m}{(\sqrt[4]{m}(1.05^{\frac{2}{m}})+\sqrt[4]{(\sqrt[4]{m}(1.05^{\frac{1}{m}}))^2-\sqrt{m}})^4-(\sqrt[4]{m}(1.05^{\frac{1}{m}})-\sqrt{(\sqrt[4]{m}(1.05^{\frac{1}{m}}))^2-\sqrt{m}})^4} \tag{22}$$

Let S_G represent the sum of Goldbach Partition primes Now

$$R(2m) = \frac{S_G}{2m} \tag{23}$$

Therefore

$$\frac{S_G}{2m} \geq \frac{m}{(\sqrt[4]{m}(1.05^{\frac{1}{m}}) + \sqrt{(\sqrt[4]{m}(1.05^{\frac{1}{m}})^2 - \sqrt{m})})^4 - (\sqrt[4]{m}(1.05^{\frac{1}{m}}) - \sqrt{(\sqrt[4]{m}(1.05^{\frac{1}{m}}))^2 - \sqrt{m})})^4} \tag{24}$$

This means that

$$S_G \geq \frac{2m^2}{(\sqrt[4]{m}(1.05^{\frac{1}{m}}) + \sqrt{(\sqrt[4]{m}(1.05^{\frac{1}{m}})^2 - \sqrt{m})})^4 - (\sqrt[4]{m}(1.05^{\frac{2}{m}}) - \sqrt{(\sqrt[4]{m}(1.05^{\frac{1}{m}}))^2 - \sqrt{m}})^4} \tag{25}$$

Substituting (23) into (24) we establish that

$$S_G \geq m \tag{26}$$

Therefore

$$R(2m) \geq \frac{1}{2} \tag{27}$$

This confirms the Goldbach conjecture to be true.

Again it should be noted that

$$R(2m) = \frac{S_G}{2m} = \frac{m}{d_g} \tag{28}$$

This means that

$$S_G = \frac{2m^2}{d_g} \tag{29}$$

since

$$d_g \leq m \tag{30}$$

then $S_G \geq 2m$ and $R(2m) \geq 1$ Again it is noted

$$m \approx \frac{(\sqrt[4]{m}(1.05^{\frac{1}{m}}) + \sqrt{(\sqrt[4]{m}(1.05^{\frac{1}{m}}))^2 - \sqrt{m}})^2 + (\sqrt[4]{m}(1.05^{\frac{1}{m}}) - \sqrt{(\sqrt[4]{m}(1.05^{\frac{2}{m}}))^2 - \sqrt{m}})^4}{2} \tag{31}$$

This result means that the minimum interval one can find Goldbach partition primes of $2m$ is

$$((\sqrt[4]{m}(1.05^{\frac{2}{m}})-\sqrt{(\sqrt[4]{m}(1.05^{\frac{1}{m}}))^2-\sqrt{m}}))^4, \sqrt[4]{m}(1.05^{\frac{1}{m}})+(\sqrt{(\sqrt[4]{m}(1.05^{\frac{1}{m}}))^2-\sqrt{m}}))^4) \tag{32}$$

Thus for $2m = 140$, the minimum interval for Goldbach partition of 140 for by the above equation is $(60.3, 81.3)$ or conveniently $(60, 80)$ In this interval the Goldbach partition pairs are $(61, 79)$ and $(67, 73)$

The minimum interval that can be taken to confirm that 4000 has a Goldbach partition is $(1944, 2056)$. In this interval the Goldbach partition pairs are $(1973, 2027)$ and $(1997, 2003)$.

The minimum interval that can be taken to confirm that 128 has a Goldbach partition is $(54, 74)$. In this interval the Goldbach partition pair is $(61, 67)$.

The minimum interval that can be taken to confirm that 32 has a Goldbach partition is $(11, 21)$. In this interval the Goldbach partition pair is $(13, 19)$.

