Proof of Collatz Conjecture Using Division Sequence V

Masashi Furuta

Abstract

This paper is positioned as a sequel edition of [1]. First, as in [1], define "division sequence", "complete division sequence", "star conversion", and "extended star conversion". Next, applying tail of division sequence to prove the Collatz conjecture. Also, Theorem Proving is not used in this paper.

Keywords

Collatz conjecture, Division sequence, Tail

1 Introduction

1.1 Collatz Conjecture

The Collatz conjecture poses the question: "What happens if one repeats the operations of taking any positive integer n,

- Divide n by 2 if n is even, and
- Multiply n by 3 and then add 1 if n is odd

The Collatz conjecture affirms that "for any initial value, one always reaches 1 (and enters a loop of 1 to 4 to 2 to 1) in a finite number of operations."
We call "**(one) Collatz operation**" an operation of performing (3x+1) on an odd number and dividing by 2 as many times as one can.
The "**initial value**" is the number on which the Collatz operation is performed. This initial value is called the "**Collatz value**."

1.2 Division Sequence and Complete Division Sequence

Definition 1.1 A division sequence is the sequence given by arranging the numbers of division by 2 in each operation when the Collatz operation is continuously performed with a positive odd number, n, as the initial value.
For example, in the case of 9, the arrangement of numbers given by continuously performing 3x + 1, and dividing by 2 provides
9,28,14,7,22,11,34,17,52,26,13,40,20,10,5,16,8,4,2,1 (stops when 1 is reached).
Therefore, the division sequence of 9 is [2,1,1,2,3,4].
The division sequence of 1 is an empty list []. Further, [6] is a division sequence of 21, but [6,2] and [6,2,2] ... that repeat the loop of 1 to 4 to 2 to 1 are not division sequences.
When the division sequence is finite, it is equivalent to reaching 1 in a series of Collatz operations.
When the division sequence is infinite, it does not reach 1 in a series of Collatz operations.
It is equivalent to entering a loop other than 4-2-1 or increasing the Collatz value endlessly.
Definition 1.2 A complete division sequence is a division sequence of multiples of 3.

- 9[2,1,1,2,3,4] is a complete division sequence of 9.
- 7[1,1,2,3,4] is a division sequence of 7.

Definition 1.3 Supposing that only one element exists in the division sequence of n, no Collatz operation can be applied to n.

Theorem 1.1 When the Collatz operation is applied to x in the complete division sequence of x (two or more elements), (some) y and its division sequence are obtained.
Proof: This follows the Collatz operation and definition of a division sequence. □
Theorem 1.2 When the Collatz operation is applied to y in the division sequence of y (two or more elements), (some) y and its division sequence are obtained.
Proof: It is self-evident from the Collatz operation and definition of a division sequence. □

1.3 One Only Looks at Odd Numbers of Multiples of 3

There is no need to look at even numbers.
By continuing to divide all even numbers by 2, one of the odd numbers is achieved.
Therefore, it is only necessary to check "whether all odd numbers reach 1 by the Collatz operation."
One only needs to look at multiples of 3.
For a number x that is not divisible by 3, the Collatz inverse operation is defined as obtaining a positive integer by $(x \times 2^k - 1)/3$. Multiple numbers can be obtained using the Collatz reverse operation.
Here, we consider the Collatz reverse operation on x.
The remainder of dividing x by 9 is one of 1,2,4,5,7,8, i.e.:
$1 \times 2^6 \equiv 1$
$2 \times 2^5 \equiv 1$
$4 \times 2^4 \equiv 1$
$5 \times 2^1 \equiv 1$
$7 \times 2^2 \equiv 1$
$8 \times 2^3 \equiv 1 \pmod 9$
This indicates that multiplying any number by 2 appropriate number of times provides an even number with a reminder of 1 when divided by 9.
By subtracting 1 from this and dividing by 3, we get an odd number that is a multiple of 3.
Performing the Collatz reverse operation once from x provides an odd number y that is a multiple of 3.
If y reaches 1, then x, which was once given by the Collatz operation of y, also reaches 1. Therefore, the following can be stated.
Theorem 1.3 One only needs to check "whether an odd number that is a multiple of 3 reaches 1 by the Collatz operation." □