From reference paper [2], the minimum interval of primes of Goldbach partition is

$$(m+\sqrt{m^2-s_{g_{max}}}, m-\sqrt{m^2-s_{g_{max}}}) \tag{33}$$

Where $s_{g_{max}}$ largest Goldbach partition semiprime. If

$$a = (\sqrt[4]{m}(1.05^{\frac{2}{m}} - \sqrt{(\sqrt[4]{m}(1.05^{\frac{2}{m}}))^2-\sqrt{m}})^2 \tag{34}$$

$$b = (\sqrt[4]{m}(1.05^{\frac{2}{m}}) + (\sqrt{(\sqrt[4]{m}(1.05^{\frac{2}{m}})^2-\sqrt{m}})^2 \tag{35}$$

and

$$c = \frac{b-a}{2} \tag{36}$$

Then

$$ab \leq s_{g_{max}} \leq ab + c^2 \tag{37}$$

Thus the maximum Goldbach partition semiprime of 128 is given by $54 \times 74 = 3996 \leq s_{g_{max}} \leq 54 \times 74 + 10^2 = 4096$ The largest Goldbach partition semiprime is actually 4087. If thecomposite even number $2m$ is not semiprime then the Goldbach partition prime pairs with a minimum gap between them is less than or equal to $2c$. That is to say also that there exists Goldbach partition primes within the interval (a, b).

Obtaining a quadratic inequality for solving the prime gap problem using the Andrica conjecture

The Andrica conjecture requires that

$$\sqrt{p}_{i+1} - \sqrt{p}_i < 1 \tag{38}$$

When we substitute (38) into (3) be obtain the quadratic inequality (39) below.

$$2(\sqrt{\sqrt{p_i p_{i+1}} + \frac{1}{4}}) > \sqrt{p}_{i+1} + \sqrt{p}_i \tag{39}$$

The gaps of inequality (13) are shorter that those of inequality (39) though comparable to those proposed in Crammer's conjecture. To achieve better results we will substitute the inequality

$$\sqrt{p}_{i+1} - \sqrt{p}_i < \sqrt{1}1 - \sqrt{7} \tag{40}$$

into (3) to obtain the quadratic inequality

$$2(\sqrt{\sqrt{p_i p_{i+1}} + (\frac{\sqrt{1}1 - \sqrt{7}}{2})^2}) \geq \sqrt{p}_{i+1} + \sqrt{p}_i \tag{41}$$

Thus the solution of

$$2(\sqrt{\sqrt{113x} + (\frac{\sqrt{1}1 - \sqrt{7}}{2})^2}) \geq \sqrt{x} + \sqrt{1}13$$

is

$99.1871054116999 \leq x \leq 127.713037038732$

The prime number after 113 is 127.

The solution of

$$2(\sqrt{\sqrt{23x} + (\frac{\sqrt{1}1 - \sqrt{7}}{2})^2}) \geq \sqrt{x} + \sqrt{2}3$$

is $17.0152788649411 \leq x \leq 29.8848635854904$

The prime number after 23 is 29. The solution of

$$2(\sqrt{\sqrt{1129x} + (\frac{\sqrt{1}1 - \sqrt{7}}{2})^2}) \geq \sqrt{x} + \sqrt{1}129$$

is

$1084.36657476504 \leq x \leq 1174.53356768539$

The prime number after 1129 is 1151. The disadvantage of formulation (41) above is that it cannot account for observed cases in which $g_i < p_i$.

A more thorough analysis of the prime gap problem

Consider the identity:

$$\sqrt{p_i p_{i+1}} + (\frac{\sqrt{p_{i+1}} - \sqrt{p}_i}{2})^2 = (\frac{\sqrt{p_{i+1}} + \sqrt{p}_i}{2})^2 \quad (42)$$

It can be reorganised to the form

$$\sqrt{p_i p_{i+1}}(\frac{\sqrt{p_{i+1}} - \sqrt{p}_i}{2})^2(\frac{1}{\sqrt{p_i p_{i+1}}} + (\frac{2}{\sqrt{p_{i+1}} - \sqrt{p}_i})^2) = (\frac{\sqrt{p_{i+1}} + \sqrt{p}_i}{2})^2 \quad (43)$$

and then simplified to

$$\sqrt{p_{i+1}} + \sqrt{p}_i = 2\sqrt{\sqrt{p_i p_{i+1}}(\frac{\sqrt{p_{i+1}} - \sqrt{p}_i}{2})^2(\frac{1}{\sqrt{p_i p_{i+1}}} + (\frac{2}{\sqrt{p_{i+1}} - \sqrt{p}_i})^2)} \quad (44)$$

$$\sqrt{p_{i+1}} + \sqrt{p}_i = 2\sqrt[4]{p_i p_{i+1}}(\sqrt{(\frac{\sqrt{p_{i+1}} - \sqrt{p}_i}{2})^2(\frac{1}{\sqrt{p_i p_{i+1}}} + (\frac{2}{\sqrt{p_{i+1}} - \sqrt{p}_i})^2)}) \quad (45)$$

Now

$$\sqrt{(\frac{\sqrt{p_{i+1}} - \sqrt{p}_i}{2})^2(\frac{1}{\sqrt{p_i p_{i+1}}} + (\frac{2}{\sqrt{p_{i+1}} - \sqrt{p}_i})^2)} > 1 \quad (46)$$

An exact prime gap relationship accounting for the various conjectures on prime gaps

Equation (45) means that

$$1 < \sqrt{(\frac{\sqrt{p_{i+1}} - \sqrt{p}_i}{2\sqrt[4]{p_i p_{i+1}}})^2 + 1} \leq 1 + (\frac{\sqrt{5} - \sqrt{3}}{2\sqrt[4]{15}})^2 \quad (47)$$

Therefore

$$\sqrt{p_{i+1}}+\sqrt{p}_i=2\sqrt[4]{p_ip_{i+1}}(\sqrt{(\frac{\sqrt{p_{i+1}}-\sqrt{p}_i}{2\sqrt[4]{p_ip_{i+1}}})^2+1})=\frac{g_i}{\sqrt{p}_{i+1}-\sqrt{p}_i} \tag{48}$$

Now the gap between consecutive primes is given by:

$$g_i=\sqrt{p_i\pm(2k_i-1)} \tag{49}$$

Therefore

$$\sqrt{p_{i+1}}+\sqrt{p}_i=2\sqrt[4]{p_ip_{i+1}}(\sqrt{(\frac{\sqrt{p_{i+1}}-\sqrt{p}_i}{2\sqrt[4]{p_ip_{i+1}}})^2+1})=\frac{\sqrt{p_1\pm(2k_i-1)}}{\sqrt{p}_{i+1}-\sqrt{p}_i} \tag{50}$$

From article reference [1] and (49)

$$2\sqrt{(}m^2-s_g)=\sqrt{p_i\pm(2k_i-1)} \tag{51}$$

This means that

$$s_g=m^2-\frac{\sqrt{p_i\pm(2k-1)}}{2} \tag{52}$$

For twin primes

$$g_i=\sqrt{p_i\pm(2k_i-1)}=2 \tag{53}$$

Applying Bertrand's postulate on maximum gaps it is noted that

$$g_i=\sqrt{p_i+(2k_i-1)}<p_i \tag{54}$$

This also means that

$$2k_i<p_i^2-p_i+1=p_i(p_i-1)+1 \tag{55}$$

Another observation is that

$$p_{i+1}=p_i+\sqrt{p_i\pm(2k_i-1)} \tag{56}$$

Thus $5=3+\sqrt{3+1}$

$29=23+\sqrt{23+13}$

$3=2+\sqrt{2-1}$

By Andrica conjecture

$$g_i=\sqrt{p_i\pm(2k_i-1)}<\sqrt{2p_i}-1 \tag{57}$$

This means that either

$$2k < (\sqrt{2p_i} - 1)^2 - p_i + 1 \tag{58}$$

or

$$2k > (\sqrt{2p_i} - 1)^2 - p_i + 1 \tag{59}$$

The Riemann hypothesis dimension of the prime gap problem

From equation (49) we established that

$$g_i^2 = p_i \pi (2k_i - 1) \tag{60}$$

The Riemann Zeta function can therefore be written in rhe form:

$$\zeta(s) = \frac{\ln(-\sqrt{g_i^2})}{\ln g_i^2} = \frac{\ln(-1) + \ln g_i}{2 \ln g_i} = \frac{1}{2} + i \frac{\pi}{2 \ln g_i} \tag{61}$$

Thus the Riemann Hypothesis proving or disproving of the Riemann hypothesis will in a sense contribute to our understanding better the prime gap problem.