2 Star Conversion

A star conversion is defined for a complete division sequence.
A complete division sequence of length, n, is copied to a complete division sequence of length, n or n+1.
The remainder, which is given by dividing the Collatz value x by 9 is
$x \equiv 3 \bmod 9$
The conversion to copy a finite or infinite sequence $[a_1, a_2, a_3 \ldots]$
to a sequence $[6, a_1 - 4, a_2, a_3 \ldots]$ is described as A [6, −4].
The conversion to copy a finite or infinite sequence $[a_1, a_2, a_3 \ldots]$
to a sequence $[1, a_1 - 2, a_2, a_3 \ldots]$ is described as B [1, −2].
$x \equiv 6 \bmod 9$
The conversion to copy a finite or infinite sequence $[a_1, a_2, a_3 \ldots]$
to a sequence $[4, a_1 - 4, a_2, a_3 \ldots]$ is described as C [4, −4].
The conversion to copy a finite or infinite sequence $[a_1, a_2, a_3 \ldots]$
to a sequence $[3, a_1 - 2, a_2, a_3 \ldots]$ is described as D [3, −2].
$x \equiv 0 \bmod 9$
The conversion to copy a finite or infinite sequence $[a_1, a_2, a_3 \ldots]$
to a sequence $[2, a_1 - 4, a_2, a_3 \ldots]$ is described as E [2, −4].
The conversion to copy a finite or infinite sequence $[a_1, a_2, a_3 \ldots]$
to a sequence $[5, a_1 - 2, a_2, a_3 \ldots]$ is described as F [5, −2].
Furthermore, the conversion to copy a finite or infinite sequence $[a_1, a_2, a_3 \ldots]$
to a sequence $[a_1 + 6, a_2, a_3 \ldots]$ is described as G [+6].
Conversions in which the elements of the division sequence are 0 or negative are prohibited.
If the original first term is 0 or negative, G [+6] is performed in advance.
Example
$117 \equiv 0 \pmod 9$, 117[5,1,2,3,4]
can be converted to E [2, −4] → 9[2, 5−4, 1, 2, 3, 4] and F [5, −2] → 309[5, 5−2, 1, 2, 3, 4].
Table 1 shows the functions corresponding to each star conversion.
The function represents a change in the Collatz value.

Table 1. Star conversion in mod 9.

When	star conversion 1	star conversion 2
$x \equiv 3 \bmod 9$	A [6, −4] $y = 4x/3 - 7$	B [1, −2] $y = x/6 - 1/2$
$x \equiv 6 \bmod 9$	C [4, −4] $y = x/3 - 2$	D [3, −2] $y = 2x/3 - 1$
$x \equiv 0 \bmod 9$	E [2, −4] $y = x/12 - 3/4$	F [5, −2] $y = 8x/3 - 3$
Always	G [+6] $y = 64x + 21$	none

The star conversion A for 21[6] replaces [6]-A->[6, 2] with [6]-A->[6]. The Collatz value is 21, and it does not change.

3 Extended Star Conversion and Extended Complete Division Sequence

Definition 3.1 The extended star conversion is the conversion in which the star conversion is applied multiple times to the complete division sequence of x excluding the Collatz value x of 3, 9. Table 2 shows the extended star conversion.

Table 2. Extended star conversion and got smaller.

No.	when	Extended star conversion	After conversion	Got smaller
	0 mod 9			
1	9	None because of the base case		
2	72t+45	E[2, −4] $y = x/12 - 3/4$	6t+3	$72t + 45 > 6t + 3$
3	216t+81	DE[3, 0, −4] $y = x/18 - 3/2$	12t+3	$216t + 81 > 12t + 3$
4	216t+153	AE[6, −2, −4] $y = x/9 - 8$	24t+9	$216t + 153 > 24t + 9$
5	216t+225	FE[5, 0, −4] $y = 2x/9 - 5$	48t+45	$216t + 225 > 48t + 45$
6	108t+27	CF[4, 1, −2] $y = 8x/9 - 3$	96t+21	$108t + 27 > 96t + 21$
7	108t+63	BF[1, 3, −2] $y = 4x/9 - 1$	48t+27	$108t + 63 > 48t + 27$
8	108t+99	EF[2, 1, −2] $y = 2x/9 - 1$	24t+21	$108t + 99 > 24t + 21$
	6 mod 9			
9	18t+15	C[4, −4] $y = x/3 - 2$	6t+3	$18t + 15 > 6t + 3$
	3 mod 9			
10	3	None because of the base case		
11	36t+21	B[1, −2] $y = x/6 - 1/2$	6t+3	$36t + 21 > 6t + 3$
12	108t+39	DB[3, −1, −2] $y = x/9 - 4/3$	12t+3	$108t + 39 > 12t + 3$
13	108t+75	AB[6, −3, −2] $y = 2x/9 - 23/3$	24t+9	$108t + 75 > 24t + 9$
14	108t+111	FB[5, −1, −2] $y = 4x/9 - 13/3$	48t+45	$108t + 111 > 48t + 45$

The extended star conversion copies the initial value from $6t + 3$ to $6t' + 3$.

Definition 3.2 The extended complete division sequence is the division sequence obtained by performing the extended star conversion.

Elements of the extended complete division sequence can contain 0 or negative values.

In addition, in [1] we prove that the Collatz conjecture is true if every complete divison sequence has finite terms.

4 Tail of Divison Sequence

4.1 Tail of Division Sequence

Whether a Collatz value reaches 1 or not depends on whether the divison sequence has finite or infinite terms, so we call this the **tail** of divison sequence.
Also, when two divison sequences are both finite or both infinite terms, we call the **tails of two divison sequences equal**.

4.2 One Theorem

Theorem 4.1 The tails of all 6t+3 complete divison sequences(a) and extended complete divison sequences(b) are equal.

proof: Prove it by induction. Base case, The complete divison sequence of 3 and 9 and the extended complete divison sequence have the same tails. Inductive case, When a Collatz value x is subjected to the extended star conversion, the Collatz value y<x.

Since y is less than x, by the induction hypothesis, the tails of the comlete divison sequence of y and the extended complete divison sequence(a and b) are equal. (The key point is that applying the extended star conversion multiple times does not affect the tail.)

As a result, the tails of the complete divison sequence(a) of the original Collatz value x and the extended complete divison sequence(b) are also equal. □

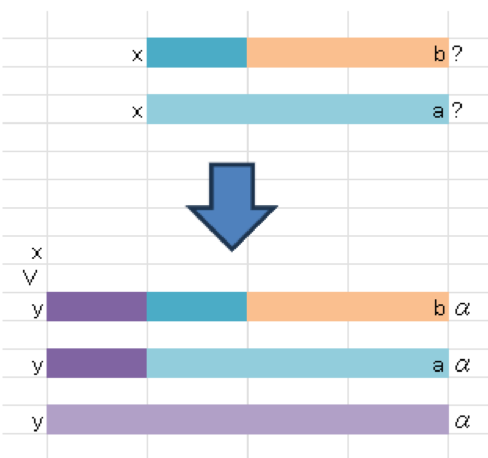

Fig 1. How Theorem 4.1 works

5 Proof of Final Theorem

Theorem 5.1 Collatz conjecture is true.

proof: By chaining extended star conversion, the tails of all extended complete divison sequences become equal to the tails of the complete divison sequences with Collatz values 3,9, which is the base case.

Since the tails of the complete divison sequence and the extended complete divison sequence are the same, the tails of all complete divison sequences are the same.

Since the base case reaches 1, it is proven that all 6t+3 and all Collatz values reach 1. □

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References

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