In the paper reference [2] it was shown that the Riemann zeta funcrion can also be written in the form

$$\zeta(s) = \frac{\ln(-\sqrt[n]{g_i^n})}{\ln g_i^n} = \frac{\ln(-1) + \ln g_i}{n \ln g_i} = \frac{1}{n} + i \frac{\pi}{n \ln g_i} \tag{62}$$

Again

$$\zeta(s) = \frac{\ln(-\sqrt[n]{g_i^{\frac{1}{m}}})}{\ln g_i^n} = \frac{nm \ln(-1) + \ln g_i}{n^2 m \ln g_i} = \frac{1}{n^2 m} + i \frac{\pi}{n \ln g_i} \tag{63}$$

In the above form nontrivial zeroes can be outside the critical strip and therefore the Riemann hypothesis was disproved as was shown in paperreference [2]

Relative size of a gap

For the purpose of this research we introduce the concept of relative size of a prime gap.

Definition: Relative size of a prime gap The relative size of a prime gap is defined as the ratio of the gap between consecutive primes to the squareroot of the smallest prime making the gap, that is:

$$r_i = \frac{g_i}{\sqrt{p}_i} = \frac{\sqrt{p_i \pm (2k_i - 1)}}{\sqrt{p}_i} = \sqrt{1 \pm \frac{2k_i - 1}{p_i}} \wedge 2k_i - 1 < p_i \tag{64}$$

A gap is of a large relative size if $r_i > 1$ otherwise its relative size is small. A prime number has a large relative gap if

$$r = \frac{g_i}{\sqrt{p}_i} = \frac{\sqrt{p_i + (2k_i - 1)}}{\sqrt{p}_i} = \sqrt{1 + \frac{2k_i - 1}{p_i}} \wedge 2k_i - 1 < p_i \tag{65}$$

It has a small relative gap if

$$r_i = \frac{g_i}{\sqrt{p}_i} = \frac{\sqrt{p_i - (2k_i - 1)}}{\sqrt{p}_i} = \sqrt{1 - \frac{2k_i - 1}{p_i}} \wedge 2k_i - 1 < p_i \tag{66}$$

A large prime gap may have a snall relative prime gap. On the other hand a small prime gap may have a small prime gap. For example

$5 = 3 + \sqrt{3+1} = 5$. Now the prime gap is small that is 2. However $r = \sqrt{1 + \frac{1}{3}} = \sqrt{\frac{5}{3}} > 1$. The gap is small but it has a large relative prime gap.

On the other hand $97 = 89 + \sqrt{89 - 25}$. The gap is 8 but the relative gap is $\sqrt{1 - \frac{25}{89}} = \sqrt{\frac{64}{89}}$. The relative prime gap. For the prime 113 the prime gap is $\sqrt{\frac{144}{113}}$
In general as prime numbers become big it reaches a point where the square of the prime gap ;g_i, becomes less than the prime, p_i, meaning the relative prime gap ratio becomes small.

Safely speaking

$$r_i = \frac{g_i}{\sqrt{p}_j} > \frac{3(\ln p_i)^2}{\sqrt{p}_i} \tag{67}$$

By the above inequality at most primes greater 4, 400,000 have a small relative prime gap ratio and are therefore subject to the inequality

$g_i < \sqrt{p}_i$. The relative prime gap can also be determined by the inequality by solution of equation (13)

+

$$r > \frac{(\sqrt[4]{p}_i(1.05^{\frac{2}{p_i}}) + \sqrt{(\sqrt[4]{p}_i(1.05^{\frac{1}{p_i}})^2 - \sqrt{p}_i)^4 - p_i}}{\sqrt{p}_i} \tag{68}$$

The relative gap ratio inequality suddenly falls to zero for primes greater than 10^24. It does not properly accurately predict properly the relative gap ratio of the very big primes. Inequality (67) succeeds in primes confirming Opperman's conjecture for primes bigger than 4, 400, 000. Now the limits of r are

$$\frac{2}{\sqrt{p}_i} \leq r_i \leq \frac{4}{\sqrt{7}} \wedge p_i > 2 \tag{69}$$

This means effectual means

$$2 \leq g_i \leq 4\sqrt{\frac{p_i}{7}} \tag{70}$$

Note here the gap between 7 and 11 is special. When it substituted into $g_i = \sqrt{p_i + 2k - 1}$ is is the the only prime number in whicb $2k - 1 > p_i$.

Take note that $11 = 7 + \sqrt{7 + 9}$. This means $2k_i - 1 = 1$. Therefor r_i of the gap between 7 and 11 forms the outermost limit of the interval of r_i. This effectvely means that the prime gap lies in the intervals This means that in the most general sense

$$1 \geq g_i \leq 4\sqrt{\frac{p_i}{7}} \tag{71}$$

This effectvely means that

$$g_j \leq 1.51185789203691\sqrt{p}_i \tag{72}$$

By the above inequality the gap between 113 and the next prime number is less than 16. The gap given in inequality (72) is shorter than that suggested by Baker, Haman and Pintz. Opperman's conjecture still needs a proof because the above result does not touch it. The gap inequality above means that both Legendre and Andrica conjectures are true.

Extension of the relative prime gap ratio to the binary Goldbach conjecture

We can define the relative gap ratio for Goldbach partition as the ratio of the gaps of Goldbach parition primes to the Goldbach partition composite even number. That is to say

$$r = \frac{g_i}{2m} = \frac{2\sqrt{m^2 - s_g}}{2m} = \sqrt{1 - \frac{s_g}{m^2}} \leq \sqrt{1 - \frac{3(2m-3)}{m^2}} \wedge 0 \geq r < 1 \tag{73}$$

Thus the Goldbach partition relative prime gap ratio is dependent on the ratio $\frac{s_g}{m^2}$. The larger it is the smaller the relative prime gap ratio. One composite even number can generate several prime gap ratios. Semiprime even numbers have one of their relative prime gap ratios equal to 0. Now take note on how semiprime even numbers are generated:

$$p_2 + p_1 + g_{2,1} = 2p_2 \tag{74}$$

again

$$p_2 + p_1 - g_{2,1} = 2p_1 \tag{75}$$

If we set $2m = p_2 + p_1$ Then it is true that for every composite even number there exists some gap $g_{2,1} \geq 0$ such

$$2m + g_{2,1} = 2p_2 \tag{76}$$

$$2m - g_{2,1} = 2p_1 \tag{77}$$

in which case $2m = p_1 + p_2$.

Conclusion

The gap between consecutive primes is given by

$$g_i = \sqrt{p_i \pm (2k_i - 1)} \geq 1 \tag{78}$$

The Legendre, Andrica, Crammer and Opperman's postulate are true. The Binary Goldbach conjecture is true. The above gap equation accounts for many of the conjectures on prime gaps.

Proof of Oppermann's conjecture

A generally accepted quotient in number theory is

$$\lim i \to \infty \frac{g_i}{p_i} = 0 \tag{79}$$

Oppermann's conjecture implies that when p_i is big then

$$\frac{g_i}{p_i} < \frac{1}{\sqrt{p}_i} < 0 \tag{80}$$

This the above inequality lies within the limits of accepted number theory and effectually means that for big prime numbers

$$p_i < \sqrt{p}_i \tag{81}$$

Thus Oppermann's conjecture is true.

References

[1]Bonaya, Samuel Buya, Confirming Buya's and Bezaleel's proof of the Binary Goldbach conjecture using Bertrand's postulate [Version 2] (May 11, 2024). Available at SSRN: https://ssrn.com/abstract=4828468 or http://dx.doi.org/10.2139/ssrn.4828468

[2] Samuel Bonaya Buya and John Bezaleel Nchima (2024). A Nessary and Sufficient Condition for Proof of the Binary Goldbach Conjecture. Proofs of Binary Goldbach, Andrica and Legendre Conjectures. Notes on the Riemann Hypothesis. International Journal of Pure and Applied Mathematics Research, 4(1), 12-27. doi: 10.51483/IJPAMR.4.1.2024.12-27.

[3] Cramér, Harald (1936). "On the order of magnitude of the difference between consecutive prime numbers". Acta Arithmetica. 2: 23–46. doi:10.4064/aa-2-1-23-46.

[4] Ingham, Albert E. (1937). "On the difference between consecutive primes" (PDF). Quarterly Journal of Mathematics. 8 (1). Oxford: 255–266. Bibcode:1937QJMat...8..255I

[5]Baker, R. C.; Harman, G.; Pintz, J. (2001). "The difference between consecutive primes, II" (PDF). Proceedings of the London Mathematical Society. 83 (3): 532–562.

[6] Hoheisel, G. (1930). "Primzahlprobleme in der Analysis". Sitzunsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin. 33: 3–11. JFM 56.0172.02.